Journal of the Nigerian Association of Mathematical Physics
Volume 57(**June – July** *2020 Issue*), **pp89 – 94**
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PROPERTIES AND APPLICATION OF A NEW THREE-PARAMETER LOMAX-EXPONENTIAL DISTRIBUTION

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Abstract

A new lifetime distribution called the Lomax-Exponential distribution is proposed in this study with its defining functions presented. Statistical properties such as quantile function, moments, inequality curves, entropy measures, measures of residual life and order statistics are discussed. Inference for point and interval estimation for parameters of the proposed distribution is presented. Application of the new distribution to lifetime data is illustrated to determine the usefulness and applicability in lifetime analysis.

Keywords: Lomax-Exponential distribution, Weibull distribution, Univariate model, Quantile function, Residual life.

1. Introduction

Competing risk method has been used to generate several flexible univariate models to analyze monotonic and non-monotonic lifetime data over the years [1]. For a series system with i^{th} (i=1,2) components which will independently failure at time z, then the probability of survival, $Pr(Z_1 > z, Z_2 > z)$ which is the survival function for the series distribution is given by

$$S(z) = \prod_{i=1}^{2} S_i(z);\, z > 0, \qquad (1)$$

where $S_i(z) = Pr(X_i > z)$ is the survival probability of the i^{th} component at time z. The corresponding cdf, pdf and hazard function to (1) are given respectively as

$$F(z) = 1 - \prod_{i=1}^{2} S_i(z), \qquad (2)$$

$$f(z) = S(z)\sum_{i=1}^{2} \frac{f_i(z)}{S_i(z)} = S(z)\sum_{i=1}^{2} h_i(z) \qquad (3)$$

and

$$h(z) = \sum_{i=1}^{2} h_i(z) \qquad (4)$$

where cdf, pdf and hazard function are denoted by is $F(z)$, $f(z)$ and $h(z)$ respectively.

Examples of some existing univariate models that have been introduced with the distribution form of a competing risk model include the log-logistic Weibull distribution [2], additive Weibull distribution [3] and modified Weibull distribution [4].

The motivation behind proposition of the Lomax-Exponential (Lom-E) distribution as a lifetime model is to generate a flexible model that combines the structural properties of the Lomax and exponential distributions, thereby enhancing the flexibility and applicability of the proposed distribution to lifetime data analysis. The flexiblility and usefulness of the new distribution is presented by its application to Vinyl chloride data.

The organization of the paper is presented as follows. Section 2 presents formulation of Lom-E distribution by presenting the defining distribution functions. Other structural properties of the proposed distribution are detailed in section 3. Section 4 presents inference of the distribution which leds to point and interval estimation of its parameters using maximum likelihood estimation (MLE) method. Section 5 presents application of the LomE, Lomax and exponential distributions to some lifetime datasets and discussion of results from the application. Section 6 gives concluding remarks to the study on the proposed distribution.

2. Lomax-Exponential distribution

2.1 Formulation of the Lom-E distribution

Let a series system be made up of two independently components that will fail at time z such that one of the component's failure follows the Lomax distribution while the failure of the other component has the exponential distribution, the defining distribution functions of Lom-E model are obtained from (1)-(4) and given as

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$S(z) = (1+\alpha z)^{-\theta} e^{-\lambda z}, z > 0, \alpha > 0, \theta > 0, \lambda > 0,$ (5)
$F(z) = 1-(1+\alpha z)^{-\theta} e^{-\lambda z},$ (6)
$f(z) = (\alpha\theta + \lambda(1+\alpha z))(1+\alpha z)^{-\theta-1} e^{-\lambda z}$ (7)
and

$$h(z) = \frac{(\alpha\theta + \lambda(1+\alpha z))}{(1+\alpha z)} \quad (8)$$

Expansion of the pdf of the Lom-E distribution can be derived from (7) for simplified manipulations to derive other structural properties of the distribution which include quantile function, moments, entropy measures, order statistics and residual lifetime.
The algebraic expansion of (7) is given as

$$f(z) = \sum_{i=0}^{\infty} \binom{\theta+i}{i} (-i)^i \alpha^i z^i (\alpha\theta + \lambda(1+\alpha z)) e^{-\lambda z} \quad (9)$$

which defines a simple expression for the pdf of Lom-E distribution.

