Journal of the Nigerian Association of Mathematical Physics
***Volume 57*(June – July *2020 Issue*), pp83 – 88**
© J. of NAMP

A SIMULATION STUDY ON THE MOMENTS AND MAXIMUM LIKELIHOOD ESTIMATION OF THE WEIBULL LOGISTIC EXPONENTIAL DISTRIBUTION

Osemwenkhae J. E. and Akata I. U.

Department of Statistics, University of Benin, Benin City, Nigeria.

Abstract

In this paper, a new three parameter statistical distribution called Weibull-Logistic-Exponential Distribution (WLED)is introduced. The simulation study on the moments, the asymptotic behavior of the maximum likelihood estimates of the distribution are examined. An estimation of the average bias and mean squared error for the estimates of the parameters of the WLED show that its maximum likelihood estimates satisfied the asymptotic properties of (unbiasedness, consistency and asymptotic normality) of estimators. The relevance of the moment shows that the kurtosis of the WLED is strictly platykurtic $(K_S < 3)$ which means, it can be used to explain the return of an investment and help to minimize the risk of large negative event.

***Keywords*:** Moments, mle, asymptotic normality, Probability Cumulative Density Function.

1. Introduction

The role of the classical logistic distribution in modelling and analyzing real life situation is well established in the literature. An extensive and detailed study has been carried out by various authors investigating the properties and applications of the classical logistic model [1, 2]. There are several methods of generalizing probability distributions as can be seen in [3-7].However, the generalization of the classical logistic distribution arises due to its short coming in handling only a symmetric data sets. Nevertheless little or no effort has been recorded on the asymptotic study of the generalized logistic distribution. In view of this, the paper focuses on the asymptotic properties of Weilbull Logistic Exponential Distribution (WLED). The remaining sections of this paper are structured as follows: In Section 2, the density function and cumulative distribution function of the WLED are presented. In Section 3, a detailed expression of the moment were studied. In Section 4, the maximum likelihood method was used to obtain the parameter estimates and simulation study was carried out to examine the performance and accuracy of the maximum likelihood estimates, and the paper concludes in Section 5.

2. THE WEIBULL LOGISTIC EXPONENTIAL DISTRIBUTION

In [8], they took T, R, and Y to be random variables from a known probability distribution with the cumulative distribution functions defined by $F_T(x) = P(T \le x)$, $F_R(x) = P(R \le x)$ and $F_T(x) = P(T \le x)$ andgave an amalgamated relation of the random variables reported in the work of [9]and define the cumulative distribution function of a random variable X as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)\,dt = P\left\{T \le Q_Y\left(F_R(x)\right)\right\} = F_T\left[Q_Y\left(F_R(x)\right)\right] \quad (1)$$

with the corresponding density function defined as

$$f_X(x) = \frac{f_R(x)}{f_Y\left\{Q_Y\left(F_R(x)\right)\right\}} f_T\left\{Q_Y\left(F_R(x)\right)\right\}$$

Following the above techniques in Equation (1), the cdf of the WLED is define by the relation

$$F(x) = 1 - e^{-\left(\left(\frac{\log\left(1+e^{\lambda x}\right)}{k}\right)^r\right)}, r, k, \lambda > 0, x \in R \quad (2)$$

where $r\ and\ k$ are the shape parameters which affect the general shape of the distribution and λ functioning as the rate parameter which is the reciprocal of the scale parameter.
The corresponding pdf of the WLED can be obtained by finding the derivative of its cdf in (2) with respect to x, this is given by:

$$f(x) = \frac{r\lambda e^{\lambda x}\left(\frac{\log\left(1+e^{\lambda x}\right)}{k}\right)^{r-1} e^{-\left(\frac{\log\left(1+e^{\lambda x}\right)}{k}\right)^r}}{k\left(1+e^{\lambda x}\right)}, r, k, \lambda > 0, x \in R \quad (3)$$

3. 0 MOMENTS OF THE WEILBULL LOGISTIC EXPONENTIAL DISTRIBUTION

Theorem 1: Let X ~ WLED, then the rth moment about the origin of X is given by

$$U'_P = \left(\frac{1}{\lambda}\right)^P \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} A_1(w_i, P, i, k, r)$$

