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A NEW FOUR-PARAMETER DISTRIBUTION WITH STRUCTURAL PROPERTIES AND APPLICATION

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Abstract

The paper considers the proposition of the exponentiated Lomax-Exponential distribution as a new four-parameter distribution which the aim of attracting applications in analysing monotonic and nonmonotonic lifetime data. Some special cases are presented as submodels. Several structural properties of the distribution are derived as explicit mathematical expressions. Inference about the parameters of the new distribution are obtained by the use of the maximum likelihood estimation. Real data sets are used to comparethe proposed distribution with some competing models existing in literature.

Keywords: Exponentiated, Lomax-Exponential, Reliability, Moments, Order statistics.

MSC2010: 60E05, 62-07

1. Introduction

Parameter induction into classical univariate distributions such as Weibull and exponential distributions to generate new distributions have been have been given great scholarly attentions by researchers in lifetime data analysis over the last twenty years. Methods for generating new distributions from classical ones have been discussed in literature with the works of [1], [2] and [3]. One method that is frequently employed in constructing new lifetime distributions is the exponentiated-G family. Suppose $F(x)$ and $f(x)$ are cumulative distribution function (cdf) and probability density function (pdf) of a baseline distribution, then the cdf and pdf for an exponentiated family are given as

$$G(x) = [F(x;\underline{\varepsilon})]^{\rho}, \rho > 0 \qquad (1.1)$$

where $\underline{\varepsilon} > \underline{0}$ defines the parameter vector of the baseline distribution.

First pioneered by [4], several classes of lifetime distributions have been generated using (1.1). Some of these classes include exponentiated T-X family by [5], exponentiated Weibull$-G$ family by [6], exponentiated half-logistic family by [7] and exponentiated generalized Topp Leone$-G$ family by [8] amongst others in literature.

Suppose a system has two independent components connected serially. Let the first component has survival function of the Lomax distribution defined by $S_L(x)$ and the second component has survival function of the exponential distribution defined by $S_{Exp}(x)$, then the cdf of the competing risk distribution for the series system is given by

$$F(x) = 1 - S_L(x)S_{Exp}(x)$$

Substituting $S_L(x) = (1+\theta x)^{-\gamma}$ and $S_{Exp}(x) = e^{-\eta x}$ into the above equation, we have

$$F(x) = 1 - (1+\theta x)^{-\gamma}e^{-\eta x}, \theta > 0, \gamma > 0, \eta > 0 \qquad (1.2)$$

which is a new three-parameter lifetime model known as the Lomax-Exponential distribution. More discussions on series (or competing risk) distributions are found in the work of [9]. The exponentiation of a baseline model given in (1.2) is considered in the paper. As a flexible distribution, it could be useful in the modelling lifetime data for prediction of survival behaviours of systems over time.

The paper is divided into the followig sections. Section 2 derives Exponentiated Lomax-Exponential (ELExp) distribution as new lifetime model with its defining structural properties. Section 3 presents other structural properties of the proposed distribution. Parameter estimation for the ELExp distribution is, also, considered in this section. Section 4 considers the application of the ELExp distribution to secondary data sets and discussion on the results is presented. Section 5 gives concluding remarks on the study of the ELExp distribution.

2. Exponentiated Lomax-Exponential (ELExp) distribution

2.1. Model and its defining properties

Substituting (1.2) into (1.1), the cdf of the ELExp distribution is obtained as

$$G(x) = [1 - (1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho}, \theta > 0, \gamma > 0, \eta > 0, \rho > 0 \qquad (2.1)$$

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The ELExp distribution generalizes some submodels such as new Lomax-exponential, exponentiated Lomax ([10]), Lomax and exponential distributions.
Differentiating (2.1) with respect to x gives the pdf of the ELExp distribution which is given as

$$g(x) = \rho(\gamma\theta + \eta(1+\theta x))(1+\theta x)^{-\gamma-1}e^{-\eta x}[1-(1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho-1} \qquad (2.2)$$

The pdf of the ELExp distribution can be expanded into a simplier form using the power series expansion. The expansion is needed for evaluation of some structural properties of the distribution. For $b > 0$ which is also an integer,

$$(1+x)^b = \sum_{k=0}^{\infty} b_k\, x^k,$$

holds provided $x < 1$.
Then, we have

$$g(x) = \rho \sum_{k,j=0}^{\infty} \binom{\rho-1}{k}\binom{\gamma(k+1)+j}{j}(-1)^{k+j}\theta^j x^j(\theta j + \eta(1+\theta x))\, e^{-(k+1)\eta x} \qquad (2.3)$$

