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ENERGY LEVELS AND ROOT-MEAN-SQUARE SPEEDS OF A SYSTEM IN PÖSCHL-TELLER TYPE POTENTIAL

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Abstract

In this work, the improved quantization rule was applied to derive closed form expression for bound state energy eigenvalues of the Pöschl-Teller type potential, normalized expression for radial eigenfunctions were obtained by ansatz solution technique. The improved Greene-Aldrich approximation scheme was used to model the centrifugal term of the Schrödinger equation, the expression for bound state energy eigenvalues and radial wave functions agrees totally with literature data for the Pöschl-Teller type potential when reduced to s-wave. With the help of the formula for energy eigenvalues. Expression for root-mean-square speed was deduced within the confines of Hellmann-Feynman theorem, it was further observed that the root-mean-square speed of the system increases monotonically with the potential strength of the potential and is smaller for small values of screening parameters.

Keywords*:* Effective potential, root-mean-square speed, proper quantization rule, Pöschl-Teller type potential, Schrödinger equation

Introduction

In quantum mechanics, it is usual to model the interaction of a system with its environment by potential energy function [1, 2], as such, any information regarding the system is obtained from the wave function. Unfortunately, wave functions for the particular system are not usually handy, they are obtained by solving the Schrödinger equation [3, 4].

For most of the potential energy functions, only approximate solutions are permissible for a given quantum state determined by the angular momentum quantum number ℓ and the principal quantum number n [3]. The approximate solutions are obtained when a suitable approximation scheme [5] is applied on the spin-orbit or centrifugal term of the Schrödinger equation, the centrifugal term is given as $L\,\hbar^2/2\,M\,r^2$ where $r \in (0, \infty)$ is the distance, $L = \ell(\ell + 1)$ is the angular momentum and M is the mass of the system.

A number solution methods have been proposed for solving the Schrödinger equation analytically, some of these methods include the supersymmetric quantum approach [6-8], Laplace transform method [9], Nikiforov-Uvarov method [10, 11], ansatz solution approach [12, 13], exact and proper quantization rule [14-17], asymptotic iteration method [18] amongst others.

Various forms of potential models have been used in the literature to solve the Schrödinger and other wave equations of quantum mechanics. The hyperbolical type potential models have particularly attracted much attention from researchers due to their numerous applications in many branches of physics [5, 7, 12, 15, 19]. However, in particular, the Pöschl-Teller type potential energy function [20, 21] given by

$$V(r) = \frac{\alpha^2\hbar^2}{2M}\lambda(\lambda+1)\tanh^2(\alpha r), \qquad (1)$$

where λ is the potential depth and α is the range of the potential. This class of hyperbolical type potential has not received much attention from researchers (at least to the best of our knowledge).

In earlier works, Chen et al. [19] obtained the s-wave (state $\ell = 0$) solution of the Schrödinger equation for this potential, the result obtained was used to study the uncertainty relations for the potential. Yahya and Oyewumi [20] also obtained the thermodynamic properties and approximate solutions for the potential via improved Greene and Aldrich approximation model, the parametric Nikiforov-Uvarov method was used to obtain the solution. However, it is noted that the expression for energy eigenvalues (Eq. (19) of ref. [20]) when reduced to s-wave does not reconcile with Eq. (8) of Chen et al. [19].

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Motivated by the successes of the improved quantization rule in deriving equations for energy eigenvalues, the present work is aimed at obtaining energy levels and root-mean-square speeds of a system in Pöschl-Teller type potential. The paper is organized as follows. In section 2, a brief review of the improved quantization rule is presented. In section 3, the improved quantization rule is employed in the derivation of bound state energy eigenvalues for the Pöschl-Teller type potential. Ansatz solution approach is considered in section 4 for the radial eigenfunctions of the potential. Section 5 deals with the derivation of root-mean-square speed of the system from the context of Hellmann-Feynman theorem. A brief conclusion of the work is presented in section 6

2. An overview of the improved quantization rule

Here, the basic ideas of improved quantization rule suitable for the present work is given, a comprehensive description of this concept is provided in the literature [14, 15]. The improved (or proper) quantization rule was proposed to overcome the difficulties arising from the quantum correctional term of exact quantization rule to obtaining eigenenergies from the effective potential of the Schrödinger equation. Initially, the exact quantization rule was proposed to solve the one-dimensional Schrödinger equation [14] given by

$$\mathrm{u}''(x)+\mathrm{k}^2(x)\,\mathrm{u}(x)=0\,, \tag{2}$$

where prime represents derivative with respect to argument *x*, $\mathrm{u}(x)$ is the one dimensional wave function and $\mathrm{k}(x)$ is the momentum, defined as

$$\mathrm{k}(x)=\frac{1}{\hbar}\sqrt{2\,\mathrm{M}\left\{E-\mathrm{V}_{\mathrm{eff}}(x)\right\}}\,, \tag{3}$$