3. Structural properties of Lom-E distribution

Other defining structural properties of Lom-E distribution will be consider in this section.

3.1 Quantile function

For $p \in (0,1)$, the quantile function for Lom-E distribution is the root, z_p, of the equation which is given by

$$\theta log(1+\alpha z_p) + \lambda z_p + log(1-p) = 0. \quad (10)$$

The root, z_p, is obtained as a unique solution for every value of $p \in (0,1)$ in (10) for different values of θ, α and λ numerically with method such as the Newton-Raphson method or the Lambert-W function. A detailed of evaluating solutions to some algebraic equations using Lambert-W function [5]. The unique algebraic solution of (10) using Lambert-W function is given as

$$z_p = \frac{W\left((1-p)^{\alpha/\lambda\theta}\right) - 1}{\alpha} \quad (11)$$

where $W(x)$ is the Lambert-W function which defines the inverse function of $W(x)e^{W(x)} = x.$
Random number generation can be achieved using (10) or (11) if $p \in U(0,1)$ where $U(0,1)$ is the Uniform distribution.

3.2 Moment, conditional moment and moment generating function

The r^{th} raw moment (μ_r) for the Lom-E distribution can be expressed as

$$E[Z^r] = \mu_r = \sum_{i=0}^{\infty} \binom{\theta+1}{i} (-1)^i \alpha^i \int_0^{\infty} (\theta\alpha z^{r+i} + \lambda(z^{r+i} + \theta z^{r+i+1}))\; e^{-\lambda z} dz.$$

Let $x = \lambda z$, then we have

$$\mu_r = \sum_{i=0}^{\infty} \binom{\theta+1}{i} (-1)^i \alpha^i \left[\frac{\theta\alpha\Gamma(r+i+1)}{\lambda^{r+i+1}} + \lambda\left(\frac{\Gamma(r+i+1)}{\lambda^{r+i+1}} + \alpha\frac{\Gamma(r+i+2)}{\lambda^{r+i+2}}\right)\right].$$

$$= \sum_{i=0}^{\infty} \binom{\theta+1}{i} (-1)^i \alpha^i \left[\frac{\theta\alpha\Gamma(r+i+1)}{\lambda^{r+i+1}} + \lambda\left(1+\alpha\left(\frac{r+i+1}{\lambda}\right)\right)\frac{\Gamma(r+i+1)}{\lambda^{r+i+1}}\right] \quad (12)$$

$$= \sum_{i=0}^{\infty} \binom{\theta+1}{i} (-1)^i \alpha^i \left(\theta\alpha + \lambda\left[1 + \frac{\alpha(r+i+1)}{\lambda}\right]\right)\frac{\Gamma(r+i+1)}{\lambda^{r+i+1}}$$

The conditional moment of Lom-E distribution denoted by μ_r^* is derived as follows

$$E[Y^r/Y>t] \quad = \mu_r^* = \frac{1}{S(t)}\int_t^{\infty} z^r f(z) dz$$

$$= \frac{1}{1-(1+\alpha t)^{-\theta} e^{-\lambda t}} \left(\sum_{i=0}^{\infty} (-1)^i \alpha^i \int_t^{\infty} \left(\theta\alpha z^{r+i+1} + \lambda(z^{r+i} + \alpha z^{r+i+1})\right) e^{-\lambda z} dz\right)$$

Evaluating the numerator of the right-hand side of the above expression for which $x = \lambda z$, the r^{th} conditional moment for Lom-E distribution is given as

$$\mu_r^* = \frac{\left(\sum_{i=0}^{\infty} (-1)^i \alpha^i \left[\frac{\theta\alpha\Gamma(r+i+1,\lambda t)}{\lambda^{r+i+1}} + \lambda\left(\frac{\Gamma(r+i+1,\lambda t)}{\lambda^{r+i+1}} + \alpha\frac{\Gamma(r+i+2,\lambda t)}{\lambda^{r+i+2}}\right)\right]\right)}{1-(1+\alpha t)^{-\theta} e^{-\lambda t}} \quad (13)$$

where $\Gamma(m,\beta) = \int_{\beta}^{\infty} x^{m-1} e^{-x} dx$ and $Y(m,\beta) = \int_0^{\beta} x^{m-1} e^{-x} dx$ are the incomplete upper and lower gamma functions such that $\Gamma(m,\beta) + Y(m,\beta) = \Gamma(m)$.