Corresponding Author**:** Akata I.U., Email: innoakata@gmail.com, Tel: +2347063333338

Proof
Let X be a random variable following the Weilbull logistic exponential distribution with parameter r, k and λthen

$$E[X^P] = U'_P = \int_{-\infty}^{\infty} x^P f(x)\,dx \quad (4)$$

where $f(x)$ is as defined in (3) above. Hence

$$E[X^P] = U'_P = \int_{-\infty}^{\infty} \frac{x^P r\lambda e^{\lambda x}\left(\frac{\log(1+e^{\lambda x})}{k}\right)^{r-1} e^{-\left(\frac{\log(1+e^{\lambda x})}{k}\right)^r}}{k(1+e^{\lambda x})}dx$$

or

$$U'_p = \frac{r\lambda}{k}\int_{-\infty}^{\infty} \frac{x^p e^{\lambda x}\left(\frac{\log(1+e^{\lambda x})}{k}\right)^{r-1} e^{-\left(\frac{\log(1+e^{\lambda x})}{k}\right)^r}}{(1+e^{\lambda x})}dx \quad (5)$$

Let

$$Y = \frac{\log(1+e^{\lambda x})}{k} \quad (6)$$

or $\quad x = \frac{\log(e^{kY}-1)}{\lambda}, \qquad \frac{dx}{dY} = \frac{ke^{kY}}{\lambda(e^{kY}-1)} \quad (7)$

Hence

$$U'_P = \frac{r\lambda}{k}\int_0^{\infty} \frac{\left(\frac{\log(e^{kY}-1)}{\lambda}\right)^P (e^{kY}-1)Y^{r-1}e^{-Y^r}}{e^{kY}} \cdot \frac{ke^{kY}}{\lambda(e^{kY}-1)}dY$$

$$U'_P = r\left(\frac{1}{\lambda}\right)^P \int_0^{\infty} (\log(e^{kY}-1))^P Y^{r-1}e^{-Y^r}\,dY \quad (8)$$

Suppose

$$w = Y^r \quad then \quad \frac{dY}{dw} = \frac{1}{r}w^{\left(\frac{1}{r}-1\right)} \quad (9)$$

Implying equation (8) becomes:

$$U'_P = r\left(\frac{1}{\lambda}\right)^P \int_0^{\infty}\left(\log\left(e^{kw^{\frac{1}{r}}}-1\right)\right)^P \left(w^{\frac{1}{r}}\right)^{r-1} e^{-W}\frac{1}{r}w^{\left(\frac{1}{r}-1\right)}dw$$

$$= \left(\frac{1}{\lambda}\right)^P \int_0^{\infty}\left(\log\left(e^{kw^{\frac{1}{r}}}-1\right)\right)^P e^{-W}\,dw$$

but $e^{-w} = \sum_{i=0}^{\infty}\frac{(-1)^i w^i}{i!}$,

then, $U'_P = \left(\frac{1}{\lambda}\right)^P \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\int_0^{\infty} w^i\left(\log\left(e^{kw^{\frac{1}{r}}}-1\right)\right)^P dw$

or $\quad U'_P = \left(\frac{1}{\lambda}\right)^P \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}A_1(w_i, P, i, k, r)$

where,

$$A_1(w_i, P, i, k, r) = \int_0^{\infty} w^i\left(\log\left(e^{kw^{\frac{1}{r}}}-1\right)\right)^P dw$$

3.1 A Demonstration on the Use of WLED

Here a study is carried out to obtain the mean, standard deviation, median, skewness and kurtosis of the Weilbull logistic exponential distribution for varying parameters values.