E is the energy eigenvalue and $\mathrm{V}_{\mathrm{eff}}(x)$ is the effective potential energy of the system. In the usual notation, Eq. (2) is expressed as a Riccati differential equation given as

$$\phi'(x)+\phi^2(x)+\mathrm{k}^2(x)=0\,, \tag{4}$$

the logarithmic derivative $\phi(x)=\mathrm{u}'(x)\,\mathrm{u}^{-1}(x)$ of the wave function $\mathrm{u}(x)$ is known as the phase angle. Yang explained that for the Sturm-Liouville problem [15, 16], the phase angle must be monotonic with respect to the energy. From Eq. (2), $\mathrm{u}(x)$ decreases monotonically with respect to *x* between two turning points where $E\geq \mathrm{V}_{\mathrm{eff}}(x)$. Specifically, as *x* increases across a node of the wave function $\mathrm{u}(x)$, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$, and then decreases again [14, 16, 17]. After a careful study of the Schrödinger equation, the exact quantization rule was proposed by Ma and Xu [14] as

$$\int_{x_A}^{x_B}\mathrm{k}(x)\,\mathrm{d}x=N\pi+\int_{x_A}^{x_B}\mathrm{k}'(x)\frac{\phi(x)}{\phi'(x)}\,\mathrm{d}x\,, \tag{5}$$

where x_{A} and x_{B} are two turning points determined by the equation $\mathrm{V}_{\mathrm{eff}}(x)=E$, $N = n+1$ is the number of nodes of $\phi(x)$ in the region $E\geq \mathrm{V}_{\mathrm{eff}}(x)$ and is larger by 1 than the number *n* of the nodes of the wave function $\mathrm{u}(x)$ [15, 16]. The term $N\pi$ in Eq. (5) is the contribution from the nodes of the phase angle while the second term is the quantum correction, it is independent of *n* for all exactly solvable quantum systems, accordingly, it can be evaluated at the ground state ($n = 0$). For a spherically symmetric potential, Eqs. (2) - (4) are expressed as

$$\left\{-\frac{\hbar^2}{2\,\mathrm{M}}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\mathrm{V}_{\mathrm{eff}}(r)\right\}\mathrm{u}_{n\ell}(r)=\mathrm{E}_{n\ell}\,\mathrm{u}_{n\ell}(r)\,, \tag{6}$$

$$\mathrm{k}_{n\ell}(r)=\frac{1}{\hbar}\sqrt{2\,\mathrm{M}\left\{\mathrm{E}_{n\ell}-\mathrm{V}_{\mathrm{eff}}(r)\right\}}\,, \tag{7}$$

$$\phi'_{n\ell}(r)+\phi^2_{n\ell}(r)+\mathrm{k}^2_{n\ell}(r)=0\,, \tag{8}$$

where

$$\mathrm{V}_{\mathrm{eff}}(r)=\mathrm{V}(r)+\frac{L\hbar^2}{2\,\mathrm{M}\,r^2}\,, \tag{9}$$

with $L=\ell(\ell+1)$. In spherical coordinates, Eq. (5) is written in compact form as

$$\mathrm{I}_{n\ell}=N\pi+\mathrm{Q}_{\mathrm{c}}\,, \tag{10}$$

where

$$\mathrm{I}_{n\ell}=\int_{r_{n\mathrm{A}}}^{r_{n\mathrm{B}}}\mathrm{k}_{n\ell}(r)\,\mathrm{d}r\,, \tag{11}$$

$$\mathrm{Q}_{\mathrm{c}}=\int_{x_{n\mathrm{A}}}^{x_{n\mathrm{B}}}\frac{\phi_{0J}(r)}{\phi'_{0J}(r)}\mathrm{k}'_{0J}(r)\,\mathrm{d}r\,. \tag{12}$$