The evaluation of standard deviation (SD), coefficients of variation (CV), skewness (CS) and kurtosis (CK) for Lom-E distribution can be obtained from (12) using moments based relations given by

$$SD = \sqrt{\mu_2 - \mu^2}, \; CV = \frac{SD}{\mu}, \; CS = \frac{(\mu_3 - 3\mu_2\mu + 2\mu^3)^2}{(\mu_2 - \mu^2)^3} \text{ and } CS = \frac{\mu_4 - 4\mu_3\mu + 6\mu_2\mu^2 - 3\mu^4}{(\mu_2 - \mu^2)^2}$$

where μ defined the first raw moment if r=1.

The moment generating (mgf), $M_Z(t)$ for LomE distribution is derived as

$$M_Z(t) = E[e^{tZ}] = \int_0^\infty e^{tz}f(z)dz = \sum_{i=0}^\infty (-1)^i\alpha^i \int_0^\infty \left(\theta\alpha z^i + \lambda(z^i + \alpha z^{i+1})\right)e^{(t-\lambda)z}dz \qquad (14)$$

Further algebraic manipulation of integral at the right-hand side of (14) results in

$$M_z(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\left[\sum_{i=0}^\infty \binom{\theta+1}{i}(-1)^i\alpha^i\left(\theta\alpha + \left[1+\frac{\alpha(r+i+1)}{\lambda}\right]\right)\frac{\Gamma(r+i+1)}{\lambda^{r+i+1}}\right]$$

3.3 Measures of entropy

The Shannon and Rényi entropies ([6]; [7]) are important information measures needed in investigation of randomness of a variable having a lifetime distribution, which have found relevance in sciencific areas such as physics, ecology, statistics, medicine and engineering. The Shannon $[H_S(f(z))]$ entropy for Lom-E distribution can be derived as follows.

Let

$$H_S(f(z)) = E[-\log f(Z)] = -\int_0^\infty f(z)\log(f(z))dz \cdot$$

The above equation can be rewritten as

$$H_S(f(z)) = -E[\log(\theta\alpha + \lambda(1+\alpha Z))] + (\theta+1)E[\log(1+\alpha Z)] + \lambda E[Z] \qquad (15)$$

We consider the expansions of the following terms

$$E[\log(\theta\alpha + \lambda(1+\alpha Z))] = \log(\theta\alpha) + \sum_{i=0}^\infty\sum_{j=0}^{i+1}\binom{i+1}{j}\frac{(-1)^{i+2}}{(i+1)}\left(\frac{\lambda}{\theta\alpha}\right)^{i+1}\alpha^j E[Z^j]$$

and

$$E[\log(1+\alpha Z)] = \sum_{i=0}^\infty \frac{(-1)^{i+2}}{(i+1)}\alpha^{i+1}E[Z^{i+1}]$$

Substituting the two expansions into the right-hand side of (15), the Shannon entropy for Lom-E distribution is given as

$$H_S(f(z)) = -\log(\theta\alpha) - \sum_{i=0}^\infty\sum_{j=0}^{i+1}\binom{i+1}{j}\frac{(-1)^{i+2}}{(i+1)}\left(\frac{\lambda}{\theta\alpha}\right)^{i+1}\alpha^j E[Z^j] + \lambda E[Z] \qquad (16)$$

$$+ (\theta+1)\sum_{i=0}^\infty \frac{(-1)^{i+2}}{(i+1)}\alpha^{i+1}E[Z^{i+1}]$$

where the explicit expressions for $E[Z^j]$, $E[Z]$ and $E[Z^{i+1}]$ are obtained from (12).

