MEAN KINETIC ENERGY OF A SYSTEM IN A MORE GENERALIZED HULTHÉN POTENTIAL WELL

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Abstract

In this paper we have obtained the analytical solutions of Schrödinger equation with the more generalized Hulthén potential well within the confines of improved quantization rule to derive closed form expressions for bound states energy eigenvalues and radial eigenfunctions, to ensure an all wave solutions satisfying $\ell \geq 0$, a Pekeris-like approximation model was applied to deal with the centrifugal term potential of the Schrödinger equation. Special case of s-wave eigen energies and eigenfunctions were deduced, the results we obtained are in perfect agreement with exact solutions of the Schrödinger equation for the deformed Hulthén potential in the literature. We have applied our expression for eigen energies to derive expression for mean kinetic energy of a system in a more generalized Hulthén potential well. For arbitrary principal and angular momentum quantum numbers and potential parameter q, we have computed bound states energy eigenvalues and mean kinetic energy in natural units, studies show that for small values of screening parameters, both eigen energy and mean kinetic energy tend to increase approximately linearly for small values of q and remains fairly constant in the region of zero for large values of q

Keywords: Kinetic energy, potential energy, expectation values, Hellmann-Feynman theorem, improved quantization rule

1. Introduction

The many benefits derivable from wave functions in quantum mechanics, statistical mechanics and many other branches of physics have been enumerated elsewhere in the literature and need not be mentioned here. Motion describing atomic and molecular vibrations of a system are represented by potential energy functions, by solving the Schrödinger equation with a particular potential energy model, wave functions representing the system can be obtained [1-2]. The greatest challenge in solving the Schrödinger equation is that the state of the system defined by the principal (n) and angular momentum (ℓ) quantum numbers plays a key role in the nature of the solution of the Schrödinger equation [3-4]. Majority of the known potential energy functions have no exact solutions with the Schrödinger equation for the non-zero angular momentum quantum state due to the presence of the spin-orbit (or centrifugal) term in the effective potential [5-7]. Approximate analytical solutions of the Schrödinger equation can be obtained by employing a suitable approximation scheme such as the Pekeris-like model or the Greene and Aldrich recipe (including the improved versions) to deal with the centrifugal term [8-10]. Different type of analytical solution methods proposed by researchers has been applied to solve the Schrödinger equation, amongst the various solution approach include: exact quantization rule [11], proper or improved quantization rule [12-13], parametric Nikiforov-Uvarov method [6], ansatz solution method [1-2], asymptotic iteration method [14] and 1/N expansion method [15]. The Hulthén potential and its modified forms have been considered in the literature, it is widely

Corresponding Author**:** Eyube E.S., Email: edwineyubes@mautech.edu.ng, Tel: +234**8036870057**

used in nucleon-heavy nucleus interaction, atomic physics, chemical physics and condensed matter physics [16-18]. The more generalized Hulthén potential well was first introduced by Eğrifes and his co-workers in 1999 [19], it is given as:

$$V(r) = -V_0 \frac{e^{-\delta r}}{1 - q\, e^{-\delta r}} \tag{1}$$

where $r \in (0, \infty)$ is the separation, $V_0 = Z\, e^2\, \delta$, Z, δ and q are positive constants. In a previous work the exact solution of the Schrödinger equation for the more generalized Hulthén potential well was obtained by the Nikiforov-Uvarov method [19]. Recently Hall *et al.* have obtained the exact solutions of Schrödinger equation with more generalized Hulthén potential, the Crum-Daboux transformation was then applied to derive exact solution for extended generalized Hulthén potential [20]. In a similar development, as an improvement on the work of Hall *et al.*, Quesne in 2020 exploited the relation between Eckart and generalized Hulthén potentials to write down bound state solution for the later, shape invariance approach of supersymmetric quantum mechanics was then applied to provide solution for extended Hulthén potential [21]. It is worthwhile to mention here that in the works so far carried out on the more generalized Hulthén potential well, eigen energies and eigenfunctions obtained were applicable only to cases of s-wave ($\ell = 0$), however, cases of $\ell \neq 0$ have not been considered. In the present work, our main focus is to apply the principles of improved quantization rule to solve the radial part of the Schrödinger equation to obtain expressions for bound state energy eigenvalues and radial eigenfunctions for the more generalized Hulthén potential well for cases in which $\ell \geq 0$ and to employ the expression for energy eigenvalues to derive relations for mean or expectation value of the kinetic energy of the system and finally, compare results with data in the literature where they exist