Letting $n = 0$ in Eq. (11) and eliminating Q_c, the expression for the improved quantization rule [16, 17] is given in compact form as

$$\mathrm{I}_{n\ell} = \mathrm{I}_{0\ell} + n\pi \,. \tag{13}$$

3. Bound state energy eigenvalues of the Pöschl-Teller type potential

Here we will employ the improved quantization rule [17] to derive closed form expression for the bound state eigen energies of the of the Pöschl-Teller type potential. Substituting Eq. (1) into (9), we have

$$\mathrm{V}_{\mathrm{eff}}(r) = \frac{\alpha^2\hbar^2}{2\,\mathrm{M}}\lambda(\lambda+1)\tanh^2(\alpha r) + \frac{L\hbar^2}{2\,\mathrm{M}\,r^2}\,. \tag{14}$$

Except for the s-wave solution, Eq. (6) has no exact solution with (14) for cases in which $\ell \neq 0$. However, approximate analytical solutions are possible by modeling the centrifugal term in Eq. (14) by a suitable approximation scheme. For very small values of α one would be able to approximate the spin-orbit term by improved Greene-Aldrich approximation recipe, it is given as [20]

$$\frac{L\hbar^2}{2\,\mathrm{M}\,r^2} \approx \frac{L\alpha^2\hbar^2}{2\,\mathrm{M}}\left\{c_0 + \cos e\mathrm{ch}^2(\alpha r)\right\}, \tag{15}$$

where $c_0 = \frac{1}{3}$. Substituting Eq. (15) in (14), we have

$$\mathrm{V}_{\mathrm{eff}}(r) = \frac{\alpha^2\hbar^2}{2\,\mathrm{M}}\lambda(\lambda+1)\tanh^2(\alpha r) + \frac{L\alpha^2\hbar^2}{2\,\mathrm{M}}\left\{c_0 + \cos e\mathrm{ch}^2(\alpha r)\right\}. \tag{16}$$

By means of the following coordinate transformation

$$x = \tanh^2(\alpha r)\,, \tag{17}$$

where $x \in (0, 1)$ Eq. (16) simplifies to

$$\mathrm{V}_{\mathrm{eff}}(r) = \frac{\alpha^2\hbar^2}{2\,\mathrm{M}}\left(\varepsilon_0^2\, x + \frac{L}{x} + \varepsilon_1^2\right), \tag{18}$$

such that

$$\varepsilon_0^2 = \lambda(\lambda+1)\,, \tag{19}$$

$$\varepsilon_1^2 = L(c_0 - 1)\,. \tag{20}$$

The turning points $r_{n\mathrm{A}} = \alpha^{-1}\tanh^{-1}\sqrt{x_{n\mathrm{A}}}$ and $r_{n\mathrm{B}} = \alpha^{-1}\tanh^{-1}\sqrt{x_{n\mathrm{B}}}$ are obtained by substituting Eq. (18) into the relation $\mathrm{V}_{\mathrm{eff}}(x) = \mathrm{E}_{n\ell}$, we find

$$\varepsilon_0^2\, x + \frac{L}{x} + \varepsilon_1^2 = \frac{2\,\mathrm{M}\,\mathrm{E}_{n\ell}}{\alpha^2\hbar^2} \equiv \varepsilon_{n\ell}^2\,. \tag{21}$$

From Eq. (21), it is easy to see that

$$x_{n\mathrm{A}} + x_{n\mathrm{B}} = \frac{\varepsilon_{n\ell}^2 - \varepsilon_1^2}{\varepsilon_0^2}\,, \tag{22}$$

$$x_{n\mathrm{A}}\, x_{n\mathrm{B}} = \frac{L}{\varepsilon_0^2}\,. \tag{23}$$

Eq. (7) and the transformation Eq. (17) yields

$$\mathrm{k}_{n\ell}(r) = \alpha\sqrt{-\varepsilon_0^2\, x - \varepsilon_1^2 + \varepsilon_{n\ell}^2 - \frac{L}{x}} \equiv \alpha\,\varepsilon_0\sqrt{\frac{(x - x_{n\mathrm{A}})(x_{n\mathrm{B}} - x)}{x}}\,. \tag{24}$$

At this point, we will consider the Riccati equation, which is an essential ingredient of the improved quantization rule. Replacing Eq. (17) in (8) and substituting Eq. (24) in the resulting expression, we have

$$\frac{2}{\alpha}x\sqrt{x}(1-x)\phi'_{n\ell}(x) + \frac{x}{\alpha^2}\phi^2_{n\ell}(x) - \varepsilon_0^2\, x^2 - (\varepsilon_1^2 - \varepsilon_{n\ell}^2)x - L = 0\,. \tag{25}$$

