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COEFFICIENT INEQUALITY OF ERROR AND BESSEL FUNCTIONS IN THE SPACE OF UNIVALENT HARMONIC FUNCTION

[1]Awolere I.T., [2]Gbolagade A.M. and [3]Fadare A.O.

[1]Department of Mathematical Science, Olusegun Agagu University of Science and Technology, Okitipupa, Ondo State, Nigeria
[2]Department of Mathematics, Emmanuel Alayande College of Education, P.M.B. 1010, Oyo, Oyo State, Nigeria
[3]Department of Mathematics, Federal College of Education (Special), Oyo, Oyo State, Nigeria

Abstract

The authors established results involving coefficient inequality, extreme points, convolution and convex combinations using convolution of error and Bessel functions for new class $\overline{T_H}(\lambda,\beta,k_1,k_2,c_1,c_2)$ ***of harmonic univalent functions in the unit disc.***

***Keywords*:** Analytic functions, univalent functions, harmonic function, Bessel function, error function.

1.0 Introduction

Error function is a special function of sigmoid shape that occurs in probability, statistics, and partial differential equations describing diffusion as defined by [1, 2].

$$er\ f(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt \tag{1}$$

The error function is an entire function. It has no singularities and its Taylor expansion always converges. Error function which is present in diffusion is part of the transport phenomena that can be applied in many disciplines such as Physics, Chemistry, Biology, Thermo mechanics and Mass flow. Abramowitz [3] transformed error function into series function as

$$er\ f(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt=\frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{(2k+1)k!}z^{2k+1} \tag{2}$$

Alzer [4] and Coman [5] studied the properties and inequality of error function while Elbert *et. al.* [6] also studied the properties and complimentary error function. Also, Ramachandran *et. al.* [7] introduced and studied modified error function written in series as

$$F(z)=(f*Erf)(z)=z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{(2k+1)(k-1)!}z^k \tag{3}$$

where *Er f* be a normalized analytic function which is obtained from (1) as

$$Er(z)=\frac{\sqrt{\pi z}}{2}e_r f(\sqrt{z})=z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{(2k-1)(k-1)!}z^k \tag{4}$$

A continuous function $f=u+iv$ is a complex-valued function in a complex domain *G* if both *u* and *v* are real and harmonic in *G*. In any simply connected domain $D\subset G$, we can write $f=h+\overline{g}$ where *h* and *g* are analytic in *D*, then for $f=h+\overline{g}\in H$, we can express the analytic function *f* and *g* as

$$h(z)=z+\sum_{k=2}^{\infty}a_k z^k,\ g(z)=z+\sum_{k=1}^{\infty}b_k z^k,\ |b_1|<1 \tag{5}$$

We refer *h* the analytic part and *g* co-analytic part of *f*. A necessary and sufficient condition for *f* to be locally univalent and orientation preserving in *D* is that $|h'(z)|>|g'(z)|$ in *U*. For details on harmonic function see [8-12]

For complex parameters $c_1,k_1,c_2,k_2(k_1,k_2\neq 0,-1,-2....)$, Porwal [13] defined generalized Bessel functions $\phi_1(z)=zu_{p1}(z)$ and $\phi_2(z)=zu_{p2}(z)$. Corresponding to these functions, the following convolution operator was introduced.

$$\Omega\equiv\Omega(c_1,k_1,c_2,k_2):H\to H \tag{6}$$

defined by

$$\Omega(c_1,k_1,c_2,k_2)f=f*(\phi_1+\overline{\phi_2})=h(z)*\phi_1(z)+\overline{g(z)*\phi_2(z)} \tag{7}$$

for any function $f=h+\overline{g}$ in *H* by letting

$$\Omega(c_1,k_1,c_2,k_2)f(z)=H(z)+\overline{G(z)} \tag{8}$$

Corresponding Author**:** Gbolagade A.M., Email: gbolagadeam@yahoo.com, Tel: +2348038063598

where

$$H(z) = z + \sum_{k=2}^{\infty} \frac{(-c_1/4)^{k-1}}{(k_1)_{k-1}(k-1)!} A_k z^k,\ G(z) = \sum_{k=1}^{\infty} \frac{(-c_2/4)^{k-1}}{(k_2)_{k-1}(k-1)!} B_k z^k \tag{9}$$