2.   Mathematical Formulation
2.1   The Concept of Improved Quantization Rule

In this section a brief outline of the basics of improved quantization rule suitable for the presented work is presented, a detailed narration is given in the literature [7, 13]. The improved quantization rule originated from the concern to address the complicated integrals confronted in the application of exact quantization rule to solve for the eigenenergies of the Schrödinger equation. At the beginning, the exact quantization rule was proposed to solve the one-dimensional Schrödinger equation, given by [11-12].

$$\psi''(x) + k^2(x)\psi(x) = 0 \tag{2}$$

where prime denotes derivative with respect to argument x, $\psi(x)$ is the radial eigenfunction and $k(x)$ is the momentum of the system, it is given by [11-12]:

$$k(x) = \sqrt{\frac{2\mu}{\hbar^2}\{E - V_{eff}(x)\}} \tag{3}$$

$V_{eff}(x)$ is the effective potential energy of the system and E being the energy eigenvalue. In the usual notation, Eq. (2) expressed as a nonlinear Riccati equation, it takes the form [7, 13]:

$$\varphi'(x) + \varphi^2(x) + k^2(x) = 0 \tag{4}$$

the logarithmic derivative $\varphi(x) = \psi'(x)/\psi(x)$ of the wave function $\psi(x)$ is the phase angle. As was explained in a talk by Yang, for the Sturm-Liouville problem, the phase angle must be monotonic with respect to the energy [11-12]. Therefore, from equation (2), $\psi(x)$ must decreases monotonically with respect to x between two turning points where $E \geq V_{eff}(x)$. Specifically, as x increases across a node of the wave function $\psi(x)$, $\varphi(x)$ decreases to $-\infty$, jumps to $+\infty$, and then decreases again. Having carefully studied the Schrödinger equation, Ma and Xu [11] proposed the exact quantization rule as:

$$\int_{x_A}^{x_B} k(x)\, dx = N\pi + \int_{x_A}^{x_B} k'(x)\frac{\varphi(x)}{\varphi'(x)}\, dx \tag{5}$$

where x_A and x_B are two turning points determined by the equation $V_{eff}(x) = E$, $N = n + 1$ is the number of nodes of $\varphi(x)$ in the region $E \geq V_{eff}(x)$ and is larger by 1 than the number n of the nodes of the wave function $\psi(x)$ [11-12]. The term $N\pi$ in Eq. (5) is the contribution from the nodes of the phase angle while the second term is the quantum correction, it is independent of n for all exactly solvable quantum systems, accordingly, it can be evaluated for the ground state ($n = 0$). For a spherically symmetric potential, Eqs. (2) – (4) are expressed as [7-8]:

$$H\psi_{n\ell}(r) = E_{n\ell}\psi_{n\ell}(r) \tag{6}$$

$$k_{n\ell}(r) = \sqrt{\frac{2\mu}{\hbar^2}\{E_{n\ell} - V_{eff}(r)\}} \qquad (7)$$

$$\varphi'_{n\ell}(r) + \varphi^2_{n\ell}(r) + k^2_{n\ell}(r) = 0 \qquad (8)$$

where

$$H = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_{eff}(r) \qquad (9)$$

is the Hamiltonian of the system and the effective potential defined by,

$$V_{eff}(r) = V(r) + \frac{L\hbar^2}{2\mu r^2} \qquad (10)$$

$L = \ell(\ell+1)$ being the angular momentum, in a spherical coordinate system, Eq. (5) takes the following form [7-8]:

$$I_{n\ell} = N\pi + Q_c \qquad (11)$$

where

$$I_{n\ell} = \int_{r_{nA}}^{r_{nB}} k_{n\ell}(r)\,dr \qquad (12)$$

$$Q_c = \int_{x_{nA}}^{x_{nB}} \frac{\varphi_{0\ell}(r)}{\varphi'_{0\ell}(r)}k'_{0\ell}(r)\,dr \qquad (13)$$