The survival, hazard and reversed hazard functions which are denoted by $\overline{G}(x)$, $h(x)$ and $\overline{h}(x)$ are given as

$$\overline{G}(x) = 1 - G(x) = 1 - [1-(1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho}, \qquad (2.4)$$

$$h(x) = \frac{f(x)}{\overline{G}(x)} = \frac{\rho(\gamma\theta+\eta(1+\theta x))(1+\theta x)^{-\gamma-1}e^{-\eta x}[1-(1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho-1}}{[1-(1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho}} \qquad (2.5)$$

and

$$\overline{h}(x) = \frac{f(x)}{G(x)} = \frac{\rho(\gamma\theta+\eta(1+\theta x))(1+\theta x)^{-\gamma-1}e^{-\eta x}[1-(1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho-1}}{1-[1-(1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho}} \qquad (2.6)$$

respectively.

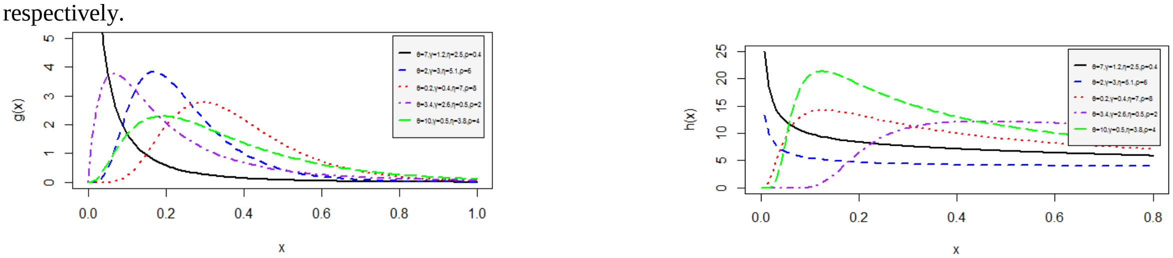

Figure 1: Plots for g(x) and h(x) of ELExp distribution

2.2. Other Structural properties

In this section, other structural properties of the ELExp distribution that are useful in lifetime data analysis will be considered.

2.2.1 Quantile function

For $0 < U < 1$, the ELExp quantile function is derived by inverting the cdf of ELExp distribution using $F(x_U) = U$. It implies that $x_U = G^{-1}(U)$.
which gives

$$\gamma log(1+\theta x_U) + \eta x_U + log(1 - U^{\frac{1}{\rho}}) = 0. \qquad (2.7)$$

x_U defines the root of (2.6) for $U \in (0,1)$ for every set of values for parameters of ELExp distribution employed. Different values of x_U are obtained for sets of values of parameters $(\theta, \gamma, \eta, \rho)$ of ELExp distribution and are listed in Table 2.1.

Table 1: Quantile function and parameter values of ELExp distribution

U	**(0.06,0.08,0.04,2)**	**(4,0.5,0.7,3)**	**(0.2,5.8,0.6,10)**	**(3,9,4.6,1.9)**
0.1	8.7101	0.2410	1.8200	0.0117
0.2	13.7090	0.3454	2.2904	0.0190
0.3	18.4753	0.4415	2.6910	0.0262
0.4	23.4387	0.5404	3.0800	0.0341
0.5	28.9023	0.6490	3.4875	0.0432
0.6	35.2366	0.7758	3.9421	0.0544
0.7	43.0604	0.9348	4.4866	0.0689
0.8	53.7083	1.1569	5.2092	0.0903
0.9	71.3879	1.5425	6.3842	0.1299

It is obvious from Table 1 that monotone increasing values are obtained for the different sets of parameter values of ELExp distribution whenever $U \in (0,1)$.

2.2.2. Moments

Moments are important quantities of distribution theory in statistics employed in the study of important characteristics of random variable, X, having a distribution. Some moments mainly discussed in distribution theory and lifetime data analysis include non-central moment, conditional moment and probability weight moment (PWM). Mathematical expressions will be presented for these moments.