In a similar manner Libra type integral operator is given as

$(S_f) = H(z) + \overline{G(z)}$

where

$$H(z) = z - \sum_{k=2}^{\infty} \frac{2(-c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!} A_k z^k,\ G(z) = \sum_{k=1}^{\infty} \frac{2(-c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!} B_k z^k \tag{10}$$

Using convolution (Hadarmard product) of error function $E_r f$ and generalized Bessel function B_f as

$$BE_r f(z) = (f * Er) \tag{11}$$

For any $z \in U$, the following convolution operator is used as

$$BE_r f(z) = BE_r H(z) + BE_r H(z) \tag{12}$$

where

$$BE_r H(z) = H(z) = z - \sum_{k=2}^{\infty} \frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2 (2k-1)} A_k z^k \tag{13}$$

and

$$\overline{BE_r G(z)} = \sum_{k=2}^{\infty} \frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2 (2k-1)} B_k z^k \tag{14}$$

Definition 1: Let $f \in T_H(\lambda, \beta, k_1, k_2, c_1, c_2), 0 \le \beta < 1, \lambda \ge 0$, then

$$\operatorname{Re}\left\{ \frac{zS_{\alpha*f} Er h'(z) - \overline{zS_{\alpha*f} Er g'(z)}}{S_{\alpha*f} Erh(z) + \overline{S_{\alpha*f} Erg(z)}} - \lambda \right\} \ge \beta \left| \frac{zS_{\alpha*f} Er h'(z) - \overline{zS_{\alpha*f} Er g'(z)}}{S_{\alpha*f} Erh(z) + \overline{S_{\alpha*f} Erg(z)}} - 1 \right|$$

for $c_1, k_1, c_2, k_2 \in C$ and $k_1, k_2 \ne 0, -1, -2 \ldots\ldots$

For more details about generalized Bessel function see [14-17].

Motivated are the results of analytic and harmonic univalent functions defined by Powal [13] and Ramachandran et al [7] using generalized Bessel and modified error functions hence we established coefficient inequality, distortion and extreme bounds, convolution and convex combinations using convolution of error and Bessel functions.

We begin with the statement of the following lemma by Ahuja and Jahangiri [18].

Lemma 1: Let $f = h + \overline{g}$ with *h* and *g* of the form (5). Then, let

$$\sum_{k=2}^{\infty} \frac{k-\beta}{1-\beta} |a_k| + \sum_{k=1}^{\infty} \frac{k+\beta}{1-\beta} |b_k| \le 1 \tag{15}$$

where $0 \le \beta < 1$. Then *f* is harmonic, orientation preserving, univalent *U* and $f \in S_H^*(\beta)$.

2.0 Main Result

Theorem 1: Let $f \in \overline{T_H}(\lambda, \beta, k_1, k_2, c_1, c_2)$ then we have

$$\sum_{k=2}^{\infty} \frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2} [\beta(k-1) + k - \lambda] |Ak| + \sum_{k=1}^{\infty} \frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2} [\beta(k-1) + k + \lambda] \left|\overline{B_k}\right| \le 1 - \lambda \tag{16}$$