In order to satisfy Sturm-Liouville requirement, since the ground state wave function has no node and its logarithmic derivative has no pole and decreases monotonically as x is increased, it is reasonable to choose a trial solution for Eq. (25) as

$$\phi_{0\ell}(x) = \frac{\alpha}{\sqrt{x}}(-\gamma\, x + \omega)\,, \tag{26}$$

where γ and ω are constant coefficients. Substituting n = 0 in Eq. (25) and using (26), we have

$$\left(\gamma^2+\gamma-\varepsilon_0^2\right)x^2+\left(-\gamma+\omega-2\gamma\omega-\varepsilon_1^2+\varepsilon_{0\ell}^2\right)x+\omega^2-\omega-L=0\,. \tag{27}$$

For Eq. (27) to remain valid, the coefficients of x^2, x^0 and x must separately varnish, leading to

$$\gamma^2+\gamma=\varepsilon_0^2\,, \tag{28}$$

$$\omega^2-\omega=L\,, \tag{29}$$

$$-\gamma+\omega-2\gamma\omega=\varepsilon_1^2-\varepsilon_{0\ell}^2\,. \tag{30}$$

With the help of Eqs. (19)- (20), Eqs. (28)-(30) can be solved to obtain $\gamma = \lambda$ and $\omega = -\ell$. Therefore, the trial wave function given by (26) is completely known. Next, the momentum integral is determined by inserting Eq. (17) into Eq. (11) and using Eq. (24) to eliminate $k_{n\ell}$. This gives

$$I_{n\ell}=\frac{\varepsilon_0}{2}\int_{x_{n\mathrm{A}}}^{x_{n\mathrm{B}}}\frac{\sqrt{\left(x-x_{n\mathrm{A}}\right)\left(x_{n\mathrm{B}}-x\right)}}{x(1-x)}\,\mathrm{d}\,x\,. \tag{31}$$

By means of the following standard integral [16]

$$\int_a^b\frac{\sqrt{(x-a)(b-x)}}{x(1-x)}\,\mathrm{d}\,x=\pi\left\{-\sqrt{a\,b}+1-\sqrt{(1-a)(1-b)}\right\}. \tag{32}$$

Eq. (32) compared to Eq. (31) results in

$$I_{n\ell}=\frac{\pi}{2}\left(\varepsilon_0-\varepsilon_1-\sqrt{\lambda(\lambda+1)+L\,c_0-\varepsilon_{n\ell}^2}\right). \tag{33}$$

Corresponding to the ground state, Eq. (33), (28)- (30) gives

$$I_{0\ell}=\frac{\pi}{2}\left(\varepsilon_0-\varepsilon_1-\lambda-\ell\right). \tag{34}$$

Substituting Eqs. (33) and (34) in Eq. (13). The improved quantization rule leads to

$$E_{n\ell}=\frac{\alpha^2\hbar^2}{2M}\left\{L\,c_0+\lambda(\lambda+1)-\left(\ell+\lambda-2n\right)^2\right\}. \tag{35}$$

Unlike Eq. (19) of ref. [20] which contradicts Eq. (8) of ref. [19] for the s-wave energy eigenvalues. With the mapping $n \rightarrow n/2$, our expression for eigen energies given by Eq. (35) reproduces Eq. (8) of Chen et al. [19] when $\ell = 0$. This is an indication that Eq. (35) is the correct expression for bound state eigen energies of the Pöschl-Teller like potential for all quantum states.

Using Eq. (35), we have computed energy eigenvalues as a function of screening parameter and potential depth for arbitrary principal and angular momentum quantum numbers. The calculated eigen energies are shown in Table 1. In Figure 2, we have plotted the variation of energy eigenvalues as a function of potential depths for different screening parameters. the plot reveals that the energy decreases linearly with increase in potential depth and is higher for larger values of screening parameter. Graphical representation of the variation of energy eigenvalues as a function of principal quantum number and potential depth is shown in Figure 2. As can be seen from the plot, the eigen energies decreases monotonically when the principal quantum number is is increased and is higher for larger values of the potential depths.