(i) The τ^{th} non-central moment denoted by μ_τ is derived with the relation given by

$$\mu_\tau = E[X^\tau] = \int_0^\infty x^\tau g(x)dx$$

Substituting (2.2) into the above relation gives

$$\mu_\tau = \rho \sum_{k,j=0}^{\infty} \binom{\rho-1}{k}\binom{\gamma(k+1)+j}{j}(-1)^{k+j}\theta^j \int_0^\infty x^{\tau+j}(\gamma\theta+\eta(1+\theta x))\, e^{-(k+1)\eta x}dx. \tag{2.8}$$

Algebraic evaluation of (2.8) gives

$$\mu_\tau = \rho \sum_{k,j=0}^{\infty} \binom{\rho-1}{k}\binom{\gamma(k+1)+j}{j}(-1)^{k+j}\theta^j \left[\frac{\gamma\theta\Gamma(\tau+j+1)}{((k+1)\eta)^{\tau+j+1}} + \eta\left(\frac{\Gamma(\tau+j+1)}{((k+1)\eta)^{\tau+j+1}} + \frac{\theta\Gamma(\tau+j+2)}{((k+1)\eta)^{\tau+j+2}}\right)\right].$$

Hence, the τ^{th} non-central moment of the ELExp distribution is given as

$$\mu_\tau = \rho \sum_{k,j=0}^{\infty} \binom{\rho-1}{k}\binom{\gamma(k+1)+j}{j}(-1)^{k+j}\theta^j \left[\gamma\theta + \eta\left(1+\frac{\theta(\tau+j+1)}{((k+1)\eta)}\right)\right]\frac{\Gamma(\tau+j+1)}{((k+1)\eta)^{\tau+j+1}}. \tag{2.9}$$

(ii) The PWM of a randam variable X is defined as

$$PWM_{\tau,\varphi,\phi}(X) = E\left[X^\tau G^\varphi(X)\overline{G}^\phi(X)\right] = \int_0^\infty x^\tau G^\varphi(x)\overline{G}^\phi(x)g(x)dx \tag{2.10}$$

Substituting (2.1),(2.2) and (2.4) into (2.10), the PWM of the ELExp distribution can be derived from the integral defined as

$$PWM_{\tau,\varphi,\phi}(X) = \rho\int_0^\infty x^\tau(\gamma\theta+\eta(1+\theta x))(1+\theta x)^{-\gamma-1}[1-(1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho(\varphi+1)-1}e^{-\eta x}$$
$$(1-[1-(1+\theta x)^{-\gamma}e^{-\eta x}]^\rho)^\phi e^{-\eta x}dx.$$

Further algebraic expansion of the integrand gives the PMW of the distribution as

$$PWM_{\tau,\varphi,\phi}(X) = \rho\sum_{i=0}^{\nu}\sum_{k,j=0}^{\infty}\binom{\nu}{i}\binom{\rho(\varphi+\phi+1)-1}{k}\binom{\gamma(k+1)+j}{j}(-1)^{i+k+j}\theta^j \times\int_0^\infty x^{\tau+j}(\gamma\theta+\eta(1+\theta x))\,e^{-(k+1)\eta x}dx. \tag{2.11}$$

Evaluation of (2.11) results in

$$PWM_{\tau,\varphi,\phi}(X) = \rho\sum_{i=0}^{\nu}\sum_{k,j=0}^{\infty}\binom{\nu}{i}\binom{\rho(\varphi+\phi+1)-1}{k}\binom{\gamma(k+1)+j}{j}(-1)^{i+k+j}\theta^j \times\left[\gamma\theta+\eta\left(1+\frac{\theta(\tau+j+1)}{((k+1)\eta)}\right)\right]\frac{\Gamma(\tau+j+1)}{((k+1)\eta)^{\tau+j+1}} \tag{2.12}$$

It should be noted that the $PWM_{\tau,\varphi,\phi}(X)$ can becomes the τ^{th} non-central moment. This is true if $\varphi = \phi = 1$.

(iii) The conditional moment defined by $E[X^\tau/X > t]$ for a random variable X is defined by the relation

$$E[X^\tau/X>t] = \frac{1}{\overline{G}(t)}\int_t^\infty x^\tau g(x)dx$$

Hence, the conditional moment of the ELExp distribution is given as

$$E\left[X^\tau/X>t\right] = \frac{\rho}{1-(1-(1+\theta t)^{-\gamma}e^{-\eta t})^\rho}\sum_{k,j=0}^{\infty}\binom{\rho-1}{k}\binom{\gamma(k+1)+j}{j}(-1)^{k+j}\theta^j \times\left[\gamma\theta\frac{\Omega(\tau+j+1,(k+1)\eta t)}{((k+1)\eta)^{\tau+j+1}} + \eta\left(\frac{\Omega(\tau+j+1,(k+1)\eta t)}{((k+1)\eta)^{\tau+j+1}} + \theta\frac{\Omega(\tau+j+2,(k+1)\eta t)}{((k+1)\eta)^{\tau+j+2}}\right)\right], \tag{2.12}$$

where $\Omega(n,t) = \int_t^\infty z^{n-1}g(z)dz$ and $\Gamma(n) = \int_0^\infty z^{n-1}g(z)dz$ define the incomplete upper and complete gamma functions.

