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ON THE EXISTENCE OF PERIODIC OR ALMOST PERIODIC SOLUTIONS OF A KIND OF THIRD ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract

We present in this paper, by using the basic tool of Lyapunov's second method the existence of periodic or almost periodic solutions of a kind of third order nonlinear delay differential equations with a continuous deviating argument $\tau(t)$. We provide in a different form that is based on Routh-Hurwitz conditions, sufficient conditions which ensure the existence of periodic or almost periodic solutions of the delay differential equations considered when the forcing term p is periodic or almost periodic in t uniformly in x, x' and x''. The new result obtained extends and improves on earlier results on delay differential equations.

***Keywords*:** Periodic solutions, Almost periodic solutions, Third order nonlinear delay differential equations, Lyapunov's method
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1. Introduction

We consider the third-order nonlinear delay differential equation

$$x''' + ax'' + f(x(t-\tau(t)), x'(t-\tau(t))) = p(t, x, x', x'') \qquad (1.1)$$

where a is positive constant and f, p are continuous in their respective arguments. $0 \le \tau(t) \le \gamma$, $\tau'(t) \le \beta$, $0 < \beta < 1$, β and γ are some positive constants, γ will be determined later. The functions $f(x, y)$, $f(x(t-\tau(t)), y(t-\tau(t)))$ and $p(t, x, y, z)$ satisfy a Lipschitz condition in $x, y, x(t-r(t)), y(t-r(t))$ and z. Then, the solutions of (1.1) are unique. Throughout the paper, $x(t)$, $y(t)$ and $z(t)$ are respectively abbreviated as x, y and z.

Periodic properties of solutions play a very significant role in characterizing the behavior of solutions of sufficiently complicated nonlinear physical system like differential equations which are important tools in scientific modeling of some practical problems and often arise in many fields of science and technology such as after effect, nonlinear oscillations, biological systems and equations with deviating arguments (see [1 – 3]). Various and more general form of equation (1.1) have been studied by several authors, see for instance [4 - 6], [7], [8 - 10], [11 - 15], to mention a few. Almost all the results immediately mentioned above hold good for one or more nonlinear terms depending on the either constant, continuous or multiple deviating arguments where they obtained the stability and boundedness of solutions. A number of methods have been developed for proving the existence of forced oscillations of nonlinear ordinary differential equations. Most of these methods are naturally based on applications of fixed point theorems [16]. It should be noted that the existence of periodic solutions of some kind of nonlinear third order scalar differential equations without deviating arguments or delay being zero have been investigated by some authors, for example [16 - 19]. The methods employed by [16 - 19] were based on the Brouwer and the Lerray-Schauder fixed point theorems also referred to as the "non-Routh Hurwitz" direction in proving the existence of periodic solutions of third order differential equations. However, here, we consider a different approach that is based on Routh-Hurwitz conditions in establishing the existence of periodic or almost periodic solutions of (1.1) if p is periodic or almost periodic due to the presence of the perturbation r. Analysis of the periodic properties of solutions for nonlinear delay differential equations using this approach is quite complicated. The difficulty increases depending on the assumptions made on forced function p and the requirement for a complete Lyapunov function.(See also [20]).

Our motivation comes from the papers of [16 - 19]. With respect to our observation in the relevant literature, periodic properties of solutions of delay differential equations in various forms of (1.1) based on Routh-Hurwitz conditions is rarely scarce. We established sufficient conditions which ensure the existence of periodic or almost periodic solutions of (1.1) when the forcing term p is periodic or almost periodic in t uniformly in x, x' and x''. An example is given to illustrate the correctness and significance of the result obtained.

Now, we will state the stability criteria for the general non-autonomous delay differential system. We consider:

$$\dot{x} = f(t, x), x_t = x(t+\theta) \quad -r \le \theta \le 0, t \ge 0, \qquad (1.2)$$

where $f: \mathbf{I} \times C_H \longrightarrow \mathbb{R}^n$ is a continuous mapping,

$f(t, 0) = 0, C_H := \{\phi \in (C[-r, 0], \mathbb{R}^n) : \parallel \phi \parallel \le H\}$

and for $H_1 \le H$, there exists $L(H_1) > 0$, with

$|f(\phi)| \le L(H_1) \quad when \quad \parallel \phi \parallel \le H_1$.