Setting $n = 0$ in Eq. (11) and eliminating Q_c, the expression for the improved quantization rule is expressed as [7, 12]:

$$I_{n\ell} = I_{0\ell} + n\pi \qquad (14)$$

2.2 Bound State Eigenenergies of the More Generalized Hulthén Potential Well

Substituting Eq. (1) into Eq. (10), the expression for the effective potential is:

$$V_{eff}(r) = -\frac{V_0}{e^{\delta r} - q} + \frac{L\hbar^2}{2\mu r^2} \qquad (14)$$

Exact analytical solution of Eq. (6) with the effective potential given by Eq. (14) is possible only for the *s*-wave ($\ell = 0$). However, approximate analytical solution can be obtained by employing a Pekeris-type approximation model given by [7]:

$$\frac{1}{r^2} = d_0 + \frac{d_1}{e^{\delta r} - q} + \frac{d_2}{(e^{\delta r} - q)^2} \qquad (15)$$

where r_e is the equilibrium bond length and the constant coefficients d_{j-1} ($j = 1,2,3$) are constant coefficients:

Substituting Eq. (15) in (14), we have that:

$$V_{eff}(r) = -\frac{V_0}{e^{\delta r} - q} + \frac{L\hbar^2}{2\mu}\left\{d_0 + \frac{d_1}{e^{\delta r} - q} + \frac{d_2}{(e^{\delta r} - q)^2}\right\} \qquad (16)$$

Introducing the following change in coordinate:

$$e^{\delta r} = q + \frac{1}{t} \qquad (17)$$

Eq. (16) can be expressed in the more compact form as:

$$V_{eff}(\tau) = \frac{\hbar^2}{2\mu}\sum_{j=1}^{3}\sigma^2_{-1+j}\,\tau^{3-j} \qquad (18)$$

where

$$\sigma^2_0 = L d_2 \qquad (19)$$

$$\sigma^2_1 = L d_1 - \frac{2\mu V_0}{\hbar^2} \qquad (20)$$

$$\sigma^2_2 = L d_0 \qquad (21)$$

The turning points $\tau_{nA} = \left(e^{\delta r_{nA}} - q\right)^{-1}$ and $\tau_{nB} = \left(e^{\delta r_{nB}} - q\right)^{-1}$ are obtained by solving the equation $V_{eff}(\tau) = E_{n\ell}$ [12], inserting Eq. (18) in this relation and executing, this leads to:

$$\sum_{j=1}^{3}\sigma^2_{-1+j}\,\tau^{3-j} = \frac{2\mu E_{n\ell}}{\hbar^2} \equiv \sigma^2_{n\ell} \qquad (22)$$

From Eq. (22), the sum and product of the turning points are obtained as:

$$\tau_{nA} + \tau_{nB} = -\frac{\sigma_1^2}{\sigma_0^2} \tag{23}$$

$$\tau_{nA}\,\tau_{nB} = \frac{\sigma_2^2 - \sigma_{n\ell}^2}{\sigma_0^2} \tag{24}$$

The expression for the momentum given by Eq. (7) can be written in terms of the turning points as:

$$k_{n\ell}(\tau) = \left\{ \sigma_{n\ell} - \sum_{j=1}^{3} \sigma_{-1+j}^2 \, \tau^{3-j} \right\}^{\frac{1}{2}} \equiv \sigma_0 \sqrt{(\tau - \tau_{nA})(\tau_{nB} - \tau)} \tag{25}$$