2.2.3. Moment generating function

The moment generating function, $M_X(t)$ for random variable X can be defined as

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}g(x)dy$$

Substituting (2.2) into the above integral and performing some algebraic manipulations on the resulting integrand gives

$$M_X(t) = \rho\sum_{k,j=0}^{\infty}\binom{\rho-1}{k}\binom{\gamma(k+1)+j}{j}(-1)^{k+j}\theta^j\left\{\sum_{i=0}^{\infty}\frac{t^i}{i!}\left[\gamma\theta+\eta\left(1+\frac{\theta(\tau+j+1)}{((k+1)\eta)}\right)\right]\frac{\Gamma(\tau+j+1)}{((k+1)\eta)^{\tau+j+1}}\right\}, \tag{2.14}$$

which defines the expression of moment generating function for the ELExp distribution.

2.2.4. Measures of entropy

The Rényi and Shannon entropies are quantitive measures used to investigate the expected amount of uncertainty related to random variable having a lifetime distribution. The Rényi entropy denoted by $[\mathcal{R}_\alpha(X)]$ for a random variable X having ELExp distribution is given as

$$R_\alpha(X) = \frac{1}{(1-\alpha)} \log\left\{\int_0^\infty [g(x)]^\alpha dx\right\}; \alpha > 0,\ \alpha \neq 1 \quad (2.15)$$

Substituting (2.2) into (2.15) and performing some algebraic manipulations gives

$$R_\alpha(X) = \frac{\alpha \log \rho}{(1-\alpha)} + \frac{1}{(1-\alpha)} \log\left[\sum_{k=0}^{\alpha}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{\alpha}{k}\binom{\alpha(\rho-1)}{j}\binom{\alpha+\gamma(\alpha+j)-k+i-1}{i}(-1)^{j+i} \eta^k \gamma^{\alpha-k}\theta^{\alpha-k+i}\frac{\Gamma(i+1)}{((\alpha+j)\eta)^{i+1}}\right\}, \quad (2.16)$$

as the expression for Rényi entropy of the ELExp distribution.
The Shannon entropy denoted by $[\mathcal{S}_{\mathcal{H}}(X)]$ for a random variable X having ELExp distribution is given as

$$S_H(X) = E(-\log[g(X)]) = -\int_0^\infty g(x)\log[g(x)]dx = -\log(\rho\gamma\theta) - \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\binom{k}{j}\frac{(-1)^{k+1}}{k}\left(\frac{\eta}{\gamma\theta}\right)^k \theta^j E(X^j) \quad (2.17)$$

$$+(\gamma+1)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\theta^k E(X^k) + \eta E(X) + (1-\rho)\sum_{k=1}^{\infty}\sum_{j,i=0}^{\infty}\binom{k\gamma+j-1}{j}\frac{(-1)^{j+i+1}}{i!k}(k\eta)^i\theta^j E(X^j).$$

The mathematical expression for $E(X^\tau)$ is defined in (2.9).

2.2.5. Residual and reversed residual lifetime function

The residual lifetime function denoted by $m_{RL}^r(t)$ determines the remaining lifetime for a system beyond age $t \geq 0$ until failure time occurs while the reversed residual lifetime function (also known as the inactivity time function) denoted by $\mathcal{M}_{RL}^r(t)$ determines the elapsed lifetime for a system from its failure for which its lifetime is less than or equals to age $t \geq 0$.
For $r = 1, 2, \ldots,$ we have

$$m_{RL}^r(t) = \frac{1}{\bar{G}(t)}\int_t^\infty (x-t)^r g(x)dx \quad (2.18)$$

$$= \frac{\rho}{1-(1-(1+\theta t)^{-\gamma}e^{-\eta t})^\rho}\sum_{i=0}^{r}\binom{r}{i}(-1)^i t^i \sum_{k,j=0}^{\infty}\binom{\rho-1}{k}\binom{(k+1)\gamma+j}{j}(-1)^{k+j}\theta^k$$

$$\left[\gamma\theta\frac{\Omega(\tau+j-i+1,(k+1)\eta t)}{((k+1)\eta)^{\tau+j-i+1}} + \eta\left(\frac{\Omega(\tau+j-i+1,(k+1)\eta t)}{((k+1)\eta)^{\tau+j-i+1}} + \theta\frac{\Omega(\tau+j-i+2,(k+1)\eta t)}{((k+1)\eta)^{\tau+j-i+2}}\right)\right]$$