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Definition 1 ***([12])*** *An element $\psi \in C$ is in the ω-limit set of ϕ, say, $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,\infty)$ and there is a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, with $\| x_{t_n}(\phi) - \psi \| \to 0$ as $n \to \infty$ where*
$x_{tn}(\phi) = x(t_n + \theta, 0, \phi) \quad for \quad -r \le \theta \le 0.$
Definition 2 ***([12], [15])*** *A set $Q \in C_H$ is an invariant set if for any $\phi \in Q$, the solution of (1.2), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.*
Lemma 1 ***([12], [15])*** *An element $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (1.2) with $x_o(\phi) = \phi$ is defined on $[0,\infty)$ and $\| x_t(\phi) \| \le H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*
$dist(x_t(\phi), \Omega(\phi)) \to 0 \quad as \quad t \to \infty.$
Lemma 2 ***([12], [15])*** *Let $V(t,\phi): I \times C_H \longrightarrow \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t,\phi) \ne 0$, and such that:*
$W_1|\phi(0)| \le V(t,\phi) \le W_2 \| \phi \|$where $W_1(r)$, $W_2(r)$ are wedges
$\dot{V}_{(1.2)}(t,\phi) \le 0$ for $\phi \le C_H$.
Then the zero solution of (1.2) is uniformly stable. If we define $Z = \{\phi \in C_H: \dot{V}_{(1.2)}(t,\phi) = 0\}$, then the zero solution of (1.2) is asymptotically stable provided that the largest invariant set in Z is $Q = \{0\}$.
Definition 3 *A continuous function $f: \mathbb{R} \to x$ is called almost periodic if for each $\varepsilon > 0$ there exists $\ell(\varepsilon) > 0$ such that every interval of length $\ell(\varepsilon)$ contains a number ζ with property that*
$|f(t+\zeta) - f(t)| < \varepsilon \quad for \ each \ t \in \mathbb{R}.$
Definition 4 *A continuous function $f: \mathbb{R} \to x$ is said to be periodic with period ω for all $t \in \mathbb{R}$ such that*
$f(t+\omega) = f(t) \quad for \ all \ t \in \mathbb{R}.$
Assume now that r is the perturbation such that p the continuous function $p(t,x,x',x'')$ is separable in the form
$p(t,x,x',x'') = q(t) + r(t,x,x',x'')$,
with $q(t) + r(t,0,0,0)$ continuous in their respective arguments, where
$|q(t)| \le d_1, d_1 \ is \ a \ positive \ constant \ for \ all \quad -\infty < t < \infty.$
Equation (1.1) may be replaced with equivalent system
$x' = y,$
$y' = z - q(t),$
$z' = -az - f(x,y) - aq(t) + p(t,x,y,z)$
$+ \int_{t-\tau(t)}^{t} f'_x(x(s),y(s))z(s)ds + \int_{t-\tau(t)}^{t} f'_{y'}(x(s),y(s))z(s)ds.$ (1.3)

Our main result is the following:

2. Statement of results

Theorem 1
In addition to the basic assumptions imposed on a and functions f and p appearing in (1.1), we further suppose that the system (1.3) with $f(0,0) = 0$ and $f'_x(x,y), f'_y(x,y)$ is continuous for all x, y. We assume that there exist positive constants L, M, D_1, v, b and c, $(ab - c > 0)$, $a > 1$, $a^2 < b$ such that the following conditions hold:
(i) $\frac{f(x,y)}{y} \ge b$, for all $x, y \ne 0$
(ii) $\frac{f(x,y)}{x} \ge v$, for all $x \ne 0, y$
(iii) $f'_x(x,0) \le c$, $f'_y(x,0) \le 0$, $f'_z(x,0) \le 0$ for all x.
(iv) $y\int_0^y f'_x(x,\sigma)d\sigma \le 0$
(v) $|f'_x(x,y)| \le L$, $|f'_y(x,y)| \le M$, for all x, y
(vi) $p(t,x,y,z)$ satisfies
$r(t,x,y,z+q) \equiv r(t,x_1,y_1,z_1+q) - r(t,x_1,y_1,z_2+q) \le \rho(t)\{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\}$,
for arbitrary t and $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ holds and $\rho(t)$ satisfy
$$\int_{-\infty}^{\infty} \rho^m(t)dt < \infty,$$
for some constant m in the range $1 \le m \le 2$.
Furthermore, suppose that γ satisfies
$$\gamma < min\{\frac{\alpha\beta v}{2(L+M)}; \frac{(b-\delta c)(1-\beta)}{2(a+1)(L+M)(1-\beta) + [\alpha\beta + (a+1) + (1+\delta)]L}; \frac{(a\delta - 1)(1-\beta)}{2(1+\delta)(L+M)(1-\beta) + [\alpha\beta + (a+1) + (1+\delta)]m}\}.$$
Then, there exists a constant D_1 and the solutions of (1.1) satisfying
$\{x^2(t) + x'^2(t) + x''^2(t)\}^{\frac{1}{2}} \le D_1, \quad for \quad -\infty < t < \infty,$ (2.1)
and having the properties that