Table 1: Bound state energies in atomic unit ($\hbar$ = M = 1)

state	α	$E_{n\ell}$ (λ)				
		λ = 10	λ = 20	λ = 30	λ = 40	λ = 50
2p	0.025	-0.00323	-0.00635	-0.00948	-0.01260	-0.01573
	0.050	-0.01292	-0.02542	-0.03792	-0.05042	-0.06292
	0.075	-0.02906	-0.05719	-0.08531	-0.11344	-0.14156
	0.100	-0.05167	-0.10167	-0.15167	-0.20167	-0.25167
	0.150	-0.11625	-0.22875	-0.34125	-0.45375	-0.56625
	0.200	-0.20667	-0.40667	-0.60667	-0.80667	-1.00667
	0.250	-0.32292	-0.63542	-0.94792	-1.26042	-1.57292
	0.300	-0.46500	-0.91500	-1.36500	-1.81500	-2.26500
	0.350	-0.63292	-1.24542	-1.85792	-2.47042	-3.08292
3p	0.025	0.00927	0.01865	0.02802	0.03740	0.04677
	0.050	0.03708	0.07458	0.11208	0.14958	0.18708
	0.075	0.08344	0.16781	0.25219	0.33656	0.42094
	0.100	0.14833	0.29833	0.44833	0.59833	0.74833
	0.150	0.33375	0.67125	1.00875	1.34625	1.68375
	0.200	0.59333	1.19333	1.79333	2.39333	2.99333
	0.250	0.92708	1.86458	2.80208	3.73958	4.67708
	0.300	1.33500	2.68500	4.03500	5.38500	6.73500
	0.350	1.81708	3.65458	5.49208	7.32958	9.16708
3d	0.025	-0.01000	-0.01938	-0.02875	-0.03813	-0.04750
	0.050	-0.04000	-0.07750	-0.11500	-0.15250	-0.19000
	0.075	-0.09000	-0.17438	-0.25875	-0.34313	-0.42750
	0.100	-0.16000	-0.31000	-0.46000	-0.61000	-0.76000
	0.150	-0.36000	-0.69750	-1.03500	-1.37250	-1.71000
	0.200	-0.64000	-1.24000	-1.84000	-2.44000	-3.04000
	0.250	-1.00000	-1.93750	-2.87500	-3.81250	-4.75000
	0.300	-1.44000	-2.79000	-4.14000	-5.49000	-6.84000
	0.350	-1.96000	-3.79750	-5.63500	-7.47250	-9.31000

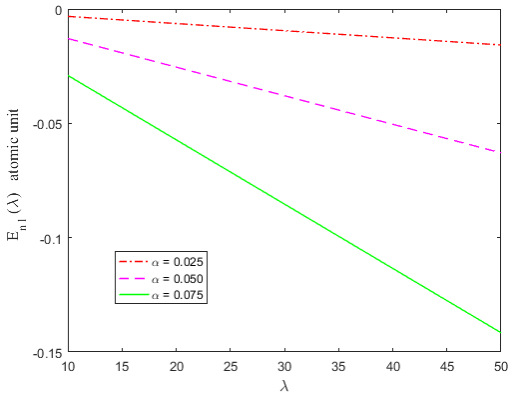

Figure 1: Variation of energy eigenvalues with potential depth for different screening parameters

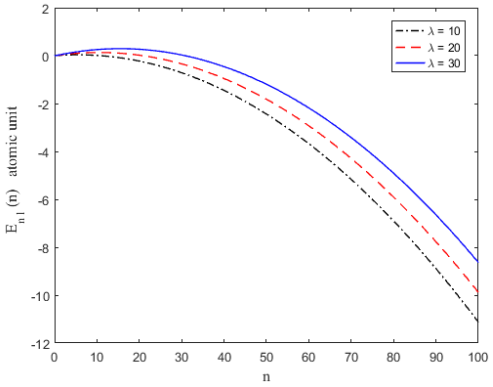

Figure 2: Graphical representation of energy eigenvalues as a function of principal quantum number for different values of potential depths

4. Eigenfunctions of the Pöschl-Teller type Potential

Having obtained the energy eigenvalues, the radial wave function of the Schrödinger equation can easily be obtained from Riccati equation. If we implore the phase angle-wave function relationship, Eq. (25) with the help of Eqs. (19)-(21) can be written as

$$2x(1-x)u''_{n\ell}(x)+(1-3x)u'_{n\ell}(x)+\left\{\frac{\varepsilon_0^2}{2}-\frac{L}{2}\frac{1}{x}-\frac{\lambda(\lambda+1)+Lc_0-\varepsilon_{n\ell}^2}{2}\frac{1}{1-x}\right\}u_{n\ell}(x)=0\,. \tag{36}$$

