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THE DIGRAPH OF THE SYMMETRIC INVERSE TRANSFORMATION SEMIGROUP

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Abstract

Enumeration of a combinatorial nature in the study transformation semigroup is inevitable and a digraph of transformation is naturally formed given any transformation semigroup.
Let $X_n = \{1, 2, \dots, n\}$ be a finite set on which the domain and range of the partial injective (symmetric inverse) transformations is defined. We characterized its strongly, unilateral and weakly connected digraphs and found its cardinality. Also cardinality of its hamiltonian, eulerian and acyclic digraphs were characterized.

Keywords: Digraph of transformation, Symmetric Inverse transformation semigroup, Connected, Hamiltonian, Eulerian, Acyclic Digraphs.
MSC:
20M20 Semigroups of transformations, relations, partitions etc.
05C30 Enumeration in graph theory

1 Introduction

A map α with domain and range as subsets of a natural ordering $X_n = \{1, 2, \dots, n\}$ is called a *transformation*. A transformation is termed *partial* if domain of α ($Dom\ \alpha \subseteq X_n$). A collection of all partial transformations is then called the *partial transformation semigroup* (P_n). A subset of P_n is called *full* or *total* if $Dom\ \alpha = X_n \longrightarrow X_n$. A collection of all full transformation is the *full transformation semigroup* (T_n). Another subset of P_n is the semigroup of all partial one-one mapppings on X_n commonly referred to as the *symmetric inverse semigroup* (I_n). It is well known that the number of elements of I_n is $\sum_{k=0}^{n} \binom{n}{k}^2 . k!$.

Let $\Gamma = (V, A)$ be a *directed graph (digraph)* where V is the *vertex set* and A a set of ordered pair of distinct vertices called the *arc set* i.e. $A \subseteq V \times V$. The digraph $\Gamma_\alpha = (V_\alpha, A_\alpha)$ of a transformation α is naturally formed when given a transformation semigroup.

Several results have emanated from the algebraic and combinatorial properties of digraphs of semigroups. In [1] it was proved that for $n \geq 3$, the set of idempotents of defect 1 of the singular mappings ($Sing_n$) of T_n is a generating set if and only if its associated digraph is strongly connected and is a complete (undirected) graph and $\frac{1}{2}n(n-1)$ is the number of strong labelled tournaments generated by the idempotents of $Sing_n$.Given two digraphs Γ_1 and Γ_2, its widened digraphs $w(\Gamma_1)$ and $w(\Gamma_2)$ was introduced in [2] and proved that for any two digraphs Γ_1 and Γ_2, its Γ_α and $\Gamma_\alpha{}'$ respectively are isomorphic if and only if $w(\Gamma_1)$ and $w(\Gamma_2)$ are isomorphic. The characterization of the Green's equivalence digraphs of the semigroup of Boolean matrices was studied in [3].

Necessary and sufficient conditions were stated for cayley graphs of α to be strongly connected, unilaterally connected and weakly connected. See [4], while in [5], enumeration of the acyclic and tree Γ_α of $T_n$was given as $\sum_{k=1}^{n} \binom{n-1}{k-1} n^{n-k}$ and n^{n-1} respectively. It is in this direction that we present the results of this paper.

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2 Preliminaries

We review some basic terms, definitions and results on digraph. For an authoritative study on digraphs, see [6] and [7].
Let (u, v) be an arc in the arc set A of $\Gamma = (V, A)$where $u, v \in V$. From the arc, u is adjacent to v and v is adjacent from u. The arc (u, v) is a *loop*if $u = v$.
The *indegree*$id(v)$ of v is the number of arcs from which v is adjacent and its *outdegree*$od(v)$ is the number of arcs to which v is adjacent. A vertex v is a *sink* when $od(v) = 0$ and a *source*when $id(v) = 0$.
A *walk* in Γ is an alternating sequence of vertices and arcs $(v_0 a_1 v_1 a_2 v_2 \ldots v_{n-1} a_n v_n)$ where an arc a_i is represented as $(v_{i-1} v_i)$. The number of arcs in a walk is its *length*. A *closed walk* has the first and last vertices as same while a *spanning walk* contains all the vertices in Γ. A *path* is a walk with distinct vertices and a *cycle* is a nontrivial closed path except that the first and last vertices are same. A loop is admitted as a cycle of length 1. A digraph without cycles is *acyclic*.
A *trivial* digraph is one containing only a vertex while the digraph of isolated vertices has more than one vertex each with$id(v) = od(v) = 0$.

Definition 2.1: A digraph Γ is *strongly connected* or *strong* if for any pair of vertices u and v in Γ, there is a path (u, v) and (v, u). It is *unilaterally connected* or *unilateral* if there is either a path (u, v) or (v, u) and *weakly connected or weak* if there is a semipath between any pair of vertices (i.e.) there is a path between any two vertices when viewed as a graph. Obviously strongly connected implies unilaterally connected which implies weakly connected. Also Γ is *strictly unilaterally connected* if it is unilateral but not strong and *strictly weakly connected* but not unilateral.

Definition 2.2: An *Eulerian* digraph Γ is a connected digraph in which every arc of Γ occurs exactly once.
Definition 2.3: A digraph Γ is *Hamiltonian* if Γcontains a spanning cycle.
Theorem 2.4: Theorem 7.6 of [8] Let D be a nontrivial connected digraph. Then D is Eulerian if and only if $od(v) = id(v)$for every vertex v of D.
Definition 2.5: A $\Gamma_\alpha = (V_\alpha, A_\alpha)$ is called a *tree with a sink* if for all $v \in V_\alpha$, the path beginning from vertex v is unique and terminates at the sink. It contains all vertices of Γ_α.