i) If $q(t)$ and $r(t,x,y,z)$ are uniformly almost periodic in t, for $\{x^2(t) + y^2(t) + z^2(t)\}^{\frac{1}{2}} \le D_1$, then the solutions of (1.1) are almost periodic in t.

ii) If $q(t)$ and $r(t,x,y,z)$ are periodic in t, with period ω, for $\{x^2(t) + y^2(t) + z^2(t)\}^{\frac{1}{2}} \le D_1$, then the solutions of (1.1) periodic with period ω.

3. Some preliminary results

Our tool in the proof of Theorem 1 is the following scalar function given by

$$V(x_t,y_t,z_t)=V_1(x_t,y_t,z_t)+V_2(x_t,y_t,z_t) \tag{3.1}$$

where V_1 and V_2 are defined by

$$2V_1=2\int_0^x f(\vartheta,0)d\vartheta+ay^2+2\delta\int_0^y f(x,\sigma)d\sigma+\delta z^2$$
$$+2yz+2\delta yf(x,0)-\alpha\beta y^2$$

and

$$2V_2=\alpha\beta bx^2+2a\int_0^x f(\vartheta,0)d\vartheta+a^2y^2+2\int_0^y f(x,\sigma)d\sigma+z^2$$
$$+2a\alpha\beta xy+2ayz+2yf(x,0)+2\alpha\beta xz$$
$$+2\lambda_1\int_{-\tau(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds$$
$$+2\lambda_2\int_{-\tau(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds,$$

with

$$\frac{1}{a}<\delta<\frac{b}{c}$$

and

$$\alpha<min\{\frac{ab-c}{\beta[a+v^{-1}(\frac{f(x,y)}{y}-b)^2]},\frac{b}{\beta},\frac{a\delta-1}{\delta\beta},\frac{b(1+\delta)}{\beta}\} \tag{3.2}$$

where λ_1 and λ_2 are positive constants which will be determined later.

Lemma 3.1 *Clearly $V(0,0,0)=0$ and there exists finite constants $D_2>0$ and $D_3>0$ such that*

$$D_2(x^2+y^2+z^2)\le V(x_t,y_t,z_t)\le D_3(x^2+y^2+z^2). \tag{3.3}$$

Proof: (3.1) can be re-arranged as follows

$$2V_1=[2\int_0^x f(\vartheta,0)d\vartheta-\frac{\delta}{b}f^2(x,0)]+\delta b[y+\frac{f(x,0)}{b}]^2$$
$$+ay^2-\delta^{-1}y^2-\alpha\beta y^2+\delta(z+\delta^{-1}y)^2+\delta[2\int_0^y f(x,\sigma)d\sigma-by^2].$$

The term

$$2\int_0^x f(\vartheta,0)d\vartheta-\frac{\delta}{b}f^2(x,0),$$

by the hypothesis of the Theorem 1 and fact that $f^2(0,0)=0$, we have that the term

$$2\int_0^x f(\vartheta,0)d\vartheta-\frac{\delta}{b}f^2(x,0)=2[\int_0^x f(\vartheta,0)d\vartheta-\frac{\delta}{b}\int_0^x f(\vartheta,0)\frac{df(\vartheta,0)}{d\vartheta}d\vartheta-\frac{\delta}{b}f^2(0,0)]$$
$$=2[\int_0^x f(\vartheta,0)d\vartheta-\frac{\delta}{b}\int_0^x f(\vartheta,0)f'_\vartheta(\vartheta,0)d\vartheta]$$
$$=2\int_0^x(1-\frac{\delta}{b}f'_\vartheta(\vartheta,0))f(\vartheta,0)d\vartheta\ge(1-\frac{\delta}{b}c)vx^2.$$