Table 1. The Mean, Standard Deviation (SD) and the Median of the WLED

		$\lambda = 0.5$			$\lambda = 2$			$\lambda = 4$		
R	κ	Mean	S.D	Median	Mean	S.D	Median	Mean	S.D	Median
0.8	3	4.3787	8.9554	2.8375	1.0947	2.2388	0.7094	0.5473	1.1194	0.3547
	5	8.8155	13.4132	5.3164	2.2039	3.3533	1.3291	1.1019	1.6767	0.6645
	8	15.1576	20.4350	8.6975	3.7894	5.1087	2.1744	1.8947	2.5544	2.5544
2	3	4.4594	3.4368	4.5073	1.1148	0.8592	1.1268	0.5574	0.4296	0.5634
	5	8.0177	5.0854	7.8051	2.0044	1.2714	1.9513	1.0022	0.6357	0.9756
	8	13.1117	7.7563	12.5485	3.2779	1.9391	3.1371	1.6390	0.9695	1.5686
5	3	5.1024	1.5839	5.3089	1.2756	0.3940	1.3272	0.6378	0.1980	0.6636
	5	8.7867	2.4166	9.0533	2.1967	0.6042	2.2633	1.0983	0.3021	1.1317
	8	14.1338	3.7683	14.5183	3.5334	0.9421	3.6296	1.7667	0.4710	1.8148

Table 2. Skewness and Kurtosis of the WLED

		$\lambda = 0.5$		$\lambda = 2$		$\lambda = 4$	
R	κ	Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis
0.8	3	0.8213	0.6715	0.8213	0.6715	0.8213	0.6715
	5	1.2279	1.1203	1.2279	1.1203	1.2279	1.1279
	8	1.4318	1.4410	1.4318	1.4410	1.4318	1.4410
2	3	-0.1320	-0.2269	-0.1320	-0.2269	-0.1320	-0.2269
	5	0.1603	-0.4626	0.1603	-0.4626	0.1603	-0.4626
	8	0.3169	-0.5454	0.3169	-0.5454	0.3169	-0.5454
5	3	-0.6830	0.3476	-0.6830	0.3476	-0.6830	0.3476
	5	-0.5324	0.0077	-0.5324	0.0077	-0.5324	0.0077
	8	-0.4620	-0.1501	-0.4620	-0.1501	-0.4620	-0.1501

From *Table* 2 we observed that when the shape parameter r and the rate parameter λ are held constant, the mean, standard deviation, and median increases as the scale parameter k increase. We also observed from *Table*3, that the WLED exhibited a right-skewed $(S_k \geq 0)$, left- skewed $(S_k \leq 0)$ and approximately symmetric $(S_k \approx 0)$ shapes while the kurtosis are strictly platykurtic $(K_S < 3)$. This assertion clearly supports and maintains the ideal behinds the graphical illustration of the density function of the WLE distribution. It is evident to note that the kurtosis of the WLED are strictly platykurtic and this have a vital role to play most especially in area of finance.

4.0 Parameter Estimation

In this section, we shall estimate the parameters of the WLED in Equation (3), we shall use the mle method and demonstrate its usefulness using a simulation study in section 4.2.

4.1 Maximum likelihood estimates

Let $x_1, x_2, x_3, \ldots x_n$ be a random samples from the Weilbull logistic exponential distribution with density function defined in Equation (3), then the likelihood function is given by

$$L(\mathrm{x},\phi) = \prod_{i=1}^{n} f\left(x_i\right) = \prod_{i=1}^{n} \frac{\left[r\lambda \ell^{\lambda x}\right]\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r-1} \ell^{-\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r}}}{k\left(1+\ell^{\lambda x}\right)}$$

and the log likelihood is

$$\ell(\mathrm{x},\phi) = \sum_{i=1}^{n} \log\left(f\left(x_i\right)\right) = \sum_{i=0}^{n} \log r + \sum_{i=0}^{n} \log \lambda + \sum_{i=0}^{n} \lambda x_i + \sum_{i=0}^{n} \log\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r-1}$$

$$-\sum_{i=0}^{n}\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r} - \sum_{i=0}^{n} \log k - \sum_{i=0}^{n} \log\left(1+\ell^{\lambda x}\right)$$

$$\sum_{i=1}^{n} \log\left(f\left(x_i\right)\right) = n\log r + n\log\lambda + \lambda\sum_{i=0}^{n} x_i + (r-1)\sum_{i=0}^{n} \log\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]$$

$$-\sum_{i=0}^{n}\left[\frac{\log\left(1+\ell^{\lambda x}\right)}{k}\right]^{r} - n\log k - \sum_{i=0}^{n} \log\left(1+\ell^{\lambda x}\right)$$

