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ANALYTICAL SHOOTING TECHNIQUE FOR THE SOLUTION OF TWO POINT NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract

In this paper, an analytical approach was used in the scheme of shooting technique to solve two point boundary value problems. The analytical method used was Adomian decomposition method in place of the usual numerical methods which are prone to discretization errors. Boundary value problems were simplified into initial value problems by the technique of shooting and the method of Adomian decomposition was applied to the initial value problems. The slope of the initial condition (t_0) is calculated as the initial shooting angle which was updated as many times as possible by the secant and Newton's method. The updated **slope (t_k) is repeated in the process of shooting until the result is closed enough to hitting the target.**

***KEYWORDS*:** Shooting angle, Adomian decomposition method, Numerical Solution, Approximate Analytical Solution, Newton's Method, Secant Method.

1. Introduction

Ordinary differential equation (ODE) frequently occurs in initial value problems (IVP's) and boundary value problem (BVP's). This usually occurs in mathematical models that arise in many branches of science, engineering, and economics with the specified value called the initial condition for IVP and boundary conditions for BVP. Ordinary differential equations may be classified into two large classes: linear ordinary differential equations and nonlinear ordinary differential equations [1].

A few nonlinear differential equations have known exact solutions, but many which are important in application do not. For this reason, it calls for an approximate solution to be derived from a numerical method, and sometimes these equations may be linearized. Then be solved using a numerical approach.

Two-point boundary value problems have gained the attention of scientists and researchers in recent years, owing to this, several techniques have been developed to obtain a solution to two-point boundary value problems for example in [1] to [6] Euler, Runge-Kutta, or Taylor has been used in the scheme of shooting technique which gives a discretized solution that is prone to discretization error as well its accuracy depending greatly on the step size chosen.

In recent year Adomian decomposition method has been an analytical approach applied by researcher like [7]-[9] in solving different equation form.

This work is aimed at using an analytical approach in the scheme of shooting technique to solve boundary value problems, by applying shooting technique [10] to reduce boundary value problems (B.V.P.) to initial value problems (I.V.P.) and applying the Adomian decomposition method (ADM) to solve I.V.P and the shooting angle is updated by both secant's and Newton's methods.

2. Overview Of Shooting Technique For Nonlinear Differential Equation

Consider the differential equation of the form

$$y'' = f(x, y, y'), \qquad a \le x \le b \tag{1}$$

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With the boundary conditions $y(a)=\alpha, \quad y(b)=\beta$
the procedure of shooting technique discussed in [10], equation (1) alongside the boundary conditions is simplified into;

$$y'' = f(x, y(x,t), y'(x,t)) \quad a \le x \le b \tag{2}$$

With the initial conditions $y(a)=\alpha, \quad y'(a)=t_k$
And

$$z''(x,t) = \frac{\partial f}{\partial y}(x,y,y')z(x,t) + \frac{\partial f}{\partial y'}(x,y,y')z'(x,t) \tag{3}$$

With initial conditions $z(a,t)=0, \quad z'(a,t)=1$
Where a and α are known from the original problem and t_k which is the shooting angle for $k=0,1,2,3,...$ are calculated from either secant or Newton's method. If secant method will be applied to adjust the slope t_k in equation (2) then;

$$t_0 = \frac{\beta-\alpha}{b-a} \tag{4}$$

$$t_1 = t_0 + \frac{\beta - y(b,t_0)}{b-a} \tag{5}$$

$$t_k = t_{k-1} - \frac{(y(b,t_{k-1})-\beta)(t_{k-1}-t_{k-2})}{y(b,t_{k-1})-y(b,t_{k-2})}, \quad k \ge 2 \tag{6}$$

Such that equation (2) is solved repeatedly until the desired result is achieved.
An alternative to the secant is Newton's method with $t_0 = \dfrac{\beta-\alpha}{b-a}$
And

$$t_k = t_{k-1} - \frac{y(b,t_{k-1})-\beta}{z(b,t_{k-1})}, \qquad k \ge 1 \tag{7}$$

When using secant method to determine the slope, only equation (2) is required to be solved and when using Newton's method both equations (2) and (3) will solved simultaneously, the result derived from equations (2) and (3) can be combined with equation (8) given below to form a solution for equation (1).

$$Y(x) = y(x) + \frac{\beta - y(b)}{z(b)} \cdot z(x) \tag{8}$$

3. Adomian Decomposition Method

Consider the differential equation of the form

$$Ly + Ry + Ny = g(x). \tag{9}$$

Where

L is the linear operator which is the highest order derivative that is , $L = \dfrac{d^n}{dx^n}$,