Also, we derive the Rényi $[I_R(\eta)]$ entropy for Lom-E distribution.

Let

$$I_R(\eta) = \frac{1}{(1-\eta)}\log\left(\int_0^\infty f^\eta(z)dz\right);\ \eta > 1,\ \eta \neq 1$$

$$= \frac{1}{(1-\eta)}\log\left(\int_0^\infty (\theta\alpha + \lambda(1+\alpha z))^\eta (1+\alpha z)^{-\eta(\theta+1)}e^{-\eta\lambda z}dz\right).$$

Further expressed as

$$I_R(\eta) = \frac{1}{(1-\eta)}\log\left(\sum_{i=0}^\eta \binom{\eta}{i}(\theta\alpha)^{\eta-i}\lambda^i\int_0^\infty (1+\alpha z)^{i-\eta(\theta+1)}e^{-\eta\lambda z}dz\right).$$

Using the expression for the definite integral in Gradshteyn and Ryzhik (2007) which is

$$\int_0^\infty (ax+b)^m e^{-px}dx = \frac{a^m e^{\frac{pb}{a}}}{p^{m+1}}\Gamma\left(m+1, \frac{pb}{a}\right);\ p > 0, \left|\arg\left(\frac{pb}{a}\right)\right| < \pi,$$

the explicit expression for Rényi entropy of the Lom-E distribution is given as

$$I_R(\eta) = \frac{1}{(1-\eta)}\log\left\{\sum_{i=0}^\eta \binom{\eta}{i}\theta^{\eta-i}\alpha^{-\eta\theta}\lambda^{\eta(\theta+1)+1}\frac{\Gamma\left(i-\eta(\theta+1)+1, {\eta\lambda}/{\alpha}\right)}{\eta^{i-\eta(\theta+1)+1}}\right\}. \qquad (17)$$

3.4 Mean deviations of Lom-E distribution about its mean and median

Mean deviation (MD) measures total variations from the mean and median of a set of data. The mean deviations of the Lom-E distribution about the mean (μ) and median (M) are derived as follow.

$$MD_\mu = E(|Z-\mu|) = \int_0^\infty |z-\mu|\, f(z)dz = -2\int_0^\mu zf(z)dz + 2\mu F(\mu) = 2(1-(1+\alpha\mu)^{-\theta}e^{-\lambda\mu})$$

$$-2\left\{\sum_{i=0}^\infty (-1)^i\alpha^i\left[\frac{\theta\alpha\, Y(i+2,\lambda\mu)}{\lambda^{i+2}} + \lambda\left(\frac{Y(i+2,\lambda\mu)}{\lambda^{i+2}} + \alpha\frac{Y(i+3,\lambda\mu)}{\lambda^{i+3}}\right)\right]\right\}$$

and

$$MD_M = E(|Z - M|) = \int_0^{\infty} |z - M| f(z)dz = \mu - 2\int_0^{M} zf(z)dz$$

$$= \mu - 2\left\{\sum_{i=0}^{\infty} (-1)^i \alpha^i \left[\frac{\theta\alpha\, Y(i+2, \lambda M)}{\lambda^{i+2}} + \lambda\left(\frac{Y(i+2, \lambda M)}{\lambda^{i+2}} + \alpha\frac{Y(i+3, \lambda M)}{\lambda^{i+3}}\right)\right]\right\}.$$

3.5 Order statistics for Lom-E distribution

Let $Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}$ be ordered variables from n continuous independent variables $Z_1, Z_2, \ldots, Z_n$, then pdf of k^{th} $(1 \le k \le n)$ order statistics is derived as

$$f_{k:n}(z) = \frac{n!\, f(z)}{(k-1)!(n-k)!}[F(z)]^{k-1}[1-F(z)]^{n-k}$$

$$= \frac{n!\, f(z)}{(k-1)!(n-k)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j [S(z)]^{n+j-k}$$

$$= \frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j (\theta\alpha + \lambda(1+\alpha z))(1+\alpha z)^{-(n+j-k+1)\theta-1} e^{-(n+j-k+1)\lambda z}$$