and

$$M_{RL}^r(t) = \frac{1}{G(t)}\int_0^t (t-x)^r g(x)dx \quad (2.19)$$

$$= \frac{\rho}{(1-(1+\theta t)^{-\gamma}e^{-\eta t})^\rho}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r+i} t^i \sum_{k,j=0}^{\infty}\binom{\rho-1}{k}\binom{(k+1)\gamma+j}{j}(-1)^{k+j}\theta^k$$

$$\left[\gamma\theta\frac{\Phi(\tau+j-i+1,(k+1)\eta t)}{((k+1)\eta)^{\tau+j-i+1}} + \eta\left(\frac{\Phi(\tau+j-i+1,(k+1)\eta t)}{((k+1)\eta)^{\tau+j-i+1}} + \theta\frac{\Phi(\tau+j-i+2,(k+1)\eta t)}{((k+1)\eta)^{\tau+j-i+2}}\right)\right]$$

respectively, where $\Omega(n,t) = \int_t^\infty z^{n-1}g(z)dz$ and $\Phi(n,t) = \int_0^t z^{n-1}g(z)dz$ define incomplete upper and lower gamma functions. It is worth noting that the mean residual lifetime (mrl) and reversed mean residual lifetime ($rmrl$) can be obtained if $r = 1$ in (2.18) and (2.19) respectively.

2.2.6. Order statistics, its cdf and raw moment

The pdf, cdf and r^{th} non-central moments for the order statistics $\{X_{(i)}\}_{i=1}^n$ of random variable X following the ELExp distribution will be presented. Suppose $X_1, X_2, \ldots, X_n$ is a n-sized random sample for random variable X, then its ordered sample is given as $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. The pdf for the order statistics $\{X_{(i)}\}_{i=1}^n$ is defined as

$$g_{n:m}(x) = \frac{m!\,g(x)}{(n-1)!\,(m-n)!}[G(x)]^{n-1}[\bar{G}(x)]^{m-n}, 1 \leq n \leq m$$

Employing (2.1), (2.2) and (2.4), the pdf for the order statistics of ELExp distribution is given as

$$g_{n:m}(x) = \frac{m!\rho}{(n-1)!(m-n)!}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\binom{n-1}{k}\binom{\rho-j}{j}(-1)^{k+j}(\gamma\theta+\eta(1+\theta x)) \quad (2.20)$$

$$\times(1+\theta x)^{-\gamma(m-n+k+j+1)-1}e^{-(m-n+k+j+1)\eta x}$$

The cdf and r^{th} non-central moments forthe order statistics of ELExp distribution are given as

$$G_{n:m}(x) = \sum_{k=n}^{m}\binom{m}{k}[G(x)]^k[\bar{G}(x)]^{m-k} = \sum_{k=n}^{m}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(-1)^j\left[(1+\theta x)^{-\gamma}e^{-\eta x}\right]^{n+j-k} \quad (2.21)$$

and

$$E[X_{n:m}^r] = \int_0^\infty x^r g_{n:m}(x)dx = \frac{m!\rho}{(n-1)!(m-n)!}\sum_{k=0}^{n-1}\sum_{j,i=0}^{\infty}\binom{n-1}{k}\binom{\rho-j}{j}\binom{\gamma(m-n+k+j+1)+i}{i} \quad (2.22)$$

$$\times(-1)^{k+j+i}\theta^i\left[\gamma\theta+\eta\left(1+\theta\frac{(r+i+1)}{((m-n+k+j+1)\eta)}\right)\right]\frac{\Gamma(r+i+1)}{((m-n+k+j+1)\eta)^{\tau+j+1}}$$

2.2.7 Stress-Strength reliability for ELExp distribution

An important event which occurs in lifetime analysis is the measure of stress-strength reliability of a system with strenght X_1 subjected to some random stress X_2. For independent variables X_1 and X_2, the reliability of a system denoted by $R = Pr(X_2 < X_1)$ signifies deterioration in the performance of the system occurs when the stress exprencienced exceeds its strenght. More discussion of stress-strenght analysis of systems can be found in[11]. The reliability for a system for which X_1 and X_2 follows the same probability distribution is given as

$$R = \int_0^\infty g_1(x;\aleph)G_2(x;\aleph)dx \qquad (2.23)$$

where $\aleph$ denotes the parameter vector for the distribution of random variables X_1 and X_2. Two cases will be given presented for the reliability of a system with with strenght X_1 experiencing stress X_2.

Case 1: If $X_1 \backsim ELExp(\theta,\gamma,\eta,\rho_1)$ and $X_2 \backsim ELExp(\theta,\gamma,\eta,\rho_2)$, the reliability of the system is given as

$$\begin{aligned} R &= \rho_1 \int_0^\infty (\gamma\theta + \eta(1+\theta x))(1+\theta x)^{-\gamma-1}e^{-\eta x}[1-(1+\theta x)^{-\gamma}e^{-\eta x}]^{\rho_1+\rho_2-1}dx \\ &= \frac{\rho_1}{\rho_1+\rho_2} \end{aligned} \qquad (2.24)$$

which shows the parameters θ,γ,η has insignificant effect on the reliability of the system.