Proof:

$$\operatorname{Re}\left\{ \frac{zS_{\alpha*f} Er h'(z) - \overline{zS_{\alpha*f} Er g'(z)}}{S_{\alpha*f} Erh(z) + \overline{S_{\alpha*f} Erg(z)}} - \lambda \right\} \ge \beta \left| \frac{zS_{\alpha*f} Er h'(z) - \overline{zS_{\alpha*f} Er g'(z)}}{S_{\alpha*f} Erh(z) + \overline{S_{\alpha*f} Erg(z)}} - 1 \right|$$

$$\frac{1 - \sum_{k=2}^{\infty} \frac{2k(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2 (2k-1)} A_k z^{k-1} - \sum_{k=2}^{\infty} \frac{2k(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2 (2k-1)} \overline{B_k} z^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2 (2k-1)} A_k z^{k-1} - \sum_{k=2}^{\infty} \frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2 (2k-1)} \overline{B_k} z^{k-1}} - \lambda$$

$$\ge \beta \left(1 - \frac{1 - \sum_{k=2}^{\infty} \frac{2k(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2 (2k-1)} A_k z^{k-1} - \sum_{k=2}^{\infty} \frac{2k(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2 (2k-1)} \overline{B_k} z^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2 (2k-1)} A_k z^{k-1} - \sum_{k=2}^{\infty} \frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2 (2k-1)} \overline{B_k} z^{k-1}} \right)$$

Thus, we have

$$\frac{1 - \lambda - \sum_{k=2}^{\infty} \frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2 (2k-1)} [k-\lambda] A_k z^{k-1} - \sum_{k=2}^{\infty} \frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2 (2k-1)} [k+\lambda] \overline{B_k} z^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2 (2k-1)} A_k z^{k-1} + \sum_{k=2}^{\infty} \frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2 (2k-1)} \overline{B_k} z^{k-1}}$$

$$\geq \frac{\sum_{k=2}^{\infty}\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2\,(2k-1)}[\beta k-\beta]A_k z^{k-1}-\sum_{k=2}^{\infty}\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2\,(2k-1)}[\beta k+\beta]\overline{B_k}z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2\,(2k-1)}A_k z^{k-1}+\sum_{k=2}^{\infty}\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2\,(2k-1)}\overline{B_k}z^{k-1}}$$

Upon further simplification, we have

$$1-\lambda \geq \sum_{k=2}^{\infty}\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2\,(2k-1)}[\beta k-\beta+k-\lambda]A_k z^{k-1}+\sum_{k=2}^{\infty}\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2\,(2k-1)}[\beta k-\beta+k+\lambda]\overline{B_k}z^{k-1}$$

Using triangular inequality

$$\sum_{k=2}^{\infty}A_1[\beta(k-1)+k-\lambda]\,|A_k|+\sum_{k=1}^{\infty}A_2[\beta(k-1)+k+\lambda]\,|B_k|\leq 1-\lambda$$

where $A_1=\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2\,(2k-1)}$ *and* $A_2=\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2\,(2k-1)}$

Which complete the proof.

Theorem 2: Let $f\in\overline{T_H}(\lambda,\ \beta,\ k_1,k_2,c_1,c_2)$, then we have

$$|f(z)|\leq(1+|b_1|)r+18\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)}-\frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]r^2 \qquad (17)$$

$$|f(z)|\geq(1+|b_1|)r-18\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)}-\frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]r^2 \qquad (18)$$

Proof:

$$|f(z)|\leq(1+|b_1|)r+\sum_{k=2}^{\infty}\left(|A_k|+|\overline{B_k}|\right)r^2\leq(1+|b_1|)r+r^2\sum_{k=2}^{\infty}\left(|A_k|+|\overline{B_k}|\right)$$

$$=(1+|b_1|)r+\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]}\sum_{k=2}^{\infty}\frac{(c_1/18k_1)[2+\beta-\lambda]}{1-\lambda}\left(|A_k|+|\overline{B_k}|\right)r^2$$

and so

$$|f(z)|\leq(1+|b_1|)r+\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]}\sum_{k=2}^{\infty}\left[A_1\frac{\beta(k-1)+k-\lambda}{1-\lambda}|A_k|+A_2\frac{\beta(k-1)+k+\lambda}{1-\lambda}|\overline{B_k}|\right]r^2$$

$$=(1+|b_1|)r+\left[\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]}-\frac{1-\lambda}{(c_2/18k_2)[2+\beta-\lambda]}\left(\frac{1+\beta+\lambda}{1-\lambda}\right)|b_1|\right]r^2$$

$$\leq(1+|b_1|)r+18\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)}-\frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]r^2$$