Letting

$$u_{n\ell}(x)=N_{n\ell}\,x^{\frac{1}{2}\delta}(1-x)^{\frac{1}{2}\sigma}\,G_{n\ell}(x)\,, \tag{37}$$

where η and σ constants, $N_{n\ell}$ is the normalization constant and $G_{n\ell}(x)$ is an unknown function. Substitution of Eq. (37) into (36) leads to the following Gauss hypergeometric ordinary differential equation

$$x(1-x)G''_{n\ell}(x)+\left\{\ell+\tfrac{1}{2}-(\ell+\sigma+\tfrac{3}{2})x\right\}G'_{n\ell}(x)-\left\{\tfrac{1}{4}(\ell+\sigma)^2+\tfrac{1}{4}(\ell+\sigma)-\tfrac{1}{4}\varepsilon_0^2\right\}G_{n\ell}(x)=0\,, \tag{38}$$

whose solution is the hypergeometric function given by

$$G_{n\ell}(x)={}_2F_1\left(-n,\ n+\ell+\sigma+\tfrac{1}{2};\ \ell+\tfrac{1}{2};\ x\right), \tag{39}$$

in which parameter σ is constrained as

$$\sigma=\left\{\lambda(\lambda+1)+Lc_0-\varepsilon_{n\ell}^2\right\}^{\frac{1}{2}}, \tag{40}$$

$$\delta=\ell+1\,. \tag{41}$$

The normalization condition of wave functions [5, 20] requires that

$$\int_0^\infty\left|u_{n\ell}(r)\right|^2 dr=1\,. \tag{42}$$

Using Eqs. (17), (37) and (39) in (42), we have

$$N_{n\ell}=\left(\frac{2\alpha}{S_{n\ell}}\right)^{\frac{1}{2}}, \tag{43}$$

where

$$S_{n\ell}=\int_0^1 x^{\delta-\frac{1}{2}}(1-x)^{\sigma-1}\left|{}_2F_1\left(-n,\ n+\delta+\sigma+\tfrac{1}{2};\ \delta+\tfrac{1}{2};\ x\right)\right|^2 dx\,. \tag{44}$$

Using Eq. (37), we have plotted normalized radial wave function of the Pöschl-Teller like potential for some few quantum states as shown in figure 3. This result is an affirmation that the wave function given by Eq. (37) is physically realizable for the Pöschl-Teller like potential.

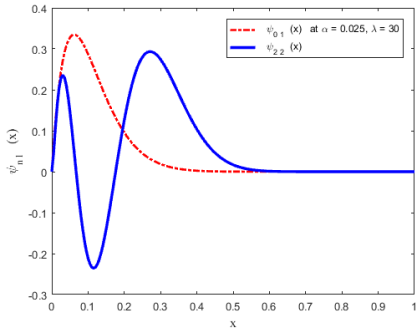

Figure 3: Normalized radial wave functions ψ_{01} and ψ_{22} of the Pöschl-Teller type potential

5. Root-mean-square-speeds of a system in Pöschl-Teller type potential

Using the Hellmann-Feynman theorem (HFT) [3, 7, 21], important relations of mean (or expectation) values of physical observables of the Pöschl-Teller type potential is derived. The HFT states that if the Hamiltonian H(q) of a given quantum mechanical system is a function of some parameter, q, then H(q, r), $E_{n\ell}(q)$ and $u_{n\ell}(r, q)$ are connected by

$$\frac{\partial E_{n\ell}(q)}{\partial q} = \left\langle u_{n\ell}(r, q) \left| \frac{\partial \mathrm{H}}{\partial q} \right| u_{n\ell}(r, q) \right\rangle . \tag{45}$$

In order to use Eq. (45) to obtain expressions for expectation values, first, the Hamiltonian of the system given by the term in curly parenthesis of Eq. (6) is required. Upon substituting Eq. (14) in this expression we have that

$$\mathrm{H} = -\frac{\hbar^2}{2\,\mathrm{M}}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\alpha^2\hbar^2}{2\,\mathrm{M}}\lambda(\lambda+1)\tanh^2(\alpha r) + \frac{L\hbar^2}{2\,\mathrm{M}\,r^2} . \tag{46}$$

5.1 Mean value of centrifugal term potential

Differentiating Eqs. (39) with respect to L, we have

$$\frac{\partial E_{n\ell}}{\partial L} = \frac{\alpha^2\hbar^2 c_0}{2\,\mathrm{M}} . \tag{47}$$