Case 2: If $X_1 \backsim ELExp(\theta,\gamma,\eta_1,\rho)$ and $X_2 \backsim ELExp(\theta,\gamma,\eta_2,\rho)$, then the reliability of the system is given as

$$\begin{aligned} R &= \rho \int_0^\infty (\gamma\theta + \eta_1(1+\theta x))(1+\theta x)^{-\gamma-1}e^{-\eta_1 x}[1-(1+\theta x)^{-\gamma}e^{-\eta_1 x}]^{\rho-1} \\ &[1-(1+\theta x)^{-\gamma}e^{-\eta_2 x}]^{\rho}dx \end{aligned}$$

Several algebraic manipulation of above expression gives

$$\begin{aligned} R &= \rho \sum_{k,j,i=0}^{\infty} \binom{\rho-1}{k}\binom{\rho}{i}\binom{(k+1)\gamma+j}{j}(-1)^{k+j+i}\theta^{j}\left[\gamma\theta + \eta_1\left(1+\theta\frac{(j+1)}{((k+1)\eta_1+\eta_2)}\right)\right] \\ &\times \frac{\Gamma(j+1)}{((k+1)\eta_1+\eta_2)^{j+1}}. \end{aligned} \qquad (2.25)$$

Remarks

For $0 \le R \le 1$, the following remarks can be drawn from the two cases.

Case 1

1. if $\rho_1 < \rho_2$, then $R \to 0$ which implies deterioration in the performance of the system.
2. if $\rho_2 < \rho_1$, then $R \to 1$ which implies consistent performance of the system.

Case 2

1. if $\eta_1 < \eta_2$, then $R \to 0$ which implies reduction in the performance of the system.
2. if $\eta_2 < \eta_1$, then $R \to 1$ implying consistency in the performance of the system.

3. Maximum Likelihood Estimation (MLE)

Point estimation of parameters of lifetime distributions have mostly been implemented using MLE method because it possesses several advantages over other estimation methods. For a random sample, $x_1, x_2, \ldots, x_n$, from variable $X \backsim ELExp(\theta,\gamma,\eta,\rho)$, the total log-likelihood function is presented as

$$\begin{aligned} Log\mathcal{L}_n &= nln(\rho) + \sum_{i=1}^n ln(\gamma\theta + \eta(1+\theta x_i)) - (\gamma+1)\sum_{i=1}^n ln(1+\theta x_i) - \eta\sum_{i=1}^n x_i \\ &+(\rho-1)\sum_{i=1}^n ln(1-(1+\theta x_i)^{-\gamma}e^{-\eta y_i}) \end{aligned} \qquad (3.1)$$

The partial derivatives of (3.1) with respect to the parameters of interest equated to zero result in the system of nonlinear equations given as

$$\begin{aligned} \frac{\partial Log\mathcal{L}_n}{\partial\theta} &= -(\gamma+1)\sum_{i=1}^n \frac{x_i}{(1+\theta x_i)} + (\rho-1)\sum_{i=1}^n \left(\frac{\gamma x_i(1+\theta x_i)^{-(\gamma+1)}e^{-\eta x_i}}{1-(1+\theta x_i)^{-\gamma}e^{-\eta x_i}}\right) \\ &+\sum_{i=1}^n \left(\frac{\gamma+\eta x_i}{\gamma\theta+\eta(1+\theta x_i)}\right) = 0, \end{aligned} \qquad (3.2)$$

$$\begin{aligned} \frac{\partial Log\mathcal{L}_n}{\partial\gamma} &= -\sum_{i=1}^n log(1+\theta x_i) + (\rho-1)\sum_{i=1}^n \left(\frac{(1+\theta x_i)^{-\gamma}e^{-\eta x_i}ln(1+\theta x_i)}{1-(1+\theta x_i)^{-\gamma}e^{-\eta x_i}}\right) \\ &+\sum_{i=1}^n \left(\frac{\theta}{\gamma\theta+\eta(1+\theta x_i))}\right) = 0, \end{aligned} \qquad (3.3)$$

$$\frac{\partial Log\mathcal{L}_n}{\partial\eta} = -\sum_{i=1}^n x_i + (\rho-1)\sum_{i=1}^n \left(\frac{x_i(1+\theta x_i)^{-\gamma}e^{-\eta x_i}}{1-(1+\theta x_i)^{-\gamma}e^{-\eta x_i}}\right) + \sum_{i=1}^n \left(\frac{1+\theta x_i}{\gamma\theta+\eta(1+\theta x_i)}\right) = 0, \qquad (3.4)$$

and

$$\frac{\partial Log\mathcal{L}_n}{\partial\rho} = \frac{n}{\rho} + \sum_{i=1}^n ln(1-(1+\theta x_i)^{-\gamma}e^{-\eta x_i}) = 0. \qquad (3.5)$$