Similarly

$$|f(z)|\geq(1+|b_1|)r-\sum_{k=2}^{\infty}\left(|A_k|+|\overline{B_k}|\right)r^2\leq(1+|b_1|)r+r^2\sum_{k=2}^{\infty}\left(|A_k|+|\overline{B_k}|\right)$$

$$=(1+|b_1|)r-\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]}\sum_{k=2}^{\infty}\frac{(c_1/18k_1)[2+\beta-\lambda]}{1-\lambda}\left(|A_k|+|\overline{B_k}|\right)r^2$$

and so

$$|f(z)|\geq(1+|b_1|)r-\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]}\sum_{k=2}^{\infty}\left[A_1\frac{\beta(k-1)+k-\lambda}{1-\lambda}|A_k|+A_2\frac{\beta(k-1)+k+\lambda}{1-\lambda}|\overline{B_k}|\right]r^2$$

$$=(1+|b_1|)r-\left[\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]}-\frac{1-\lambda}{(c_2/18k_2)[2+\beta-\lambda]}\left(\frac{1+\beta+\lambda}{1-\lambda}\right)|b_1|\right]r^2$$

$$\geq(1+|b_1|)r+18\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)}-\frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]r^2$$

The upper bound given for $f\in\overline{T_H}(\lambda,\ \beta,\ k_1,k_2,c_1,c_2)$ is sharp and equality occurs for the function

$$f(z)=z+|B_1|\overline{z}+18\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)}-\frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]\overline{z}^{2}\quad(z=r)\ |B_1|\leq\frac{1-\lambda}{1+\beta+\lambda} \qquad (19)$$

Theorem 3: Let $f \in \overline{T}_H(\lambda, \beta, k_1, k_2, c_1, c_2)$, then we have

$$|f'(z)| \le (1+|b_1|) + 36\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)} - \frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]r \quad (20)$$

$$|f'(z)| \ge (1+|b_1|) - 36\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)} - \frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]r \quad (21)$$

Proof:

$$|f'(z)| \le (1+|b_1|) + \sum_{k=2}^{\infty} k(|A_k| + |\overline{B_k}|) \le (1+|b_1|) + 2r\sum_{k=2}^{\infty}(|A_k| + |\overline{B_k}|)$$

$$= (1+|b_1|) + 2\frac{(1-\lambda)}{(c_1/18k_1)[2+\beta-\lambda]}\sum_{k=2}^{\infty}\frac{(c_1/18k_1)[2+\beta-\lambda]}{1-\lambda}(|A_k| + |\overline{B_k}|)r$$

and so

$$|f'(z)| \le (1+|b_1|) + 2\frac{(1-\lambda)}{(c_1/18k_1)[2+\beta-\lambda]}\sum_{k=2}^{\infty}\left[A_1\frac{\beta(k-1)+k-\lambda}{1-\lambda}|A_k| + A_2\frac{\beta(k-1)+k+\lambda}{1-\lambda}|\overline{B_k}|\right]r$$

$$= (1+|b_1|) + 2\left[\frac{(1-\lambda)}{(c_1/18k_1)[2+\beta-\lambda]} - \frac{1-\lambda}{(c_2/18k_2)[2+\beta-\lambda]}\left(\frac{1+\beta+\lambda}{1-\lambda}\right)|b_1|\right]r$$

$$\le (1+|b_1|) + 36\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)} - \frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]r$$