R is the remainder of the linear operator of order less than L ,
N is the nonlinear term and
g is the source term.

$$y(x) = L^{-1}(g(x)) - L^{-1}(Ry) - L^{-1}(Ny) \tag{10}$$

Where L^{-1} is n- fold definite integration from [0 to x]. If L it is a second-order operator, then L^{-1} is a twofold integral hence equation (2) gives

$$y(x) = \psi_0 + g(x) - L^{-1}(Ry) - L^{-1}(Ny) \tag{11}$$

The non-linear operator Ny is represented by an infinite series of specifically generated Adomian polynomials for the specific nonlinear part present in the equation. Assuming Ny is analytic. Where

$$\psi_0 = \begin{cases} y(0), & for\ \ L = \dfrac{d}{dx} \\ y(0) + xy' & for\ \ L = \dfrac{d^2}{dx^2} \\ y(0) + xy'(0) + \dfrac{1}{2!}x^2 y''(0) & for\ \ L = \dfrac{d^3}{dx^3} \\ y(0) + xy'(0) + \dfrac{1}{2!}x^2 y''(0) & \\ \quad + \dfrac{1}{3!}x^3 y'''(0) + \cdots + & \\ \dfrac{1}{(n-1)!}x^{n-1}\dfrac{d^{n-1}y}{dx^{n-1}}(0), & for\ \ L = \dfrac{d^n}{dx^n} \end{cases} \tag{12}$$

And decomposing the nonlinear operator Ny into a series, that is

$$Ny = \sum_{k=0}^{\infty} A_k \tag{13}$$

A_k is generated by a formula in equation(14) for all kinds of nonlinearity so that they depend only on A_0 and A_k [11] and [12]

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N\left(\sum_{n=0}^{k} \lambda^n y_n \right) \right]_{\lambda=0} \tag{14}$$

Thus (11) becomes

$$\sum_{n=0}^{\infty} y_n(x) = \psi_0 + L^{-1}g(x) - L^{-1}R(\sum_{k=0}^{\infty} y_k) - L^{-1}(\sum_{k=0}^{\infty} A_k) \tag{15}$$

Thus, the solution to the given problem can be written in an infinite series as

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{16}$$

4 NUMERICAL EXAMPLE

Example 1: Consider the ordinary differential equation

$$y'' = 2y^2 + 4xyy', \qquad y(0) = \frac{1}{4}, \quad y(1) = \frac{1}{3} \tag{17}$$

This problem has been solved by [1] using Runge-Kutta of order 4 method, this same problem will be solved using the proposed method.

The two IVP's are:

$$y'' = 2y^2 + 4xyy', \qquad y(0) = \frac{1}{4}, \quad y'(0) = t_k \tag{18}$$

$$z'' = 4yz + 4xy'z + 4xyz' \quad z(0) = 0, \ \ z'(0) = 1 \tag{19}$$

Solving (18) using Adomian decomposition method and applying Newton's method to update the slope with the first iteration, $t_0 = \frac{1}{12}$; This gives

$$y(x) = 0.25 + 0.08333333333x + 0.0625x^2 + 0.02777777777x^3 - 0.01909722222x^4 + 0.009722222220x^5 + 0.006028163580x^6 + 0.003290343915x^7 + 0.0009717399689x^8 + 0.0001398278790x^9 + 0.000007922729275x^{10} \quad (20)$$

Solution of equation (19) also with Adomian decomposition method gives

$$z(x) = x + 0.3333333333x^3 + 0.08333333332x^4 + 0.1166666667x^5 + 0.05092592592x^6 + 0.04345238095x^7 + 0.02332175925x^8 + 0.005033803643x^9 + 0.0003802910052x^{10} \quad (21)$$

So that the approximate analytical solution at first iterate t_0 is;

$$y = 0.25 + 0.00513271377x + 0.0625x^2 + 0.00171090459x^3 + 0.01258050392x^4 + 0.0005988166x^5 + 0.00204572462x^6 - 0.00010765919x^7 - 0.00085203605x^8 - 0.00025381868x^9 - 0.00002181626x^{10} \quad (22)$$

Since $y(b, t_0) = 0.4628685536$ which is a value greater than the target ($\beta = 0.3333333333$), then there is need to update the shooting angle, from t_0 to t_1. t_1 is calculated from equation (7) $t_1 = 0.00513271377$, repeating the process of Adomian decomposition method the approximate analytical solution as

$$y = 0.25 + 0.000886382994x + 0.0625x^2 + 0.000295460998x^3 + 0.01561637718x^4 + 0.0001034113491x^5 + 0.003900980496x^6 + 0.000033182109x^7 - 0.0000024131934x^8 \quad (23)$$