Taking the derivatives with respect to the various parameters (r, k, λ), the followings are obtained:

$$\frac{\partial L}{\partial r} = \frac{n}{r} + \sum_{i=1}^{n} \log\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right) - \sum_{i=1}^{n} \log\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{\ell^{\lambda x_i} x_i}{1+\ell^{\lambda x_i}} + (r-1)\sum_{i=1}^{n} \frac{\ell^{\lambda x_i} x_i}{\left(1+\ell^{\lambda x_i}\right)\log\left[1+\ell^{\lambda x_i}\right]}$$

$$-\sum_{i=1}^{n} \frac{\ell^{\lambda x_i} r\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1} x_i}{\left(1+\ell^{\lambda x_i}\right)k}$$

$$\frac{\partial L}{\partial k} = -\frac{n}{k} - \frac{n(r-1)}{k} + \sum_{i=1}^{n} \frac{r \log\left[1+\ell^{\lambda x_i}\right]\left(\frac{\log\left[1+\ell^{\lambda x_i}\right]}{k}\right)^{r-1}}{k^2}$$

The maximum likelihood estimator $\hat{\varphi}$ of φ can be derivedusing Newton Raphson's iterative method given by the relation:

$$\hat{\varphi} = \varphi_q - H^{-1}(\varphi_q)U(\varphi_q), \qquad \hat{\varphi} = (\hat{r}, \hat{k}, \hat{\lambda})^T$$

where $H(\varphi_q)$ is the hessian matrix denoted by:

$$H(\varphi_q) = \begin{pmatrix} \frac{\partial^2 \ell}{\partial r^2} & \frac{\partial^2 \ell}{\partial r \partial \lambda} & \frac{\partial^2 \ell}{\partial r \partial k} \\ \frac{\partial^2 \ell}{\partial \lambda \partial r} & \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial k} \\ \frac{\partial^2 \ell}{\partial k \partial r} & \frac{\partial^2 \ell}{\partial k \partial \lambda} & \frac{\partial^2 \ell}{\partial k^2} \end{pmatrix}$$

4.2 Simulation Study

This section investigates the performance and accuracy of the maximum likelihood estimates of the parameters via a simulation study for different choice of parameter values as well as different sample sizes. Random sample was generated from the WLED and the Monte Carlo simulation study was repeated 1000 times for different sample sizes of n = 75,100,150,200,250, 300 and parameter values (r = 2, k = 2, lamda= 1) (r= 2, k = 2, lamda= 0.5) and (r = 1.5, k = 1.5, lamda= 0.2) respectively. It is imperative to note that the choice of values for the parameter estimate is subjective. An algorithm for the simulation study is given by the following steps.

1. Choose the value N(i.e. number of Monte Carlo Simulation)
2. Choose the value $\varphi_0 = (r_o, \lambda_o, k_o)$ corresponding to the Weibull-Logistic Exponential Distribution(r, λ, k);
3. Generate a sample of size n fromWLED
4. Compute the maximum likelihood estimate $\hat{\varphi}_o$ of φ_o
5. Repeat steps (3-4), N-times
6. Compute the average bias $(\text{AVB}) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\phi}_1 - \phi_o\right)$ and the mean squared error $(MSE) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\varphi}_1 - \varphi_o)^2$

Table 3: Monte Carlo Simulation Results for Average Bias and Mean Squared Error of the MLE of the WLED.

