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ON CONNECTEDNESS OF THE DIGRAPH OF PARTIAL TRANSFORMATION SEMIGROUP

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Abstract

Let $X_n = \{1, 2, \dots, n\}$ be a natural ordering on which the partial transformation semigroup and its digraph are defined. We state a sufficient condition for a partial digraph of transformation to be strongly, unilaterally and weakly connected and enumerated it. We also found the cardinality of its Hamiltonian and Eulerian digraphs which rely greatly on connectedness of a digraph.

***Keywords*:** Digraph, Digraph of transformation, Partial transformation semigroup, Connectedness, Hamiltonian and Eulerian Digraphs.
MSC: 20M20 Semigroups of transformations, relations, partitions etc.
05C30 Enumeration in graph theory

1. Introduction

A transformation is a function α on which domain and image are defined on a natural order $X_n = \{1, 2, \dots, n\}$. A transformation α is partial if domain of α ($Dom\ \alpha \subseteq X_n$). The collection of all partial transformations is termed the partial transformation semigroup (P_n). It is well known that the number of elements of P_n is given as $(n+1)^n$.
A *directed graph (digraph)* is an ordered pair $\Gamma = (V, A)$, where V is the set of vertices of Γ and $A \subseteq V \times V$ is called the arcs of Γ. Let $\Gamma_\alpha = (V_\alpha, A_\alpha)$ be the *digraph of transformation* α constructed naturally from a digraph, where the vertex set V_α represents the $Dom(\alpha)$, and for any $x \in X_n$, $(x, y) \in A_\alpha$ is an arc if and only if $\alpha(x) = y$.
Many authors have considered enumeration of a combinatorial nature on transformation semigroups, e.g. see [1], [2], [7] and [8]. It's quite natural that we consider enumeration on $\Gamma_\alpha = (V_\alpha, A_\alpha)$.
Necessary and sufficient conditions for connectedness of cayley graphs of a semigroup were investigated in [10] and in [3], necessary and sufficient conditions for strongly, unilateral and weakly connected cayley graphs of finite transformation semigroups were stated. The numbers of acyclic and tree with a sink Γ_α for $\alpha \in P_n$ was given in [9] as $\sum_{k=1}^{n} \binom{n-1}{k-1} n^{n-k}$ and n^{n-1} respectively. We bring the results of this paper in same direction.

2. Preliminaries

We give a review of some basic terms, definitions and theorem germane to our purpose, For a detailed study on digraph see [5] and [6].
Let (u, v) be an arc in the arc set of the digraph, $\Gamma = (V, A)$,. u is *adjacent to*v and v is *adjacent from* u. The arc (u, v) is a *loop* when $u = v$.
The *indegree*, $id(v)$, of a vertex v is the number of vertices adjacent to v and the *out degree*, $od(v)$, is the number of vertices adjacent from v. The vertex v is a source if $id(v) = 0$ and a sink if $od(v) = 0$.
A walk is an alternating sequence $(v_0, a_1, v_1, a_2, v_2, \dots, v_{n-1}, a_n, v_n)$ of vertices and arcs and a path is a walk with distinct vertices. A cycle is a closed path where $v_0 = v_n$. The length of a path or a cycle is determined by the number of arcs it contains. A loop is admitted as a cycle of length 1. The trivial digraph is the digraph with only one vertex.

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Definition 2.1: Given a vertex set $V = (v_1, v_2, \ldots, v_{n-1}, v_n)$of $\Gamma = (V, A)$. The digraph is strongly connected or strong if for any $v_i, v_j \in V$, there is a path (v_i, v_j) and a path (v_j, v_i). It is unilaterally connected if there is a path (v_i, v_j) or a path (v_j, v_i) and weakly connected or weak when there is a semipath between v_i and v_j i.e. there is a path when viewed as a graph.
Weakly connected digraphs implies unilaterally connected which implies strongly connected. Strictly weak digraphs implies weak but not unilateral and strictly unilaterally connected implies unilateral but not strong.
Definition 2.2: A cycle C is a *Hamiltonian cycle* if $V(C) = V(\Gamma)$. A digraph Γ is *Hamiltonian* if D contains a Hamiltonian cycle. A *trail* $C = v_1, v_2, \ldots, v_{n-1}, v_n$ is an *Eulerian trail* if $A(C) = A(\Gamma), V(C) = V(\Gamma)$ and $v_1 = v_n$.
Theorem 2.3: ([4] Theorem 7.6) Let Γ be a nontrivial connected digraph. Then Γ is Eulerian if and only if $od(v) = (id)v$ for every vertex v of Γ.