The third and fourth iterates when $t_2 = 0.000886382994$ and $t_3 = 0.0008748634797$ are given in (24) and (25) respectively,

$$y = 0.25 + 0.000874863479x + 0.0625x^2 + 0.0002916211599x^3 + 0.01562538263x^4 + 0.000102067406x^5 + 0.00390648382x^6 + 0.00003280757170x^7 \quad (24)$$

$$y = 0.25 + 0.000874863278x + 0.0625x^2 + 0.0002916210927x^3 + 0.01562538269x^4 + 0.000102067382x^5 + 0.003906483868x^6 + 0.00003280756427x^7 \quad (25)$$

Solving equation (18) with the same approach while secant method is used to update the slope, the analytical result at first, second, third, fourth and fifth iterate are as follows; s

$$y = \frac{1}{4} + \frac{x}{12} + \frac{x^2}{16} + \frac{x^3}{36} + \frac{11x^{\wedge 4}}{576} + \frac{7x^5}{720} + \frac{125x^6}{20736} + \frac{199x^7}{60480} + \frac{403x^8}{414720} + \frac{137x^9}{979776} + \frac{23x^{10}}{2903040} \quad (27)$$

$$y = 0.25 - 0.04620187x + 0.0625x^2 - 0.015401x^3 + 0.0166923x^4 - 0.0053902x^5 + 0.0045585x^6 - 0.00117607x^7 + 0.000298698x^8 - 0.00002383x^9 + 0.0000007486x^{10} \quad (28)$$

$$y = 0.25 - 0.0015844x + 0.065x^2 - 0.00053x^3 + 0.01562623x^4 - 0.0001848x^5 + 0.00391x^6 - 0.00005941633x^7 + 0.000000351273x^8 \tag{29}$$

$$y = 0.25 + 0.000948x + 0.0625x^2 + 0.000316x^3 + 0.0156255x^4 + 0.0001107x^5 + 0.0039065x^6 + 0.000035578x^7 + 0.000000125953x^8 \tag{30}$$

$$y = 0.25 + 0.000945x + 0.0625x^2 + 0.000315x^3 + 0.0156254x^4 + 0.0001102x^5 + 0.00039065x^6 + 0.000035425x^7 + 0.0000012487x^8 \tag{31}$$

Example 2: Consider the Troesch's problem

$$y'' = \lambda \sinh(\lambda y), \qquad 0 \le x \le 1 \tag{32}$$

With boundary conditions $y(0) = 0, y(1) = 1.$

Troesch's problem has been in existence for a long period as a result of an investigation carried out on the confinement of a plasma column under radiation pressure [13]. To solve equation (32) with the proposed method involves simplifying (32) to two IVP's

The two IVP's are:

$$y'' = \lambda \sinh(\lambda y), \quad y(0) = 0, \quad y'(0) = t_k \tag{33}$$

and

$$z'' = \lambda^2 z \cosh(\lambda y), \quad z(0) = 0, \quad z'(0) = 1 \tag{34}$$

The first iterate of the approximate analytical solution for Troesch's problem (32) when $\lambda = 0.5$ Newton's method gives

$$y_0 = 0.065130 - 5.590824x - 0.021212x^3 + 22.699470\sinh(0.5x) - 6.11134x\cosh(0.5x) - 0.946543\cosh(0.5)^2 x + \cdots \tag{35}$$

The approximate solution for the second, third, and fourth iterates are also calculated respectively;

$$y_1 = 0.001567 - 5.686746x - 0.022198x^3 + 23.208420\sinh(0.4896x) - 0.001378\cosh(0.4896x) - 0.837578x\cosh(0.4896x)^2 - 6.076601x\cosh(0.4896x) + \cdots \tag{36}$$

$$y_2 = 0.000032 - 5.688004x - 0.02222x^3 + 23.216071\sinh(0.4894x) - 0.000028\cosh(0.4894x) - 0.834822x\cosh(0.4894x)^2 - 6.074969x\cosh(0.4894x) + \cdots \tag{37}$$

$$y_3 = 0.0000007 - 5.68803x - 0.02222x^3 + 23.216228\sinh(0.4894x) - 0.0000006\cosh(0.4894x) - 0.834764x\cosh(0.4894x)^2 - 6.074935x\cosh(0.4894x) + \cdots \tag{38}$$