Using Eq. (17) to transform Eq. (8) into one involving coordinate τ, it is obvious that:

$$-\delta\,\tau\,(1 + q\,\tau)\,\varphi'_{n\ell}(\tau) + \varphi^2_{n\ell}(\tau) - \sigma_0^2\,\tau^2 - \sigma_1^2\,\tau - \sigma_2^2 + \sigma_{n\ell}^2 = 0 \tag{26}$$

To satisfy the Sturm-Liouville requirement, we choose the ground state wave function as a trial solution, *viz*:

$$\varphi_{0\ell}(\tau) = -\alpha\,\tau + \beta \tag{27}$$

where α and β are constants, with $n = 0$ and replacing Eq. (27) in (26), we have the following relation:

$$\left(\alpha\,\delta\,q + \alpha^2 - \sigma_0^2\right)\tau^2 + \left(\alpha\,\delta - 2\,\alpha\,\beta - \sigma_1^2\right)\tau + \beta^2 - \sigma_2^2 - \sigma_{0\ell}^2 = 0 \tag{28}$$

For Eq. (28) to be valid, the separate coefficients of τ^2, τ and τ^0 must each varnish, leading to:

$$\alpha^2 + \delta\,q\,\alpha = \sigma_0^2 \tag{29}$$

$$\alpha\,\delta - 2\,\alpha\,\beta = \sigma_1^2 \tag{30}$$

$$\beta^2 = \sigma_2^2 - \sigma_{0\ell}^2 \tag{31}$$

Clearly, Eq. (29) gives:

$$\alpha = -\delta\,q\,\gamma \tag{32}$$

where, for simplicity we have used:

$$\gamma = \frac{1}{2} + \left(\frac{1}{4} + \frac{\sigma_0^2}{\delta^2\,q^2}\right)^{\frac{1}{2}} \tag{33}$$

Using Eq. (17) to express the momentum integral in terms of coordinate τ, and with the help of Eq. (25), Eq. (12) leads to:

$$I_{n\ell} = -\frac{\sigma_0}{\delta} \int_{\tau_{nA}}^{\tau_{nA}} \frac{\sqrt{(\tau - \tau_{nA})(\tau_{nB} - \tau)}}{\tau\,(1 + q\,\tau)}\,d\tau \tag{34}$$

The definite integral appearing in Eq. (34) can be evaluated with the following standard integral adopted from ref. [8] viz:

$$\int_{a}^{b} \frac{\sqrt{(x - a)(b - x)}}{x\,(1 + q\,x)}\,dx = \pi\left\{ \frac{\sqrt{(1 + q\,a)(1 + q\,b)}}{q} - \frac{1}{q} - \sqrt{a\,b} \right\} \tag{35}$$

Comparison of Eqs. (34) and Eq. (35) yields:

$$I_{n\ell} = \frac{\pi}{\delta}\left\{ \frac{\sigma_0}{q} + \sqrt{\sigma_2^2 - \sigma_{n\ell}^2} - \sqrt{\frac{\sigma_0^2}{q^2} - \frac{\sigma_1^2}{q} + \sigma_2^2 - \sigma_{n\ell}^2} \right\} \tag{36}$$

For the ground state ($n = 0$), Eq. (36) gives:

$$I_{0\ell} = \frac{\pi}{\delta}\left\{ \frac{\sigma_0}{q} + \sqrt{\sigma_0^2 - \sigma_{0\ell}^2} - \sqrt{\frac{\sigma_0^2}{q^2} - \frac{\sigma_1^2}{q} + \sigma_2^2 - \sigma_{0\ell}^2} \right\} \tag{37}$$

Substituting Eqs. (29) – (31) into Eq. (37) and simplifying, we have that:

$$I_{0\ell} = \frac{\pi}{\delta}\left(\frac{\sigma_0}{q} + \delta\,\gamma\right) \tag{38}$$