The system of equations given by (3.2), (3.3), (3.4) and (3.5) are solved simultaneously or numerical iterative methods. The point estimates, $\hat{\theta}, \hat{\gamma}, \hat{\eta}, \hat{\theta}$ are the unique solutions obtained as the maximum likelihood estimates for the random sample $x_1, x_2, \ldots, x_n$.

4. Applications of ELExp distribution to real lifetime data

The paper considers the application of the ELExp distribution to the percentage (%) of the body fat for the 202 athletes as reported in

[12]. Comparative analysis will be carried out among the new four-parameter distribution and some well-established distributions in literature using goodness-of-fit tests such Cramér von Mises (W^*), Anderson darling (A^*) and Kolmogorov-Smirnov (KS) tests. The parameter estimates (with standard errors in parentheses) and the log-likelihood values ($-ll$) of the competing distributions are presented with the comparison criteria. All computations are achieved using AdequacyModel package in R software. The data are given as
19.75, 21.30, 19.88, 23.66, 17.64, 15.58, 19.99, 22.43, 17.95, 15.07, 28.83, 18.08, 23.30, 17.71, 18.77, 19.83, 25.16, 18.04, 21.79, 22.25, 16.25, 16.38, 19.35, 19.20, 17.89, 12.20, 23.70, 24.69, 16.58, 21.47, 20.12, 17.51, 23.70, 22.39, 20.43, 11.29, 25.26, 19.39, 19.63, 23.11, 16.86, 21.32, 26.57, 17.93, 24.97, 22.62, 15.01, 18.14, 26.78, 17.22, 26.50, 23.01, 30.10, 13.93, 26.65, 35.52, 15.59, 19.61, 14.52, 11.47, 17.71, 18.48, 11.22, 13.61, 12.78, 11.85, 13.35, 11.77, 11.07, 21.30, 20.10, 24.88, 19.26, 19.51, 23.01, 8.07, 11.05, 12.39, 15.95, 9.91, 16.20, 9.02, 14.26, 10.48, 11.64, 12.16, 10.53, 10.15, 10.74, 20.86, 19.64, 17.07, 15.31, 11.07, 12.92, 8.45, 10.16, 12.55, 9.10, 13.46, 8.47, 7.68, 6.16, 8.56, 6.86, 9.40, 9.17, 8.54,9.20, 11.72, 8.44, 7.19, 6.46, 9.00, 12.61, 9.03 ,6.96, 10.05, 9.56, 9.36, 10.81, 8.61, 9.53, 7.42, 9.79, 8.97, 7.49, 11.95, 7.35, 7.16, 8.77, 9.56, 14.53, 8.51, 10.64, 7.06, 8.87, 7.88, 9.20, 7.19, 6.06,5.63, 6.59, 9.50, 13.97, 1.66, 6.43, 6.99, 6.00, 6.56, 6.03, 6.33, 6.82, 6.20, 5.93, 5.80, 6.56, 6.76, 7.22, 8.51, 7.72, 19.94, 13.91, 6.10, 7.52, 9.56, 6.06,7.35, 6.00, 6.92, 6.33, 5.90, 8.84, 8.94, 6.53, 9.40, 8.18, 17.41, 18.08, 9.86,7.29, 18.72, 10.12, 19.17, 17.24, 9.89, 13.06, 8.84, 8.87, 14.69, 8.64, 14.98, 7.82, 8.97, 11.63, 13.49, 10.25, 11.79, 10.05, 8.51, 11.50, 6.26.
The pdfs of the competing distributions are given as
Kumarawamy Fréchet (KFr) distribution [13];

$$f(x) = \rho\gamma\eta x^{-\gamma-1}e^{-\eta(\frac{\theta}{x})^{\gamma}}\left(1 - e^{-\eta(\frac{\theta}{x})^{\gamma}}\right)^{\rho-1}.$$

Extended log-logistic (ELL) distribution [14];

$$f(x) = \frac{\rho\gamma\theta}{\eta^{\gamma}}x^{\gamma-1}\left(1 - \frac{x^{\gamma}}{\eta^{\gamma} + x^{\gamma}}\right)^{\theta+1}\left(1 - \left(1 - \frac{x^{\gamma}}{\eta^{\gamma} + x^{\gamma}}\right)^{\theta}\right)^{\rho-1}.$$