Solving the same problem using secant method to determine the slope required only equation (33), the result of the solution in polynomial at different shooting angle when $\lambda = 0.5$ as follows

$$y_0 = -5.083333x - 0.020833x^3 + 20.66667\sinh(0.5x) - 5.5x\cosh(0.5x) - 0.75x\cosh(0.5x)^2 + \cdots \tag{39}$$

$$y_1 = -5.7023x - 0.022248x^3 + 23.277183\sinh(0.4892x) - 6.088549x\cosh(0.4892x) - 0.836776x\cosh(0.4892x)^2 + \cdots \tag{40}$$

$$y_2 = -5.688030x - 0.0222157x^3 + 23.216232\sinh(0.4894x) - 6.074934x\cosh(0.4894x) - 0.834763x\cosh(0.4894x)^2 + \cdots \tag{41}$$

$$y_3 = -5.688030x - 0.022216x^3 + 23.216230\sinh(0.4849x) - 6.074933x\cosh(0.4894x) - 0.834763x\cosh(0.4894x)^2 + \cdots \quad (42)$$

Example 3: consider the nonlinear boundary value problem given in [14] which is

$$y'' = -2yy', \quad y(0) = 1, \quad y(1) = \frac{1}{2} \quad (43)$$

Simplifying equation (43) into IVP's gives:

$$y'' = -2yy' \quad y(0) = 1, \; y'(0) = t_k \quad (44)$$

And

$$z'' = -2y'z - 2yz' \qquad z(0), \;\; z'(0) = 1 \quad (45)$$

The solution to the problem at different shooting angles when using Newton's method are:

$$y_0 = 1 - 0.951513x + 0.951513x^2 - 0.868180x^3 + 0.784846x^4 - 0.576261x^5 + 0.175151x^6 - 0.015558x^7 \quad (46)$$

$$y_1 = 1 - 0.930467x + 0.930467x^2 - 0.908753x^3 + 0.887039x^4 - 0.741814x^5 + 0.303851x^6 - 0.040324x^7 \quad (47)$$

$$y_2 = 1 - 0.930469x + 0.930469x^2 - 0.908904x^3 + 0.887339x^4 - 0.742311x^5 + 0.304329x^6 - 0.040453x^7 \quad (48)$$

$$y_3 = 1 - 0.930469x + 0.930469x^2 - 0.908904x^3 + 0.887339x^4 - 0.742311x^5 + 0.304329x^6 - 0.040453x^7 \quad (49)$$

The equation was also solved with the secant method, the following approximate analytical solution are gotten at first, second, third, and fourth iterate as:

$$y_0 = 1 - \frac{1}{2}x - \frac{1}{2}x^2 - \frac{5}{12}x^3 - \frac{1}{3}x^4 - \frac{1}{5}x^5 + \frac{1}{360}x^6 - \frac{17}{360}x^7 \quad (50)$$

$$y_1 = 1 + 021032x - 0.021032x^2 + 0.013874x^3 - 0.006716x^4 - 0.000323x^5 + 0.000004x^6 \quad (51)$$

$$y_2 = 1 - 0.939637x + 0.939637x^2 - 0.920730x^3 + 0.901824x^4 - 0.758089x^5 + 0.313413x^6 - 0.042071x^7 \quad (52)$$

$$y_3 = 1 - 0.9290394x + 0.929039x^2 - 0.907064x^3 + 0.885089x^4 - 0.739866x^5 + 0.302928x^6 - 0.040205x^7 \quad (53)$$

5 RESULTS

This section shows differences between the exact solution and approximate numerical solution of each of the problems solved with Newton's and secant methods and the tolerance value between each successive shooting angle.

Table 1a: Difference between exact solution and Adomian decomposition with Newton's method for Example one

x	Error at t_0	Error at t_1	Error at t_2	Error at t_3
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.0010354251	0.0001796572	0.0001773369	0.0001773369
0.4	0.0020816852	0.0003736654	0.0003690524	0.0003690522
0.6	0.0029763371	0.0005851905	0.0005788011	0.0005788010
0.8	0.0030466404	0.0007008547	0.0006947148	0.0006947147
1.0	0.0000000001	0.0000000001	0.0000000000	0.0000000000

Table 1b: Difference between exact solution and Adomian decomposition with Secant method for Example 1

X	Error at t_0	Error at t_1	Error at t_2	Error at t_3	Error at t_4
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.0168977336	0.0093635821	0.0003211666	0.0001923124	0.0001914837
0.4	0.0353136455	0.0194949581	0.0006701841	0.0004002740	0.0003985462
0.6	0.0573968624	0.0313605904	0.0010985776	0.0006292057	0.0006264163
0.8	0.0867307723	0.0465220563	0.0018052343	0.0007698724	0.0007657134
1.0	0.1295352203	0.0680604008	0.0036565147	0.0001100209	0.0001039322