Inserting the results given by Eqs. (36) and (38) in the expression for improved quantization rule, we have that:

$$\sigma_{n\ell}^2 = \sigma_2^2 - \delta^2\left\{ \frac{\sigma_0^2 - q\,\sigma_1^2}{2\,\delta^2\,q^2\,(n + \gamma)} - \frac{n + \gamma}{2} \right\}^2 \tag{39}$$

Upon substituting Eqs. (19) – (22) in Eq. (39), we obtained the energy eigenvalues of the More Generalized Hulthén Potential well as:

$$E_{n\ell} = \frac{L\hbar^2 d_0}{2\mu} - \frac{\delta^2 \hbar^2}{2\mu} \left\{ \frac{\frac{2\mu V_0}{\delta^2 \hbar^2 q} + \frac{L}{\delta^2 q^2}(d_2 - q d_1)}{2n+1+\sqrt{1+\frac{4Ld_2}{\delta^2 q^2}}} - \frac{2n+1+\sqrt{1+\frac{4Ld_2}{\delta^2 q^2}}}{4} \right\}^2 \tag{40}$$

2.3 Bound State Eigenfunctions of the More Generalized Hulthén Potential Well

The radial eigenfunctions can be obtained by solving the Riccati equation given by Eq. (26), however, a much easier approach is to use the wave function – phase angle relationship to recover the radial eigenfunction $\psi_{n\ell}(\tau)$. Therefore, substituting a second coordinate transformation $z = 1 + q\tau$ in Eq. (26) and eliminating $\varphi_{n\ell}(\tau)$ we have:

$$z(1-z)\psi''_{n\ell}(z) + (1-2z)\psi'_{n\ell}(z) + \left\{ \frac{\sigma_0^2}{\delta^2 q^2} - \frac{1}{\delta^2}\left(\frac{\sigma_0^2}{q^2} - \frac{\sigma_1^2}{q} + \sigma_2^2 - \sigma_{n\ell}^2 \right)\frac{1}{z} - \frac{\sigma_2^2 - \sigma_{n\ell}^2}{\delta^2}\frac{1}{1-z} \right\}\psi_{n\ell}(z) = 0 \tag{41}$$

In order to solve Eq. (41) we consider the ansatz [18].

$$\psi_{n\ell}(z) = N_{n\ell}(qz)^\upsilon (1-qz)^\eta \Xi_{n\ell}(z) \tag{42}$$

where υ and η are parameters and $\Xi_{n\ell}(z)$ is an unknown function of z and $N_{n\ell}$ is the normalization constant. Substituting Eq. (42) into (41), we obtained the Gauss hypergeometric differential equation given by:

$$z(1-z)\Xi''_{n\ell}(z) + \{2\upsilon+1-(2\upsilon+2\eta+2)z\}\Xi'_{n\ell}(z) - \left\{(\upsilon+\eta)^2 + \upsilon+\eta - \frac{\sigma_0^2}{\delta^2 q^2}\right\}\Xi_{n\ell}(z) = 0 \tag{43}$$

subject to the following constraints:

$$\upsilon = \frac{1}{\delta}\left(\frac{\sigma_0^2}{q^2} - \frac{\sigma_1^2}{q} + \sigma_2^2 - \sigma_{n\ell}^2 \right)^{\frac{1}{2}} \tag{44}$$

and

$$\eta = \frac{1}{\delta}\left(\sigma_2^2 - \sigma_{n\ell}^2\right)^{\frac{1}{2}} \tag{45}$$

Following [18], Eq. (43) has a polynomial solution given by:

$$\Xi_{n\ell}(z) = {}_2F_1\left(-n, n+2\upsilon+2\eta+1; 2\upsilon+1; qz\right) \tag{46}$$

${}_2F_1\left(-n, n+2\upsilon+2\eta+1; 2\upsilon+1; qz\right)$ being the hypergeometric function