Also the term,

$$(ay^2-\delta^{-1}y^2-\alpha\beta y^2)=(a-\delta^{-1}-\alpha\beta)y^2\ge 0$$

since δ and α satisfy (3.2)
and the term,

$$\delta[2\int_0^y f(x,\sigma)d\sigma-by^2]$$

by the hypothesis of the Theorem 1, we have that

$$\delta[2\int_0^y f(x,\sigma)d\sigma-by^2]=\delta[\frac{f(x,y)}{y}-b]y^2\ge 0.$$

Combining all the estimates for $2V_1$, we have that

$$2V_1\ge(1-\frac{\delta}{b}c)vx^2+(a-\delta^{-1}-\alpha\beta)y^2+\delta(z+\delta^{-1}y)^2+\delta b[y+\frac{f(x,0)}{b}]^2.$$

Similarly, V_2 can be arranged as follows:

$$2V_2=\alpha\beta(b-\alpha\beta)x^2+a[2\int_0^x f(\vartheta,0)d\vartheta-\beta^{-1}f^2(x,0)]$$
$$+\beta[a^{-\frac{1}{2}}y+\beta^{-1}a^{\frac{1}{2}}f(x,0)]^2+[2\int_0^y f(x,\sigma)d\sigma-\beta a^{-1}y^2]$$
$$+(\alpha\beta x+ay+z)^2$$

$+2\lambda_1 \int_{-\tau(t)}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds$
$+2\lambda_2 \int_{-\tau(t)}^{0} \int_{t+s}^{t} z^2(\theta) d\theta ds,$
the term
$2a[\int_0^x f(\vartheta, 0) d\vartheta - \beta^{-1} f^2(x, 0)],$
by the hypothesis of the Theorem 1, the above term becomes
$= 2a[\int_0^x f(\vartheta, 0) d\vartheta - \beta^{-1} \int_0^x f(\vartheta, 0) \frac{df(\vartheta, 0)}{d\vartheta} d\vartheta - \frac{\delta}{b} f^2(0,0)]$
$= 2a[\int_0^x f(\vartheta, 0) d\vartheta - \beta^{-1} \int_0^x f(\vartheta, 0) f'_\vartheta(\vartheta, 0) d\vartheta]$
$= 2a \int_0^x (1 - \beta^{-1} f'_\vartheta(\vartheta, 0)) f(\vartheta, 0) d\vartheta \geq a(1 - \frac{c}{\beta}) \nu x^2.$
Also the term
$2\int_0^y f(x, \sigma) d\sigma - \beta a^{-1} y^2,$
using the hypothesis of the Theorem 1, the above term becomes
$= [y^2 \frac{f(x,y)}{y} - \beta a^{-1} y^2] \geq (b - \beta a^{-1}) y^2.$
Thus,
$2V_2 \geq [\alpha\beta(b - \alpha\beta) + a(1 - \frac{c}{\beta})\nu] x^2 + (b - \beta a^{-1}) y^2$
$+(\alpha\beta x + ay + z)^2$
$+2\lambda_1 \int_{-\tau(t)}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds$
$+2\lambda_2 \int_{-\tau(t)}^{0} \int_{t+s}^{t} z^2(\theta) d\theta ds.$
Combining the estimates V_1 and V_2 for V in (3.1) , we have
$V \geq [(1 - \delta b^{-1} c)\nu + \alpha\beta(b - \alpha\beta) + a(1 - \frac{c}{\beta})\nu] x^2$
$+[(a - \delta^{-1} - \alpha\beta) + (b - \beta a^{-1})] y^2 + \delta(z + \delta^{-1} y)^2$
$+(\alpha\beta x + ay + z)^2$
$+2\lambda_1 \int_{-\tau(t)}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds$
$+2\lambda_2 \int_{-\tau(t)}^{0} \int_{t+s}^{t} z^2(\theta) d\theta ds,$
if we choose $\beta = ab$, the constants $(1 - \delta b^{-1} c)\nu, \alpha\beta(b - \alpha\beta), a(1 - \frac{c}{\beta}), (a - \delta^{-1} - \alpha\beta)$ and $(b - \beta a^{-1})$ are positive by the inequalities in (3.2) and the integrals $2\lambda_1 \int_{-\tau(t)}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds$ and $2\lambda_2 \int_{-\tau(t)}^{0} \int_{t+s}^{t} z^2(\theta) d\theta ds$ are non-negative.
So that
$V(x_t, y_t, z_t) \geq \xi_1(x^2 + y^2) + \delta(z + \delta^{-1} y)^2 + (\alpha\beta x + ay + z)^2 + \lambda_1 r^2(t) y^2 + \lambda_2 r^2(t) z^2,$
where $\xi_1 = \min\{(1 - \delta^{-1} bc)\nu + \alpha\beta(b - \alpha\beta) + a(1 - \frac{c}{\beta})\nu, (a - \delta^{-1} - \alpha\beta) + (b - \beta a^{-1})\}.$
Thus, it is evident from the terms contained in the above inequality that there exists a constant $D_2 > 0$ small enough such that
$V(x_t, y_t, z_t) \geq D_2(x^2 + y^2 + z^2).$
To prove the right side of inequality (3.3), the hypotheses (i) - (iii) of Theorem 1 and using the fact that
$2|x||y| \leq x^2 + y^2$
yields from V, term by term
$|2xy| \leq 2|x||y| \leq x^2 + y^2$
$|2yz| \leq 2|y||z| \leq y^2 + z^2$
$|2xz| \leq 2|x||z| \leq x^2 + z^2$
$2\int_0^x f(\vartheta, 0) d\vartheta \leq \nu x^2$
$2\int_0^y f(x, \sigma) d\sigma \leq by^2$
$2yf(x, 0) \leq \nu|x||y| \leq \nu(x^2 + y^2)$
and
$2\lambda_1 \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds = \lambda_1 r^2(t) y^2$
$\leq \lambda_1 \gamma^2 y^2.$