Similarly

$$|f'(z)| \ge (1+|b_1|) - \sum_{k=2}^{\infty} K\ (|A_k| + |\overline{B_k}|)\ r^2 \le (1+|b_1|) + 2r\sum_{k=2}^{\infty}\ (|A_k| + |\overline{B_k}|)$$

$$= (1+|b_1|) - 2\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]}\sum_{k=2}^{\infty}\frac{(c_1/18k_1)[2+\beta-\lambda]}{1-\lambda}(|A_k| + |\overline{B_k}|)\ r^2$$

and so

$$|f'(z)| \ge (1+|b_1|) - 2\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]}\sum_{k=2}^{\infty}\left[A_1\frac{\beta(k-1)+k-\lambda}{1-\lambda}|A_k| + A_2\frac{\beta(k-1)+k+\lambda}{1-\lambda}|\overline{B_k}|\right]r$$

$$= (1+|b_1|) - 2\left[\frac{1-\lambda}{(c_1/18k_1)[2+\beta-\lambda]} - \frac{1-\lambda}{(c_2/18k_2)[2+\beta-\lambda]}\left(\frac{1+\beta+\lambda}{1-\lambda}\right)|b_1|\right]r$$

$$\ge (1+|b_1|) + 36\left[\frac{c_1(1-\lambda)}{k_1(2+\beta-\lambda)} - \frac{c_2(1+\beta+\lambda)}{k_2(2+\beta-\lambda)}\right]r$$

The upper bound given for $f \in \overline{T_H}(\lambda, \beta, k_1, k_2, c_1, c_2)$ is sharp and inequality occurs for the function.

Theorem 4: Let $f = h + \overline{g}$, where *h*and *g* are given by (1).

Then $f \in clco\ \overline{T_H}(\lambda, \beta, k_1, k_2, c_1, c_2)$ if and only if

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k) \quad (22)$$

where $h_1(z) = z$

$$h_k(z) = z + \frac{1-\lambda}{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2(2k-1)}[\beta(k-1)+k-\lambda]}z^k \ (k = 2,3,\ldots),$$

$$g_k(z) = z + \frac{1-\lambda}{\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2(2k-1)}[\beta(k-1)+k+\lambda]}z^k \ (k = 1,2,3,\ldots),$$

$\sum_{k=1}^{\infty}(X_k + Y_k)$, $X_k \ge 0$ and $Y_k \ge 0$. In particular, the extreme points of the class $f \in clco\ \overline{T_H}(\lambda, \beta, k_1, k_2, c_1, c_2)$ are $\{h_k\}$ and $\{g_k\}$ respectively.

Proof: For a function of the form (9), we have

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k)$$

$$= \sum_{k=1}^{\infty}(X_k + Y_k)(z) + \sum_{k=1}^{\infty}\frac{1-\lambda}{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2(2k-1)}[\beta(k-1)+k-\lambda]}X_k z^k +$$

$$\sum_{k=1}^{\infty}\frac{1-\lambda}{\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2(2k-1)}\left[\beta(k-1)+k+\lambda\right]}Y_k\bar{z}^k$$

$$= z+\sum_{k=1}^{\infty}\frac{1-\lambda}{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k+1)(k-1)!^2(2k-1)}\left[\beta(k-1)+k-\lambda\right]}X_k z^k+\sum_{k=1}^{\infty}\frac{1-\lambda}{\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k+1)(k-1)!^2(2k-1)}\left[\beta(k-1)+k+\lambda\right]}Y_k\bar{z}^k$$

Implies that

$$\sum_{k=2}^{\infty}\frac{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k-\lambda\right]}{1-\lambda}\left[\frac{1-\lambda}{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k-1)!^2(k+1)(2k-1)}}\left[\beta(k-1)+k+\lambda\right]X_k\right]+$$

$$\sum_{k=2}^{\infty}\frac{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k-\lambda\right]}{1-\lambda}\left[\frac{1-\lambda}{\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k+\lambda\right]}Y_k\right]$$

$$\sum_{k=1}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\le 1$$

Thus $f\in\overline{T_H}(\lambda,\beta,k_1,k_2,c_1,c_2)$

Conversely, suppose that $f\in\overline{T_H}(\lambda,\beta,k_1,k_2,c_1,c_2)$. Set

$$X_k=\frac{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k-\lambda\right]}{1-\lambda}|a_k|,\quad (k=2,3,...)$$

and

$$Y_k=\frac{\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k+\lambda\right]}{1-\lambda}|b_k|,\quad (k=1,2,3,...)$$

Then by the inequality theorem 1, we have $0\le X_k\le 1(k=2,3,...)$ and $0\le Y_k\le 1\,(k=1,2,3,...)$

Define $X_1=1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k$ and note that $X_1\ge 0$. Thus, we obtain $f(z)=\sum_{k=2}^{\infty}X_kh_k+Y_kg_k$. This completes the proof of Theorem 4.