Similarly, differentiating Eq. (46) with respect to L gives

$$\frac{\partial \mathrm{H}}{\partial L} = \frac{\hbar^2}{2\,\mathrm{M}\,r^2} . \tag{48}$$

Substituting Eqs. (47) and (48) in (45) corresponding to $q = L$, we have that

$$\left\langle r^{-2} \right\rangle = \alpha^2 c_0 . \tag{49}$$

The expectation value of the spin-orbit term, given as $\left\langle \mathrm{V}_L \right\rangle \equiv \frac{L\hbar^2}{2\,\mathrm{M}}\left\langle r^{-2} \right\rangle$ [3] is

$$\left\langle \mathrm{V}_L \right\rangle \equiv \frac{L\alpha^2\hbar^2 c_0}{2\,\mathrm{M}} . \tag{50}$$

5.2 Mean of the Pöschl-Teller type potential energy function

Taking the mean of Eq. (1) gives the expectation value of the Pöschl-Teller type potential energy function as

$$\left\langle \mathrm{V} \right\rangle = \frac{\alpha^2\hbar^2}{2\,\mathrm{M}}\lambda(\lambda+1)\left\langle \tanh^2(\alpha r) \right\rangle . \tag{51}$$

The expression for $\left\langle \tanh^2(\alpha r) \right\rangle$ appearing in Eq. (51) can be obtained via the Hellmann-Feynman theorem. In other to do this, we differentiate Eqs. (35) and (46) with respect to λ, this leads to

$$\frac{\partial E_{n\ell}}{\partial \lambda} = \frac{\alpha^2\hbar^2}{2\,\mathrm{M}}(2n+1) , \tag{52}$$

$$\frac{\partial \mathrm{H}}{\partial \lambda} = \frac{\alpha^2\hbar^2}{2\,\mathrm{M}}(2\lambda+1)\tanh^2(\alpha r) , \tag{53}$$

Replacing Eqs. (52) and (53) in (45), yields

$$\left\langle \tanh^2(\alpha r) \right\rangle = \frac{2n+1}{2\lambda+1} . \tag{54}$$

It follows that with Eq. (54) substituted into Eq. (51), the mean of the Pöschl-Teller type potential function is given by

$$\left\langle \mathrm{V} \right\rangle = \frac{\alpha^2\hbar^2}{2\,\mathrm{M}}\frac{\lambda(\lambda+1)(2n+1)}{2\gamma+1} . \tag{55}$$

5.3 The mean kinetic energy of the system

The expectation value of kinetic energy $\langle \mathrm{T} \rangle$ is given as [3]

$$\langle \mathrm{T} \rangle \equiv \left\langle \mathrm{u}_{n\ell}(r,q) \left| -\frac{\hbar^2}{2\,\mathrm{M}} \frac{\mathrm{d}^2}{\mathrm{d}\,r^2} \right| \mathrm{u}_{n\ell}(r,q) \right\rangle . \tag{56}$$

To obtain closed form explicit relation for $\langle \mathrm{T} \rangle$ by the Hellmann-Feynman theorem, if we differentiate Eqs. (35) and (46) with respect to M, we obtained

$$\frac{\partial E_{n\ell}}{\partial \mathrm{M}} = -\frac{\alpha^2 \hbar^2}{2M^2}\left(L c_0 + 2\lambda n + \lambda - 4n^2 \right), \tag{57}$$

$$\frac{\partial \mathrm{H}}{\partial \mathrm{M}} = \frac{\hbar^2}{2\,\mathrm{M}^2} \frac{\mathrm{d}^2}{\mathrm{d}\,r^2} - \frac{\alpha^2 \hbar^2}{2\,\mathrm{M}^2} \lambda(\lambda+1) \tanh^2(\alpha r) - \frac{L\hbar^2}{2\,\mathrm{M}^2 r^2} . \tag{58}$$

Inserting Eqs. (57) and (58) in (45) and substituting Eqs. (49), (54) and (56) in the resulting expression, we obtained the mean kinetic energy of the system as

$$\langle \mathrm{T} \rangle = \frac{\alpha^2 \hbar^2 \lambda^2}{2\,\mathrm{M}} \left(\frac{2n+1}{2\lambda+1} - \frac{4n^2}{\lambda^2} \right). \tag{59}$$

The kinetic energy of the system is given by $T = ½\ \mu\ v^2$, where v is the speed. Taking the mean of this expression, we obtained

$$\langle \mathrm{v}^2 \rangle = \frac{2}{\mathrm{M}} \langle T \rangle . \tag{60}$$