3. Main results

Lemma 3.1: Given $\alpha \in P_n$ and Γ_α a digraph with n vertices

i. Γ_α is strongly connected if and only it consists of exactly one cycle of length n
ii. Γ_α is unilaterally connected, if and only if there is either a cycle of length $(n-k)$ and a path of length k from a vertex not in the cycle to a vertex in the cycle or there is a path of length $(n-1)$
iii. Γ_α is weakly connected if and only if it has exactly one cycle and more than one path from vertices not in the cycle to any of the vertices in the cycle

Proof: i. Clearly for any two vertices $v_i, v_j : v_i \neq v_j$, v_i is reachable to v_j and v_j is reachable from v_i when $v_0 = v_n$ in a closed spanning path $P = \left(v_{0,} v_1, v_2, v_3, \ldots, v_n\right)$. Therefore, we have a cycle of maximum length n.
ii. Any two vertices on the cycle of length $(n-k)$ is reachable from and to the other. The path of length k beginning from a vertex v_i is a source and not reachable from any other vertex in Γ_α. Therefore there exist at least an arc without converse.
Let $P = (v_1, v_2, \ldots, v_n)$ be a path with distinct vcertices of length n-1. If the path begins with v_1, then it is a source and not reachable from v_2, v_2 in turn is not reachable from other vertices except v_1 etc. Therefore, for any two vertices v_i, v_j, if there exist the arc $(v_i, v_j) \in P$ then $(v_j, v_i) \notin P$ and vice versa.
iii. Having at least two paths beginning from vertices v_i and $v_j$not in the cycle to meet any of the vertices on the cycle shows that v_i is not reachable from and tov_j . Therefore there is at least two vertices not reachable from and to the other. ■

From the above, we deduce a necessary condition for connectedness in P_n

Lemma 3.2: Let $\alpha \in P_n$, the image and rank of α are given as $im(\alpha) = \{x\alpha : x \in \{1, \ldots, n\}\}$ and $rank(\alpha) = |im(\alpha)|$. The digraph Γ_α is

i. strong if $rank(\alpha) = |im(\alpha)| = n$
ii. strictly unilateral if $rank(\alpha) = |im(\alpha)| = n - 1$
iii. strictly weak if $rank(\alpha) = |im(\alpha)| = \{1, 2, \ldots, n-2\}$

Proof: i. It is easy to see that a cycle of length n has $|im(\alpha)| = n$
ii. Given n vertices, a cycle C of length $(n-k)$ indicates $|im(C)| = n - k$ and a path P of length k from a vertex not in C to a vertex in C shows the existence of a source which implies $|im(P)| = k - 1$. Therefore $|im(\alpha)| = (n-k) + (k-1) = n - 1$
iii. at least two paths P_1, P_2 with lengths k, r respectively meet any vertices on C implies $|im(\alpha)| = (n-k-r) + (k-1) + (r-1) = n - 2$ or less. $\therefore$ $|im(\alpha)| = \{n-2, n-3, \ldots, 2, 1\}$

Proposition 3.3: Given $\alpha \in T_n$, the number of α for which Γ_α is

i. strong is $(n-1)!; n \geq 1$
ii. strictly unilateral is $n!\,(n-1) + n!; n \geq 2$

Proof: i. We observe that Γ_α is a cycle of maximum length with n distinct vertices and arcs. It is not difficult to see that the permutations of vertices in the cycle is$(n-1)!$. Also the trivial Γ_α is strongly connected, hence the result.

ii. Given n vertices, with a cycle of length $(n-k)$ and a path of length k from a vertex not in the cycle to a vertex in the cycle. There are $n-1$ different $(n-k)$ lengths of cycles. For each of the different lengths of cycles, there are$(n-1)!\,n$ ways of arranging the path of length k to meet any of the vertices of the cycle. Therefore for such Γ_α, there are $(n-1)(n-1)!\,n = n!\,(n-1)$.
Also, the digraphs with only a path of length $(n-1)$ is of n vertices, there are $n!$ ways of different arrangement of the vertices on the path. The digraph Γ_1 is not strictly unilateral, hence $n \geq 2$. ■

We state other results related to connectedness of P_n. From the definition of Hamiltonian and Eulerian Digraphs, it is obvious they are strong digraphs and we enumerate them.

Corollary 3.4: Let$\alpha \in P_n$, the number of α for which Γ_α is eulerian is $(n-1)!\colon n \geq 1$
Proof: Let $P = (v_{0,}v_1, v_2, v_3, \dots, v_n)$ be a closed path with n distinct arcs. Theorem 2.3 is satisfied since $od(v_i) = id(v_i) = 1$ for $i = 1, 2, \dots, n$ and it is strongly connected. We fix an arc $(v_i, v_j)\colon i \neq j$and find the permutations of the other arcs in $(n-1)!$ Ways. Thus the result.

Corollary 3.5: Let$\alpha \in P_n$, the number of α for which Γ_α is Hamiltonian is $(n-1)!\colon n \geq 1$
Proof: As proved in Proposition 3.3(i).

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