2.4 Mean Kinetic Energy of the More Generalized Hulthén Potential Well

Armed with the expression for the energy eigenvalues of the more generalized Hulthén potential well given by Eq. (40), we are now in the position to derive a formula for the mean kinetic energy of the system in this potential model from the point of view of the Hellmann-Feynman theorem [6, 9]. The Hellmann-Feynman theorem states that for a given quantum mechanical system, if the Hamiltonian H, is a function of some parameter λ, then the $H(\lambda)$, $\psi_{n\ell}(\lambda)$ and $E_{n\ell}(\lambda)$ are related by [6, 9]:

$$E_\lambda(\lambda) = \langle \psi_{n\ell}(\lambda)| H_\lambda(\lambda) |\psi_{n\ell}(\lambda)\rangle \tag{47}$$

where $E_\lambda(\lambda) = \partial E_{n\ell}(\lambda)/\partial\lambda$ and $H_\lambda(\lambda) = \partial H(\lambda)/\partial\lambda$. Generally, in what follows henceforth, the notation f_λ will be taken to mean $\partial f/\partial\lambda$. To apply the Hellmann-Feynman theorem, we begin by substituting Eq. (14) in (9) to obtain explicit expression for the Hamiltonian of the system, *viz*:

$$H = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{L\hbar^2}{2\mu r^2} - \frac{V_0}{e^{\delta r} - q} \tag{48}$$

If we choose $\lambda = L$, Eq. (47) gives:

$$E_L = \langle \psi_{n\ell}| H_L |\psi_{n\ell}\rangle \tag{49}$$

Eqs. (48) and (40) respectively gives:

$$H_L = \frac{\hbar^2}{2\mu r^2} \tag{50}$$

$$E_L = \frac{\hbar^2 d_0}{2\mu} - \frac{\hbar^2 \Lambda^{\frac{1}{2}}}{\mu q^2}\left(\frac{d_2 - q d_1}{\phi_2} - \frac{2 d_2}{\phi_1 \phi_4} \right) \tag{51}$$

where

$$\Lambda = \left(\frac{\phi_3}{\phi_2} - \frac{\phi_2}{4} \right)^2 \tag{52}$$

$$\phi_4^{-1} = \frac{\phi_3}{\phi_2^2} + \frac{1}{4} \tag{53}$$

$$\phi_3 = \frac{2\mu V_0}{\delta^2 \hbar^2 q} + \frac{L}{\delta^2 q^2}(d_2 - q d_1) \tag{54}$$

$$\phi_2 = 2n + 1 + \phi_1 \tag{55}$$

$$\phi_1 = 2\gamma - 1 \tag{56}$$

By replacing Eqs. (50) and (51) in (49), we have that:

$$\langle r^{-2} \rangle = d_0 - \frac{2\Lambda^{\frac{1}{2}}}{q^2} \left(\frac{d_2 - q d_1}{\phi_2} - \frac{2 d_2}{\phi_1 \phi_4} \right) \tag{57}$$

Setting $\lambda = \mu$ in Eqs. (47) and (48) respectively, we obtained:

$$E_\mu = \langle \psi_{n\ell} | H_\mu | \psi_{n\ell} \rangle \tag{58}$$

$$H_\mu = \frac{\hbar^2}{2\mu^2} \frac{d^2}{d r^2} - \frac{L \hbar^2}{2\mu^2 r^2} \tag{59}$$

Substitute Eq. (59) in (58) to get:

$$\langle T \rangle \equiv \langle \psi_{n\ell} \left| -\frac{\hbar^2}{2\mu} \frac{d^2}{d r^2} \right| \psi_{n\ell} \rangle = -\mu E_\mu - L E_L \tag{60}$$

where $\langle T \rangle \equiv \langle \psi_{n\ell} \left| -\frac{\hbar^2}{2\mu} \frac{d^2}{d r^2} \right| \psi_{n\ell} \rangle$ is the expectation value of the kinetic energy and E_μ is obtained from Eq. (40) as:

$$E_\mu = -\frac{L \hbar^2 d_0}{2\mu^2} + \frac{\delta^2 \hbar^2 \Lambda}{2\mu^2} - \frac{2 V_0 \Lambda^{\frac{1}{2}}}{\mu q \phi_2} \tag{61}$$