Parameters	N	Average Bias (r)	Average Bias (K)	Average Bias (Lamda)	Mean Squared Error(r)	Mean Squared Error(K)	Mean Squared Error (Lamda)
r = 2 K =2 Lamda = 1	75	0.1919	0.09735	0.06042	0.4413	0.4457	0.1897
	100	0.1092	0.0399	0.0334	0.2461	0.2186	0.1017
	150	0.1072	0.0203	0.0131	0.1711	0.1642	0.0734
	200	0.0833	0.0071	0.0041	0.1159	0.1192	0.0532
	250	0.0551	0.00121	0.00319	0.0734	0.0863	0.0398
	300	0.0468	0.0202	0.0109	0.0572	0.0743	0.0336
r = 2 K = 2 Lamda = 0.5	75	0.2192	0.0176	0.00513	0.4019	0.2850	0.0326
	100	0.1977	0.0244	0.0065	0.4298	0.2772	0.0298
	150	0.0820	0.0255	0.0078	0.1378	0.1447	0.0156
	200	0.0865	-0.0013	-0.0022	0.0960	0.1046	0.0115
	250	0.0509	0.0256	0.0096	0.0661	0.0913	0.0107
	300	0.0219	0.0225	0.0074	0.0417	0.0613	0.0072
r = 1.5 K = 1.5 Lamda = 0.2	75	0.2302	0.0099	0.0067	1.1808	0.0979	0.00510
	100	0.1400	0.0162	0.0054	0.2922	0.0769	0.0041
	150	0.0612	0.00271	0.0022	0.0917	0.0405	0.0019
	200	0.0739	0.0022	0.00035	0.0889	0.0322	0.0017
	250	0.0584	-0.0066	-0.00177	0.0486	0.0279	0.0014
	300	0.0323	0.00595	0.00195	0.04154	0.0221	0.00113

From Table 3, we clearly observe that the values of the average bias and the mean squared error of the parameter estimates decreases as the sample size n increases. This implies that as the sample size increases, the MLE becomes more efficient and hence validates the consistency property of the MLE.

5. Conclusion

In this paper, the moments and the maximum likelihood estimates of the parameters of the weilbull logistic exponential distribution are presented, some mathematical properties which includes the probability density function and cumulative density function are obtained. The demonstration on the relevance of the moment shows that the kurtosis of the WLED is strictly platykurtic which means, it can beused to explain the return of an investment and help to minimize the risk of large negative event. finally a simulation study was carried out to validatethe asymptotic properties (unbiasedness, consistency and asymptotic normality) of the maximum likelihood estimates and the results clearly demonstrated that the average bias and the mean squared error of the parameter estimates of the WLED decreases as the sample size increases which shows that the maximum likelihood estimates is more efficient and hence validate the consistency property of the MLE of WLED.

REFERENCE

[1] Cox, D.R. (1970). Analysis of binary data. Methuen, London.

[2] Johnson, N.L. and Kotz, S. (1970). Continues Univariate Distributions, 2, Houghton Mifflin, Boston.

[3] Gauss, M., Edwin, M., &Thiago, G. (2015). A new generalized Weibull family of distributions: properties and applications, *Journal of statistical distributions and applications* DOL 10. 1186 540488-015-0036-6.

[4] Mandouh, R.M. (2018). Logistic-Modified Weibull distribution and parameter estimation, international journal of contemporary mathematical sciences, Vol. 13, no 1, 11-23.

[5] Arber O., &Markela, M. (2018). A new modified generalized odd Log-logistic Distribution with three parameters, Mathematical Theory and Modeling. Vol. 8, No. 1.

[6] Oguntunde, P.E., Adejumo, A.O., Mundhur, A.K., Okagbue, I.O., & Odetunmibi, O.A. (2018). The logistic inverse exponential distribution, basic structural properties and application, proceeding of the world congress on Engineering and Computer Science. Vol 11, 23-25.

[7] Mansoor, M., Tahir, M., Cordeiro, G., Alizadeh, M., Provost, S., &Alzaatreh, A. (2018). The Marshall-Olkin logistic-exponential distribution. *Communications in Statistics - Theory and Methods*. https://doi.org/10.1080/03610926. 2017.1414254.

[8] Alzaatreh, A., Lee, C., &Famoye, F. (2014). T-normal family of distributions: a new approach to generalize the normal distribution. *Journal of statistical distributions and applications* 1(1), 16.

[9] Aljarrah, M.A., Lee, C., &Famoye, F. (2014). A method of generating T-X family of distributions using quantile functions. J. Stat. Distributions Appl. 1(2), 17.