3.0 Convolution and Convex Combinations

For two harmonic functions

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=1}^{\infty}\bar{b}_k\bar{z}^k$$

$$F(z)=z+\sum_{k=2}^{\infty}A_kz^k+\sum_{k=1}^{\infty}\bar{B}_k\bar{z}^k$$

we define their convolution

$$(f*F)(z)=z+\sum_{k=2}^{\infty}a_kA_kz^k+\sum_{k=1}^{\infty}\overline{b_kB_k}\,\bar{z}^k,$$

Using this definition, we show that the class $\bar{T}_{H,n}(\lambda)$ is close under convolution.

Theorem 5: Let $f,F\in\bar{T}_{H,n}(\lambda)$. Then $f*F\in\bar{T}_{H,n}(\lambda)$.

Proof: We note that $|A_k|\le 1$ and $|B_k|\le 1$. Now for the convolution $f*F$ we have

$$\sum_{k=2}^{\infty}\frac{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k-\lambda\right]}{1-\lambda}|A_ka_k|+\sum_{k=1}^{\infty}\frac{\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k+\lambda\right]}{1-\lambda}|B_kb_k|$$

$$\le\sum_{k=2}^{\infty}\frac{\frac{2(c_1/4)^{k-1}}{(k_1)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k-\lambda\right]}{1-\lambda}|a_k|+\sum_{k=1}^{\infty}\frac{\frac{2(c_2/4)^{k-1}}{(k_2)_{k-1}(k-1)!^2(k+1)(2k-1)}\left[\beta(k-1)+k+\lambda\right]}{1-\lambda}|b_k|\le 1$$

Therefore $f*F\in\bar{T}_{H,n}(\lambda)$ which completes the proof.

Now, we show that the case $\bar{T}_{H,n}(\lambda)$ is closed convex combination of its members.

Theorem 6: The class $\bar{T}_{H,n}(\lambda)$ is closed under convex combination.

Proof: For $(i=1,2,3,...)$, let $f_i\in\bar{T}_{H,n}(\lambda)$ where $f_i(z)$ is given by

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=1}^{\infty}\bar{b}_k\bar{z}^k$$

Then, by Theorem 1 we have

$$\sum_{k=2}^{\infty}\frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}|a_{ki}|+\sum_{k=1}^{\infty}\frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}|b_{ki}|\leq 1$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i = z+\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{ki}\right)z^k+\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i \overline{b_{ki}}\right)z^k$$

Then, by Theorem 1 we have

$$\sum_{k=2}^{\infty}\frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}\left|\sum_{i=1}^{\infty} t_i a_{ki}\right|+\sum_{k=1}^{\infty}\frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}\left|\overline{\sum_{i=1}^{\infty} t_i b_{ki}}\right|$$

$$\leq \sum_{t=1}^{\infty} t_1\left(\sum_{k=2}^{\infty}\frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}|a_{ki}|+\sum_{k=1}^{\infty}\frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}|b_{ki}|\right)$$

$$\leq \sum_{t=1}^{\infty} t_1 = 1$$

Therefore,

$$\sum_{i=1}^{\infty} t_i f_i \in \overline{T}_{H,n}(\lambda)$$

In conclusion, we established the convolution of error and Bessel functions in the space of univalent harmonic function for the purpose of coefficient inequality and extreme points as established in theorems one to four. The paper also looked at convolution and convex combinations of the function in theorems four and five.

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