The corresponding cdf for $F_{k:n}(z)$ is given as

$$F_{k:n}(z) = \sum_{i=k}^{n}\binom{n}{i}[F(z)]^i[1-F(z)]^{n-i} = \sum_{i=k}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j[S(z)]^{n+j-i}$$

$$= \sum_{i=k}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j(1+\alpha z)^{-(n+j-i)\theta} e^{-(n+j-i)\lambda z}$$

The r^{th} raw moments of $f_{k:n}(z)$ is given as

$$E[Z_{k:n}^r] = \int_0^{\infty} z^r f_{k:n}(z)dz = \frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j$$

$$\times\int_0^{\infty} z^r(\theta\alpha + \lambda(1+\alpha z))(1+\alpha z)^{-(n+j-k+1)\theta-1} e^{-(n+j-k+1)\lambda z}dz$$

$$= \frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{k-1}\sum_{i=0}^{\infty}\binom{k-1}{j}\binom{(n+j-k+1)\theta}{i}(-1)^{j+i}\alpha^i$$

$$\times\int_0^{\infty} z^r(\theta\alpha + \lambda(1+\alpha z))e^{-(n+j-k+1)\lambda z}dz.$$

Therefore, the explicit expression for the r^{th} raw moments of $f_{k:n}(z)$ is given as

$$E[Z_{k:n}^r] = \frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{k-1}\sum_{i=0}^{\infty}\binom{k-1}{j}\binom{(n+j-k+1)\theta}{i}(-1)^{j+i}\alpha^i$$

$$\times\left(\theta\alpha + \lambda\left[1 + \frac{\alpha(r+i+1)}{(n+j-k+1)\lambda}\right]\right)\frac{\Gamma(r+i+1)}{[(n+j-k+1)\lambda]^{r+i+1}}$$

It is important the distribution functions for the minimum and maximum values of the variables, $Z_{(1)}$ and $Z_{(n)}$ are presented.

Remarks

(i) If k=1, then the distribution functions of $Z_{(1)}$ are

$$F_{1:n}(z) = 1 - [1-F(z)]^n = 1 - (1+\alpha z)^{-n\theta} e^{-n\lambda z}$$

with

$$f_{1:n}(z) = n(\theta\alpha + \lambda(1+\alpha z))(1+\alpha z)^{-(n\theta+1)} e^{-n\lambda z}.$$

This is n-order Lomax-Exponential distribution which is a competing risk model.

(ii) If k=n, then the distribution functions of $Z_{(n)}$ are

$$F_{1:n}(z) = [F(z)]^n = [1-(1+\alpha z)^{-\theta} e^{-\lambda z}]^n$$

with

$$f_{1:n}(z) = n(\theta\alpha + \lambda(1+\alpha z))(1+\alpha z)^{-\theta-1} e^{-\lambda z}[1-(1+\alpha z)^{-\theta} e^{-\lambda z}]^{n-1}.$$

This is a new lifetime model called the exponentiated Lomax-Exponential distribution with the exponentiated parameter, n, where n is a positive integer.

3.6 Residual lifetime

Residual lifetime function is an important measures which is used to test reliability and survival of systems that will experience failures over time, say t. Its application has been employed in diverse scientific fields for maintenance and survival analysis. The residual lifetime function defines the life remaining for a system beyond age $t \geq 0$ until failure time is known. It is known as the conditional random variable $Z_t = [(Z - t)/Z > t]$.