$2\lambda_2\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds = \lambda_2 r^2(t)z^2$
$\leq \lambda_2\gamma^2 z^2.$
$V(x_t,y_t,z_t) = 2V_1 + 2V_2 \leq ((2+a+\delta)\nu + (a+b+1)\alpha\beta)x^2$
$+(1+a(a+2)+b(1+\delta)+\nu(1+\delta)+\alpha\beta(a-1)+\lambda_1\gamma^2)y^2$
$+(2+a+\delta+\alpha\beta+\lambda_2\gamma^2)z^2.$
$\leq \xi_2(x^2+y^2+z^2),$
where $\xi_2 = \max\{(2+a+\delta)\nu + (a+b+1)\alpha\beta, 1+a(a+2)+b(1+\delta)+\nu(1+\delta)+\alpha\beta(a-1)+\lambda_1\gamma^2, 2+a+\delta+\alpha\beta+\lambda_2\gamma^2\}$.
If we choose a positive constant D_3, then we have
$V(x_t,y_t,z_t) \leq D_3(x^2+y^2+z^2).$
Thus, (3.3) of lemma 3.1 is established where D_2, D_3 are finite constants.
Now, differentiating (3.1) along the system (??) after simplification we get

$$\frac{d}{dt}V(x,y,z) = -U_1 + U_2 + U_3 + U_4, \tag{3.4}$$

where

$$U_1 = \frac{1}{2}\alpha\beta x f(x,y) - (1+\delta)y\int_0^y f'_x(x,\sigma)d\sigma - (1+\delta)y^2 f'_x(x,0) + (a\delta-1)z^2$$
$$-(1+\delta)z^2 f'_z(x,0) + y^2\frac{f(x,y)}{y} + y^2 f'_y(x,0) - a\alpha\beta y^2$$
$$+ay^2\frac{f(x,y)}{y} + \alpha\beta[\frac{f(x,y)}{y} - b]xy.$$

Using the hypothesis of Theorem 1, we have that

$$U_1 \geq \frac{1}{2}\alpha\beta\nu x^2 + [ab + b - a\alpha\beta - (1+\delta)]y^2 + (a\delta-1)z^2$$
$$+\alpha\beta[\frac{f(x,y)}{y} - b]xy.$$

It follows that

$$U_1 \geq \frac{1}{4}\alpha\beta\nu x^2 + [ab + b - a\alpha\beta - (1+\delta)]y^2 + (a\delta-1)z^2$$
$$+[ab - c - \alpha\beta(a + \frac{1}{\nu}(\frac{f(x,y)}{y} - b)^2)]y^2$$
$$+\frac{1}{4}\alpha\beta\nu[x + \frac{2}{\nu}(\frac{f(x,y)}{y} - b)y]^2.$$