3.   Results and Discussion

If we let $\ell = 0$ and choose natural units, that is ($\hbar = \mu = e = 1$), Eq. (40) is reduced to:

$$E_{n0} = -\frac{\delta^2}{2\mu} \left\{ \frac{\mu V_0}{\delta^2 \hbar^2 q(n+1)} - \frac{n+1}{2} \right\}^2 \tag{62}$$

Eqs. (62) and (42) turns out to be identical to the expressions for the energy eigenvalues and radial wave functions of the generalized Hulthén potential well obtained in ref. [20-21]. On the other hand, if we set $q = 1$, $d_0 = 0$, $d_1 = \omega \delta^2$ and $d_2 = \delta^2$ where ω is a parameter, Eq. (40) reproduces the energy eigenvalues of the Hulthén potential while Eq. (16) corresponds to the effective potential ref. [18]. These results are clear demonstrations that Eqs. (40) and (42) holds as the correct expression for the energy eigenvalues and radial eigenfunctions of the generalized Hulthén potential well. We have computed energy eigenvalues and mean kinetic energy as a function of parameters n, ℓ and q for $Z = 1$, $\delta = 0.01$ where parameter $c_0 = 0.0823058167837972$ was taken from ref [17], $d_0 = c_0 \delta^2$ and $d_1 = d_2 = \delta^2$. Results obtained using Eqs. (40) and (60) are shown in the Table. The variation of $E_{n\ell}$ with n is shown in Figure 1, evidently, as n is gradually increased from zero, the energy eigenvalues sharply increases from a large negative value to reach a maximum value of about zero, then it decreases monotonically as n is further increased. In Figure 2 we have the variation of mean kinetic energy, $\langle T \rangle_{n\ell}$ with n, the plot shows that $\langle T \rangle_{n\ell}$ has a sharp initial increase from a large negative value to zero after which it remains approximately stable at zero as n is further increased. The plots in Figures 3 and 4 are for the variation of $E_{n\ell}$ and $\langle T \rangle_{n\ell}$ respectively with q, the plots reveal the same trend as for the variation of $\langle T \rangle_{n\ell}$ with n.

Table: Bound State Energy Eigenvalues and Mean Kinetic Energy (in atomic unit: $\hbar = \mu = e = 1$) as a Function of q, n and ℓ for $Z = 1$, $\delta = 0.01$