Table 2a: Difference between exact solution and Adomian decomposition with Newton's method for Example 2

x	Error at t_0	Error at t_1	Error at t_2	Error at t_3
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.0038813536	0.0037988977	0.0037971964	0.0037971597
0.4	0.0068010063	0.0066561384	0.0066531481	0.0066530854
0.6	0.0077888952	0.0076222591	0.0076188259	0.0076187537
0.8	0.0058585672	0.0057325256	0.0057299242	0.0057298666
1.0	0.0000000006	0.0000000052	0.0000000020	0.0000000042

Table 2b: Difference between exact solution and Adomian decomposition with secant method for Example 2

x	Error at t_0	Error at t_1	Error at t_2	Error at t_3
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.0080381689	0.0037028368	0.0037971567	0.0037971607
0.4	0.0151565539	0.0064639624	0.0066530777	0.0066530864
0.6	0.0204286942	0.0073338552	0.0076187444	0.0076187575
0.8	0.0229152030	0.0053476647	0.0057298615	0.0057298745
1.0	0.0216585660	0.0004816882	0.0000000154	0.0000000030

Table 3a: Difference between exact solution and Adomian decomposition with Newton's method for Example 3

x	Error at t_0	Error at t_1	Error at t_2	Error at t_3
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.0085614993	0.0117227692	0.0117214772	0.0117214771
0.4	0.0166707330	0.0205326043	0.0205266076	0.0205266075
0.6	0.0237524414	0.0257218520	0.0257074016	0.0257074015
0.8	0.0229905343	0.0217382616	0.0217184152	0.0217184152
1.0	0.0000000001	0.0000000000	0.0000000000	0.0000000000

Table 3b: Difference between exact solution and Adomian decomposition with secant method for Example 3

x	Error at t_0	Error at t_1	Error at t_2	Error at t_3
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.0838056457	0.1701318877	0.0101787777	0.0119621124
0.4	0.1457208482	0.2914745777	0.0178133687	0.0209497000
0.6	0.1947567771	0.3821486970	0.0219816463	0.0262881682
0.8	0.2337800939	0.4520553114	0.0170014689	0.0224536291
1.0	0.2605158729	0.5068313640	0.0056534050	0.0008819764

Table 4: Tolerance Value I for Example 1

Tolerance	Newton's Method	Secant Method
$\|t_1-t_0\|$	0.07820061956	0.1295352202000
$\|t_2-t_1\|$	0.004246330776	0.0446174816800
$\|t_3-t_2\|$	0.0000115195143	0.0025331464840
$\|t_4-t_3\|$	-	0.0000040882695

6. DISCUSSION

In this section, the results of the problems considered are analysed. Three different problems considered and the error associated with each problem was calculated by finding the absolute difference between the exact solution and the approximated solution.

Table 1 to 3 show the calculated error at each iterate for both secant's and Newton's.

Table 5: Tolerance Value for Example 2

Tolerance	Newton's Method	Secant Method
$\lvert t_1 - t_0 \rvert$	0.020749428	0.021658566
$\lvert t_2 - t_1 \rvert$	0.00042889	0.000471209
$\lvert t_3 - t_2 \rvert$	0.0000088441	0.0000000151

Table 6: Tolerance Value for Example 3

Tolerance	Newton's Method	Secant Method
$\lvert t_1 - t_0 \rvert$	0.4515130674	0.5210317460
$\lvert t_2 - t_1 \rvert$	0.0210462800	0.9606685938
$\lvert t_3 - t_2 \rvert$	0.0000027018	0.0105974830

method. It was observed in all the three tables that the error reduced with increased update on the shooting angle for both secant's and Newton's method, however, the results of the Newton's gave smaller error when compared to secant's. Table 4 to 6 also shows the tolerance which is the difference between two successive shooting as a smaller tolerance guarantee a more accurate result. It was observed from table 4, 5 and 6 that Newton's method of adjusting the shooting angle converged faster than that of secant's and so gave credence to why the errors in Newton's method is smaller than secant's.
Above all, the results presented are in the form of polynomials and hence avoids discretization error.

7 CONCLUSION

Adomian decomposition method has been used effectively in the technique of shooting method in order to avoid discretization error always encountered by Euler, Rung-Kutta and Taylor's method. In order to investigate the efficiency of the technique, three problems were considered and findings revealed that the more iteration considered the closer the results to the exact solution. It was also shown that Newton's converges faster than the secant's method.

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