If we choose

$$\alpha < min\{\frac{ab-c}{\beta[a+\nu^{-1}(\frac{f(x,y)}{y}-b)^2]}, \frac{b}{\beta}, \frac{a\delta-1}{\delta\beta}, \frac{b(1+\delta)}{\beta}\},$$
$$U_1 \geq \frac{1}{4}\alpha\beta\nu x^2 + (b - \delta c)y^2 + (a\delta-1)z^2.$$

Now,

$$U_2 = (\alpha\beta x + (a+1)y + (1+\delta)z)\int_{t-\tau(t)}^{t} f'_x(x(s),y(s))y(s)ds$$
$$+(\alpha\beta x + (a+1)y + (1+\delta)z)\int_{t-\tau(t)}^{t} f'_y(x(s),y(s))z(s)ds$$
$$+\lambda_1\tau(t)y^2 + \lambda_2\tau(t)z^2 - \lambda_1(1-\tau'(t))\int_{t-\tau(t)}^{t} y^2(s)ds$$
$$-\lambda_2(1-\tau'(t))\int_{t-\tau(t)}^{t} z^2(s)ds.$$

From (v) of Theorem 1, $|f'_x(x,y)| \leq L$, $|f'_y(x,y)| \leq M$ and using $2uv \leq u^2 + v^2$, we have that

$$(\alpha\beta x + (a+1)y + (1+\delta)z)\int_{t-\tau(t)}^{t} f'_x(x(s),y(s))y(s)ds$$
$$\leq \alpha\beta\frac{L}{2}\tau(t)x^2 + \alpha\beta\frac{L}{2}\int_{t-\tau(t)}^{t} y^2(s)ds$$
$$+\frac{(a+1)}{2}L\tau(t)y^2 + \frac{(a+1)}{2}L\int_{t-\tau(t)}^{t} y^2(s)ds$$
$$+\frac{(1+\delta)}{2}L\tau(t)z^2 + \frac{(1+\delta)}{2}L\int_{t-\tau(t)}^{t} y^2(s)ds$$

and

$$(\alpha\beta x + (a+1)y + (1+\delta)z)\int_{t-\tau(t)}^{t} f'_y(x(s),y(s))z(s)ds$$
$$\leq \frac{\alpha\beta}{2}M\tau(t)x^2 + \frac{\alpha\beta}{2}M\int_{t-\tau(t)}^{t} z^2(s)ds$$

$$+\frac{(a+1)}{2}M\tau(t)y^2+\frac{(a+1)}{2}M\int_{t-\tau(t)}^{t}z^2(s)ds$$
$$+\frac{(1+\delta)}{2}M\tau(t)z^2+\frac{(1+\delta)}{2}M\int_{t-\tau(t)}^{t}z^2(s)ds.$$

Thus,

$$U_2\geq\frac{1}{2}\{\alpha\beta(L+M)\tau(t)x^2+((a+1)L+(a+1)M+\lambda_1)\tau(t)y^2$$
$$+((1+\delta)L+(1+\delta)M+\lambda_1)\tau(t)z^2\}$$
$$+\frac{1}{2}\{\alpha\beta L+(1+\delta)L-2\lambda_1(1-\tau'(t))\}\times\int_{t-\tau(t)}^{t}y^2(s)ds$$
$$+\frac{1}{2}\{\alpha\beta M+(1+\delta)M-2\lambda_2(1-\tau'(t))\}\times\int_{t-\tau(t)}^{t}z^2(s)ds.$$

We use $0\leq\tau(t)\leq\gamma,\tau'(t)\leq\beta$ from the assumption and choose

$$\lambda_1=\frac{[\alpha\beta+(a+1)+(1+\delta)]L}{2(1-\beta)}>0$$

and

$$\delta=\frac{[\alpha\beta+(a+1)+(1+\delta)]M}{2(1-\beta)}>0$$

we have,

$$U_2\geq\frac{1}{2}\alpha\beta(L+M)\gamma x^2$$
$$+\frac{1}{2}\gamma\{(a+1)L+(a+1)M+\frac{[\alpha\beta+(a+1)+(1+\delta)]L}{2(1-\beta)}\}y^2$$
$$+\frac{1}{2}\gamma\{(1+\delta)L+(1+\delta)M+\frac{[\alpha\beta+(a+1)+(1+\delta)]M}{2(1-\beta)}\}z^2.$$