The root-mean-square speed is given by

$$v_{\mathrm{rms}} \equiv \langle \mathrm{v}^2 \rangle^{\frac{1}{2}} = \frac{\alpha \hbar \lambda}{\mathrm{M}} \sqrt{\frac{2n+1}{2\lambda+1} - \frac{4n^2}{\lambda^2}} . \tag{61}$$

Using the expression (61), we have computed root-mean-square speed as a function of screening parameter and potential depth. The results obtained for arbitrary principal and angular momentum quantum numbers are shown in Table 2. In figure 4, we have plotted the variation of root-mean-square speed as a function of potential depth for different screening parameters of the Pöschl-Teller type potential. The plot reveals that the root-mean-square speed increases monotonically with increase in potential and also increases with increase in potential screening parameter.

Table 2: root-mean-square speed atomic unit ($\hbar$ = M = 1) of a system in Pöschl-Teller type potential

state	α	$v_{\mathrm{rms}}(\lambda)$				
		λ = 10	λ = 20	λ = 30	λ = 40	λ = 50
2p	0.025	0.0546	0.0781	0.0960	0.1111	0.1244
	0.050	0.1091	0.1562	0.1921	0.2222	0.2488
	0.075	0.1637	0.2343	0.2881	0.3333	0.3731
	0.100	0.2182	0.3123	0.3841	0.4444	0.4975
	0.150	0.3273	0.4685	0.5762	0.6667	0.7463
	0.200	0.4364	0.6247	0.7682	0.8889	0.9950
	0.250	0.5455	0.7809	0.9603	1.1111	1.2438
	0.300	0.6547	0.9370	1.1523	1.3333	1.4926
	0.350	0.7638	1.0932	1.3444	1.5556	1.7413
3p	0.025	0.0802	0.1257	0.1586	0.1858	0.2095
	0.050	0.1604	0.2513	0.3173	0.3717	0.4191
	0.075	0.2405	0.3770	0.4759	0.5575	0.6286
	0.100	0.3207	0.5027	0.6345	0.7434	0.8382
	0.150	0.4811	0.7540	0.9518	1.1150	1.2573
	0.200	0.6414	1.0054	1.2691	1.4867	1.6764
	0.250	0.8018	1.2567	1.5863	1.8584	2.0955
	0.300	0.9621	1.5080	1.9036	2.2301	2.5146
	0.350	1.1225	1.7594	2.2208	2.6018	2.9337
3d	0.025	0.0546	0.0781	0.0960	0.1111	0.1244
	0.050	0.1091	0.1562	0.1921	0.2222	0.2488
	0.075	0.1637	0.2343	0.2881	0.3333	0.3731
	0.100	0.2182	0.3123	0.3841	0.4444	0.4975
	0.150	0.3273	0.4685	0.5762	0.6667	0.7463
	0.200	0.4364	0.6247	0.7682	0.8889	0.9950
	0.250	0.5455	0.7809	0.9603	1.1111	1.2438
	0.300	0.6547	0.9370	1.1523	1.3333	1.4926
	0.350	0.7638	1.0932	1.3444	1.5556	1.7413

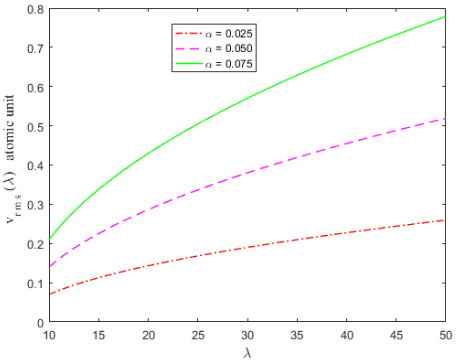

Figure 4: Variation of root-mean-square-speed as a function of potential depth for different values of screening parameters

6. Conclusion

In this paper, expressions for bound state energy eigenvalues and radial wave functions of the Pöschl-Teller type potential were obtained in compact form via the improved quantization rule and ansatz solution methods. In dealing with the effective potential of the Schrödinger equation, the improved Greene-Aldrich approximation model was employed. Equations for energy eigenvalues and normalized radial wave functions for the s-wave are in excellent agreement with available literature data for this potential. Equation for bound state energy eigenvalues and the Hellmann-Feynman theorem was subsequently used to derive expression for root-mean-square speed for the Pöschl-Teller type potential. The root-mean-square speed of the system increases monotonically when the potential depth is increased.

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