Transmuted exponentiated generalized Weibull (TExGW) distribution [15];

$$f(x) = \rho\gamma\theta x^{\gamma-1}e^{-\theta x^{\gamma}}\left(1 - e^{-\theta x^{\gamma}}\right)^{\rho-1}\left(1 + \eta - 2\eta\left(1 - e^{-\theta x^{\gamma}}\right)^{\rho}\right).$$

Lomax distribution [16];

$$f(x) = \gamma\theta(1 + \theta x)^{-\gamma-1}.$$

Exponentiated exponential (EExp) distribution [17];

$$f(x) = \eta\rho e^{-\eta x}(1 - e^{-\eta x})^{\rho-1}.$$

Exponential (Exp) distribution

$$f(x) = \eta e^{-\eta x}.$$

Table 2 list the values of the point estimates $(\hat{\theta}, \hat{\gamma}, \hat{\eta}, \hat{\rho})$ and the goodness-of-fit criteria for the competing lifetime distributions applied to the data on body fat for the 202 athletes.

Table 2: Point estimates and goodness-of-fit tests for competing distributions

Distribution	$\hat{\theta}$ (s.e)	$\hat{\gamma}$ (s.e)	$\hat{\eta}$ (s.e)	$\hat{\rho}$ (s.e)	*-ll*	*W**	*A**	*KS* (p-value)
ELExp	0.1678 (0.5351)	0.3504 (1.1497)	0.1738 (0.0550)	8.1464 (4.7819)	637.7663	0.3750	2.3096	0.0845 (0.1117)
KFr	7.5405 (141.9260)	0.5667 (0.1547)	5.6392 (60.1655)	49.0594 (58.2407)	638.4590	0.4001	2.4313	0.0871 (0.0935)
ELL	11.8564 (11.6217)	1.1975 (0.3230)	47.4959 (48.9783)	5.4343 (3.6084)	638.3432	0.3946	2.4221	0.0864 (0.0978)
TExGW	0.1871 (0.1748)	1.0016 (0.2448)	0.0782 (0.2448)	6.8746 (4.6779)	637.7694	0.3886	2.3746	0.0852 (0.1063)
Lomax	0.0039 (0.0006)	19.4308 (3.2152)	----	----	731.1652	0.4496	2.6574	0.3402 (0.0000)
EExp	----	----	0.1917 (0.0126)	6.8801 (0.9664)	637.8236	0.3935	2.3967	0.0848 (0.1089)
Exp	----	----	0.0743 (0.0052)	----	727.1127	0.4597	2.7048	0.3402 (0.0000)

The flexibility of the ELExp distribution over the six competing distributions is evident in Table 2. It is seen that the values of the goodness-of-fit criteria is smaller to the corresponding values of the competing distribution which presents the ELExp distribution better fit the data better than the other six distributions. The plot of the histogram of the data and the estimated densities of the seven distributions is presented in Figure 2.

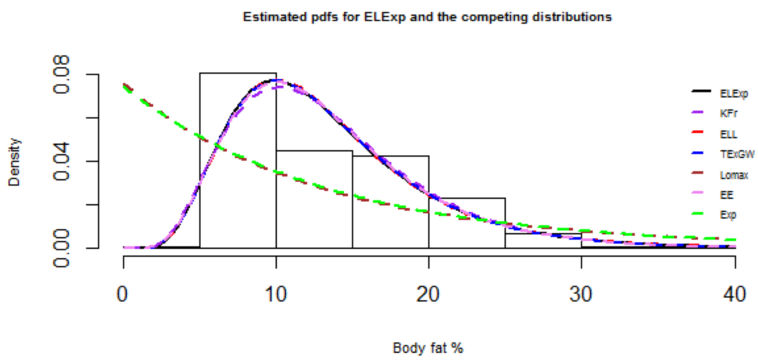

Figure 2: Plot for Histogram, estimated ELExp pdf and pdfs for competing distributions

5. Conclusion

A new four-parameter distribution known as the exponentiated Lomax-exponential (ELExp) distribution is presented. Mathematical expressions are derived for the structural properties of the new distribution. Expressions to obtain maximum likelihood stimates for the new distribution are presented in the study. Applications of the distribution to data of body fat percentage for 202 athletes demonstrate its flexiblity is considered. It shows that the new distribution provide better fit for the data over the competing distributions used in the study.

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Competing financial interest

The authors wish to declare no competing financial interest.

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