Combining the estimates for U_1 and U_2 only in $\frac{d}{dt}V(x,y,z)$ in (3.4), we obtain

$$\frac{d}{dt}V(x,y,z)\leq-\frac{1}{4}\alpha\beta\{\nu-2\gamma(L+M)\}x^2$$
$$-\frac{1}{2}\{(b-\delta c)-2\gamma[(a+1)L+(a+1)M+\frac{(\alpha\beta+(a+1)+(1+\delta)L)}{2(1-\beta)}]\}y^2$$
$$-\frac{1}{2}\{(a\delta-1)-2\gamma[(1+\delta)L+(1+\delta)M+\frac{(\alpha\beta+(a+1)+(1+\delta)M)}{2(1-\beta)}]\}z^2.$$

Choosing

$$\gamma<min\{\frac{\alpha\beta\nu}{2(L+M)};\frac{(b-\delta c)(1-\beta)}{2(a+1)(L+M)(1-\beta)+[\alpha\beta+(a+1)+(1+\delta)]L};$$
$$\frac{(a\delta-1)(1-\beta)}{2(1+\delta)(L+M)(1-\beta)+[\alpha\beta+(a+1)+(1+\delta)]m}\},$$

we have that

$$\frac{d}{dt}V(x_t,y_t,z_t)\leq-\delta_1(x^2+y^2+z^2),$$

for some $\delta_1>0$.

Thus, in view of (3.3) of Lemma 3.1 and the immediate above inequality, the conditions of Lemma 2 are immediate provided γ is satisfied and δ,α satisfy inequalities (3.2) respectively.

Also,

$$U_3=[(1+\delta)xf'_x(x,0)+(1+\delta)y\frac{f(x,y)}{y}-ay^2+(1+2a)z]q(t)$$

By the hypothesis of Theorem 1, we have

$$U_3\geq[(1+\delta)c|x|+(1+\delta b-a^2)|y|+(1+2a)|z|]q(t)$$

and finally

$$U_4\geq[\alpha\beta|x|+(1+a)|y|+(1+\delta)|z|]r(t,x,y,z+q).$$

Now, combining all the estimates U_1, U_2, U_3 and U_4 to obtain $\frac{d}{dt}V(x,y,z)$ in (3.4), yields

$$\frac{d}{dt}V(x_t,y_t,z_t)\leq-\delta_1(x^2+y^2+z^2)+[(1+\delta)c|x|+((1+\delta)b-a^2)|y|+(1+2a)|z|]q(t)$$
$$+[\alpha\beta|x|+(1+a)|y|+(1+\delta)|z|]r(t,x,y,z+q).$$

It follows that

$$\frac{d}{dt}V(x_t,y_t,z_t)\leq-\delta_1(x^2+y^2+z^2)+\delta_2(x^2+y^2+z^2)^{\frac{1}{2}}$$
$$+\delta_3(x^2+y^2+z^2)^{\frac{1}{2}}|r(t,x,y,z+q)|.$$

where $\delta_2=\max\sqrt{3}d_1\{(1+\delta)c,(1+\delta)b-a^2,(1+2a)\}$ and $\delta_3=\max\sqrt{3}\{\alpha\beta,(1+a),(1+\delta)\}$.

It follows that

$$\frac{d}{dt}V(x_t,y_t,z_t)\leq-\delta_1(x^2+y^2+z^2)+\delta_4(x^2+y^2+z^2)^{\frac{1}{2}}[r(t,x,y+q)+1],$$

where $\delta_4 = max\{\delta_2, \delta_3\}$.
So that since
$|r(t, x, y, z + q)| \le \delta_4 \rho(t)[(x^2 + y^2 + z^2)^{\frac{1}{2}} + 1]$.
Following the argument used in [8] it can further verified that

$$\frac{d}{dt} V(x_t, y_t, z_t) \le -\delta_5 (x^2 + y^2 + z^2) + \delta_6 (x^2 + y^2 + z^2)^{\frac{1}{2}} |\varphi|, \qquad (3.5)$$

where δ_5, δ_6 are finite constants and $\varphi = r(t, x_1, y_1 + z_1 + q) - r(t, x_2, y_2 + z_2 + q)$.