Suppose $m_r(t) = E[(Z - t)/Z > t]$ denotes the r^{th} moments for residual lifetime functions for a random variable Z following Lom-E distribution, then $m_r(t); r = 1, 2, ...$ is given as

$$m_r(t) = \frac{1}{S(t)}\int_t^{\infty}(z-t)^r f(z)dz = \frac{1}{(1+\alpha t)^{-\theta} e^{-\lambda t}}\sum_{j=0}^{r}\sum_{i=0}^{\infty}\binom{r}{j}\binom{\theta+i}{i}(-1)^{j+i} t^j \alpha^i \tag{18}$$

$$\times\left[\frac{\theta\alpha\,\Gamma(r-j+i+1,\lambda t)}{\lambda^{r-j+i+1}} + \lambda\left(\frac{\Gamma(r-j+i+1,\lambda t)}{\lambda^{r-j+i+1}} + \alpha\,\frac{\Gamma(r-j+i+1,\lambda t)}{\lambda^{r-j+i+2}}\right)\right].$$

4. Point and interval estimation for parameters of Lom-E distribution

Let $z_1, z_2, ..., z_n$ be n-sized random sample drawn from a random variable Z which follows the Lom-E distribution, then its total log-likelihood function is given as

$$L_n = L_n(z_i;\theta,\alpha,\lambda) = \sum_{i=1}^{n}\ln f(z_i) = \sum_{i=1}^{n}\ln(\theta\alpha + \lambda(1+\alpha z_i)) - (\theta+1)\sum_{i=1}^{n}\ln(1+\alpha z_i) - \lambda\sum_{i=1}^{n} z_i. \tag{19}$$

The first partial derivatives of equation (19) with respect to each of the parameters are equated to zero to obtain the set of nonlinear equations given as

$$\frac{\partial L_n}{\partial\theta} = -\sum_{i=1}^{n}\ln(1+\alpha z_i) + \sum_{i=1}^{n}\frac{\alpha}{\theta\alpha + \lambda(1+\alpha z_i)} = 0,$$

$$\frac{\partial L_n}{\partial\alpha} = -(\theta+1)\sum_{i=1}^{n}\frac{z_i}{1+\alpha z_i} + \sum_{i=1}^{n}\frac{\theta+\lambda z_i}{\theta\alpha + \lambda(1+\alpha z_i)} = 0$$

and

$$\frac{\partial L_n}{\partial\lambda} = -\sum_{i=1}^{n} z_i + \alpha\sum_{i=1}^{n}\frac{1+\alpha z_i}{\theta\alpha + \lambda(1+\alpha z_i)} = 0$$

To obtain the estimates for θ, α and λ, the set of equations are solved simultaneously using any known iterative method. The unique solutions from the set of nonlinear equations are the point estimates given as $\hat{\theta}, \hat{\alpha}$ and $\hat{\lambda}$ are achieved by numerical packages found in R.

For interval estimation, then the asymptotic distribution of $\sqrt{n}(\hat{\Psi} - \Psi) \xrightarrow{d} N_3(\underline{0}, I^{-1}(\Psi))$ for which $\underline{0} = (0,0,0)^T$, $\Psi = (\theta,\alpha,\lambda)$ and $I(\Psi) = [I_{\phi_l,\phi_j}]_{3\times 3} = E\left(-\frac{\partial^2 L_n}{\partial\phi_l\partial\phi_j}\right); l, j = 1,2,3$ is the expected Fisher's information. A valid asymptotic results exist with $A(\hat{\Psi})$ as observed information matrix evaluated at $\hat{\Psi}$ whenever $A(\hat{\Psi})$ replaces $I(\Psi)$. The multivariate normal distribution given as $N_3(\underline{0}, A^{-1}(\hat{\Psi}))$ is needed to construct approximate confidence intervals for θ, α, λ using the approximation of the total Fisher information matrix which is given as

$$A_n(\Psi) \approx -\begin{pmatrix} \lambda_{\theta\theta} & \lambda_{\theta\alpha} & \lambda_{\theta\lambda} \\ \cdot & \lambda_{\alpha\alpha} & \lambda_{\alpha\lambda} \\ \cdot & \cdot & \lambda_{\lambda\lambda} \end{pmatrix}$$

where $\lambda_{\theta\theta} = \frac{\partial^2 L_n}{\partial\theta^2}, \lambda_{\theta\alpha} = \frac{\partial^2 L_n}{\partial\theta\partial\alpha}, ..., \lambda_{\lambda\lambda} = \frac{\partial^2 L_n}{\partial\lambda^2}$.