3 Main results

We characterize the connected digraphs of I_n.
Lemma 3.1: Given $\alpha \in I_n$ and Γ_α a digraph with n vertices

i. Γ_α is strongly connected if and only if it consists of exactly one cycle of length n
ii. Γ_α is unilaterally connected, if and only if there exists a path of length $(n-1)$
iii. Γ_α is not weakly connected

Proof: i. Clearly for any two vertices $v_i, v_j : v_i \neq v_j$, v_i is reachable to v_j and v_j is reachable from v_i when $v_0 = v_n$ in a closed spanning path $P = (v_{0,} v_1, v_2, v_3, \ldots, v_n)$. Therefore, we have a cycle of maximum length n.
ii. Let $P = (v_1, v_2, v_3, \ldots, v_n)$be a path with distinct vertices and arcs of length $n-1$. The first vertex is a source and not reachable from any other vertex on the path. The second vertex is also not reachable from the other vertices except the first etc. Therefore for any two vertices, v_i, v_j there is the arc (v_i, v_j) but not (v_j, v_i) or vice versa.
iii. by contradiction, if Γ_α is weakly connected then it has exactly one cycle and more than one path beginning from vertices say $v_i, v_j : v_i \neq v_j$ not in the cycle to any of the vertices in the cycle. $v_i, v_j$are sources and $(v_i, v_j), (v_j, v_i) \notin \Gamma_\alpha$ but Γ_α is not disconnected. The vertices on the cycle where the paths ends have each an indegree of 2 but all vertices in Γ_α for $\alpha \in I_n$ has indegree or outdegree of at most 1. Therefore, Γ_α is not weakly connected.■

Proposition 3.2: Given $\alpha \in T_n$, the number of α for which Γ_α is

i. strong is $(n-1)!; n \geq 1$
ii. strictly unilateral is $n!: n \geq 2$

Proof:
i. We observe that Γ_α is a cycle of maximum length with n distinct vertices and arcs. It is not difficult to see that the permutations of vertices in the cycle is$(n-1)!$. Also the $\Gamma_\alpha$when $n = 1$ and is a loop is strongly connected, hence the result.

ii.$\Gamma_\alpha$is unilaterally connected, if and only if there exists a path of length $(n-1)$. The permutations of n vertices on the path is $n!$ and the $\Gamma_\alpha$when $n = 1$ is a loop and strongly connected. Hence $n \geq 2$.■

Lemma 3.3: Let$\alpha \in I_n$, the number of α for which Γ_α is eulerian is $(n-1)!: n \geq 1$
Proof: Let $P = (v_{0,}v_1, v_2, v_3, \dots, v_n)$ be a closed spanning path with n distinct arcs. Theorem 2.4 is satisfied since $od(v_i) = id(v_i) = 1$ and it is strongly connected. We can fix an arc $(v_i, v_j): i \neq j$ in $n-1$ ways with v_i as the tail and v_j as the head. For each fixed arc, there are $n-2$ ways of placing the next arc with v_j as the tail and $n-3$ for the next while maintaining the head-tail arrangement. The last arc has v_i as the head. Therefore the number of Γ_α which is eulerian is $(n-1)(n-2)(n-3) \dots 2 \cdot 1$. ■

Lemma 3.4: Let$\alpha \in I_n$, the number of α for which Γ_α is Hamiltonian is $(n-1)!: n \geq 1$.
Proof: As proved in Proposition 3.2(i).

Lemma 3.5: The number of $\alpha \in I_n$, for which Γ_α is a treewithasink is$n!$: $n \geq 1$
Proof: By definition 2.5, a tree with a sink is a path containing the all the (distinct) vertices in $\Gamma_\alpha$and contains $n-1$ arcs and therefore is in the rank of $n-1$ where image and rank of α is $im(\alpha) = \{x\alpha : x \in \{1, \dots, n\}\}$ and $rank(\alpha) = |im(\alpha)|$ respectively. The Γ_α is the permutation of n vertices on the path,hence $n!$. The trivial Γ_α is regarded as a root and admitted as a tree.■

Lemma 3.6: The number of $\alpha \in I_n$, for which Γ_α is acyclic is$a(n) = 1 + \sum_{k=2}^{n} \frac{n!}{(n-k)!} : n \geq 2, a(1) = 1$.
Proof: An acyclic digraph of I_n is a walk and $\alpha \in I_n$ is injective, hence the walk has distinct vertices. Therefore $rank(\alpha) = \{n-1, n-2, \dots, 2, 1, 0\}$. We find permutation of n vertices taking $n, n-1, n-2, \dots, 3, 2$ vertices at a time. The trivial digraph is also acyclic. ■

4 Conclusion

The Online Encycloaedia of integer sequences [9] has some of the sequences of our enumeration.
The number of strictly unilateral connected digraph of I_n for $n = 1, 2, \dots$ is the sequence in http://oeis.org/A000142 of [9].
The number of the strongly connected, Hamiltonian and Eulerian digraphs of I_n for $n = 1, 2, \dots$ is the sequence called the shifted factorial numbers and can be found in http://oeis.org/A104150 of [9].
The number of acyclic digraphs of I_n for $n = 1, 2, \dots$ is in http://oeis.org/A243014 of [9].

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