| State		$q = 1$		$q = 2$		$q = 3$	
n	ℓ	$\langle T \rangle_{n\ell}$	$E_{n\ell}$	$\langle T \rangle_{n\ell}$	$E_{n\ell}$	$\langle T \rangle_{n\ell}$	$E_{n\ell}$
0	0	-4948.630050	-0.495013	-305.908800	-0.122513	-59.725482	-0.053901
1	0	-311.875200	-0.120050	-19.375200	-0.028800	-3.788780	-0.012272
	1	-64.097161	-0.050660	-9.943269	-0.019681	-2.626316	-0.009855
2	0	-63.200697	-0.050668	-3.993706	-0.011501	-0.787783	-0.004619
	1	-21.136500	-0.026442	-2.558227	-0.008568	-0.616999	-0.003860
	2	-9.251550	-0.015288	-1.544569	-0.005994	-0.436672	-0.002955
3	0	-20.834550	-0.026450	-1.349316	-0.005513	-0.268686	-0.002006
	1	-9.144750	-0.015304	-0.972440	-0.004235	-0.224168	-0.001682
	2	-4.767317	-0.009314	-0.661520	-0.003018	-0.171896	-0.001276
	3	-2.807114	-0.005767	-0.451485	-0.002060	-0.127183	-0.000895
4	0	-9.011250	-0.015313	-0.600000	-0.002813	-0.120015	-0.000868
	1	-4.711065	-0.009331	-0.465155	-0.002162	-0.103987	-0.000710
	2	-2.774828	-0.005792	-0.341659	-0.001514	-0.083707	-0.000506
	3	-1.787436	-0.003563	-0.249361	-0.000983	-0.064898	-0.000309
	4	-1.232963	-0.002103	-0.181917	-0.000565	-0.048927	-0.000136
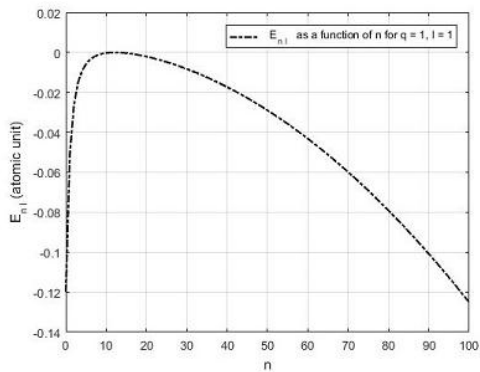
Figure 1: Plot of $E_{n\ell}$ as a function of n for $\ell = q = 1$
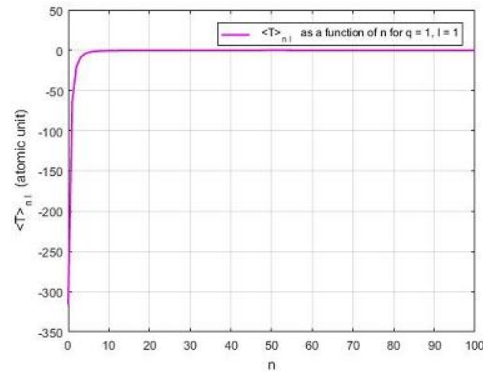

Figure 2: Plot of $\langle T \rangle_{n\ell}$ as a function of n for $\ell = q = 1$
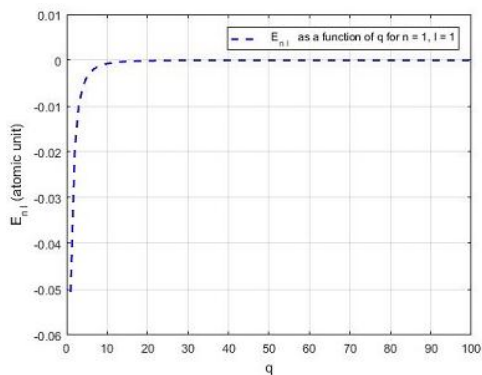

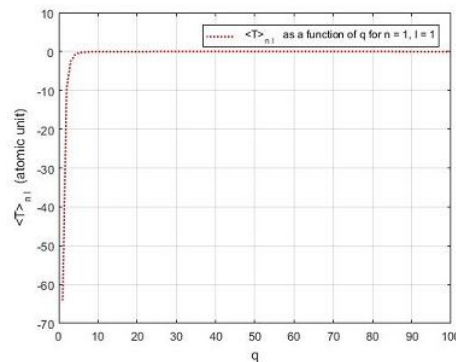**Figure 3**: Plot of $E_{n\ell}$ as a function of q for $\ell = n = 1$

Figure 4: Plot of $E_{n\ell}$ as a function of q for $\ell = n = 1$

4 Conclusion

Bound states energy eigenvalues and radial eigenfunctions of the more generalized Hulthén potential well has been obtained within the frameworks of improved quantization rule. A Pekeris-like approximation recipe was used on the centrifugal term potential of the Schrödinger equation to make equation amenable to solution. Expression for energy eigenvalues and radial wave function for the Hulthén potential were deduced, the results obtained were in total agreement with expressions for bound states energy eigenvalues and eigen functions in the literature obtained by other methods. Using the expression obtained for eigen energies, we have derived expression for mean kinetic energy for the system in more generalized Hulthén potential. We have computed energy eigenvalues and mean kinetic energy in natural units, the results we obtained show that for small values of screening parameters, both the eigen energies and mean kinetic energy increases approximately linearly for small values of potential parameter, q and remains fairly constant at zero for large values of q.

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