The $100(1-\vartheta)\%$ approximate confidence intervals for θ, α and λ are presented as $\hat{\theta} \pm c_{\frac{\vartheta}{2}}\sqrt{\hat{\lambda}_{\theta\theta}}, \hat{\alpha} \pm c_{\frac{\vartheta}{2}}\sqrt{\hat{\lambda}_{\alpha\alpha}}$ and $\hat{\lambda} \pm c_{\frac{\vartheta}{2}}\sqrt{\hat{\lambda}_{\lambda\lambda}}$, where $c_{\frac{\vartheta}{2}}$ is the critical value of standard normal distribution at $\frac{\vartheta}{2}$.

5. Vinyl chloride data analysis

The Lom-E distribution is applied to the Vinyl chloride data and compared with four distributions (namely the Marshall Olkin Fréchet (MOF)[8], Lomax-Gumbel{Fréchet} (LG{F})[9], Weibull (W)[10] and exponential (Exp) distributions) using values of the loglikelihood function, Kolmogorov-Smirnov (*KS*) test and *p-value* of *KS* test as goodness-of-fit criteria. the goodness-of-fit criteria are computed using AdequacyModel package in R software. The cdf of comparing distributions are given as

$$F_{MOF}(z) = \frac{e^{-\eta z^{-\theta}}}{\alpha + (1-\alpha)e^{-\eta z^{-\theta}}}, \theta > 0, \alpha > 0, \lambda > 0.$$

$$F_{LG\{F\}}(z) = 1-(1+\alpha e^{\lambda z})^{-\theta}, \theta > 0, \alpha > 0, \lambda > 0.$$

$$F_W(z) = 1 - e^{\lambda z^{\alpha}}, \ \alpha > 0, \lambda > 0.$$

$$F_W(z) = 1 - e^{\lambda z}, \ \ \lambda > 0.$$

The vinyl chloride data are obtained from clean upgradient monitoring wells in mg/l which is obtained from [11]. Vinyl chloride is an organic compound which is volatile and carginogenic in nature. They are given as: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2.

Table 1 presents the parameter estimates and the goodness-of-fit tests for the Lom-E and the competing distributions.

Table 1: Point estimates and goodness-of-fit criteria for five compared distributions

Model	α (std.error)	θ (std.error)	λ (std.error)	*-loglik*	*KS*	*p-value*
Lom-E	0.3571 (2.0070)	0.3744 (2.9121)	0.4432 (0.4022)	**55.4080**	**0.0856**	**0.9644**
MOF	27.2840 (44.9725)	1.4676 (0.2356)	0.0424 (0.0658)	55.7062	0.0857	0.9640
LG{F}	0.1280 (0.2454)	0.0202 (0.0179)	27.4570 (23.7631)	55.9650	0.1005	0.8819
Weibul	1.0104 (0.1327)	----	0.5262 (0.1177)	55.4496	0.0919	0.9364
Exponential	----	----	0.5321 (0.0913)	727.1127	2.7048	0.3402 (0.0000)

It should be noted that in comparing the lifetime distributions, the distribution with the smallest values of loglikelihood function and *KS* test fits the data better than the other distribution while the distribution with the largest value of *p-value* of the *KS* test, also, fits the data better the the remaining distribution. It is seen from Table 1 that that the values of the loglikelihood function and *KS* test of the Lom-E distribution is the smallest value, while the *p-value* of the *KS* test is the largest. It indicates that the Lom-E distribution better fits the vinyl data than the four competing distributions.

6. Conclusion

A new three-parameter Lomax-Exponential distribution is proposed and some structural properties are derived. Point and interval estimatations are presented for the parameters of the new distribution. The usefulness of the new distribution to lifetime data analysis is presented by application of the new distribution to Vinyl chloride data. Results obtained from the application based on some discrepancy criteria show the superiority of the proposed distribution over some competing distributions in literature. It is desire that the new three-parameter Lomax-Exponential distribution will attract wider usefulness in scientific areas where lifetime data analysis is required.

Declaration

The authors wish to declare that the paper has not been published or awaiting publication in any journal.

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