4. Proof of Theorem 1

Consider the function
$\Psi(t) = V((x(t-\zeta) - x(t)), (y(t-\zeta) - y(t)), (z(t-\zeta) - z(t)))$
where V is the function defined in (3.1) with x, y, z replaced by $(x(t+\zeta) - x(t))$, $(y(t+\zeta) - y(t))$ and $(z(t+\zeta) - z(t))$ respectively. Then, by Lemma 3.1 we have positive constants D_4 and D_5 such that

$$D_4 S(t) \le \Psi(t) \le D_5 S(t), \qquad (4.1)$$

where
$S(t) = \{|x(t+\zeta) - x(t)|^2 + |y(t+\zeta) - y(t)|^2 + |z(t+\zeta) - z(t)|^2\}$.
Differentiating Ψ along the system (1.3), we get as in (3.5),

$$\dot{\Psi}(t) \le -\delta_7 \{|x(t+\zeta) - x(t)|^2 + |y(t+\zeta) - y(t)|^2 + |z(t+\zeta) - z(t)|^2\}$$
$$+\delta_8 \{|x(t+\zeta) - x(t)|^2 + |y(t+\zeta) - y(t)|^2| + |z(t+\zeta) - z(t)|^2\}^{\frac{1}{2}} |\varphi|, \qquad (4.2)$$

where $\varphi = r((t+\zeta), x(t), y(t) + z(t) + q(t+\zeta) - r(t, x, y, z + q)$ with δ_7 and δ_8 being finite constants.
Inequality (4.2) can be arranged as

$$\dot{\Psi}(t) \le -\delta_7 \{|x(t+\zeta) - x(t)|^2 + |y(t+\zeta) - y(t)|^2 + |z(t+\zeta) - z(t)|^2\}$$
$$+\delta_9 \{|x(t+\zeta) - x(t)|^2 + |y(t+\zeta) - y(t)|^2 + |z(t+\zeta) - z(t)|^2\}^{\frac{1}{2}} |\varphi|$$
$$+\delta_{10} \{|x(t+\zeta) - x(t)|^2 + |y(t+\zeta) - y(t)|^2 + |z(t+\zeta) - z(t)|^2\}^{\frac{1}{2}}$$
$$\times |r((t+\zeta), x(t), y(t) + z(t) + q(t+\zeta) - r(t, x, y, z + q)|. \qquad (4.3)$$

Since the perturbation r is uniformly almost periodic in t. Then, given arbitrary $\varepsilon > 0$, we can find $\zeta > 0$ such that $|q(t+\zeta) - q(t)| \le \ell \varepsilon^2$,

$$|r((t+\zeta), x(t), y(t) + z(t) + q(t+\zeta) - r(t, x, y, z + q(t))| \le \ell \varepsilon^2, \qquad (4.4)$$

where ℓ is a constant whose value will be determined later. Thus, (4.3) becomes

$$\dot{\Psi}(t) \le -\delta_7 S(t) + \delta_9 S^{\frac{1}{2}}(t) |\varphi| + \delta_{10} S^{\frac{1}{2}}(t) \ell \varepsilon^2. \qquad (4.5)$$

In view of (2.1) of Theorem 1, we have that
$\{|x(t+\zeta) - x(t)|^2 + |y(t+\zeta) - y(t)|^2 + |z(t+\zeta) - z(t)|^2\}^{\frac{1}{2}} \le D_1$
Inequality (4.5) becomes,

$$\dot{\Psi}(t) + \delta_7 S(t) \le \delta_9 S^{\frac{1}{2}} |\varphi| + \delta_{10} D_1 \ell \varepsilon^2. \qquad (4.6)$$

Let m be any constant such that $1 \le m \le 2$ and set $k = 1 - \frac{1}{2} m$, so that $0 \le k \le \frac{1}{2}$.
Then, (4.6) becomes

$$\dot{\Psi}(t) + \delta_7 S(t) \le \delta_9 S^m \Psi^* + \delta_{10} D_1 \ell \varepsilon^2 \qquad (4.7)$$

and $\Psi^* = S^{(\frac{1}{2} - k)} (|\varphi| - \delta_7 \delta_9^{-1} S^{\frac{1}{2}}(t))$.
We consider two cases
1) $|\varphi| \le \delta_7 \delta_9^{-1} S^{\frac{1}{2}}$ and
2) $|\varphi| > \delta_7 \delta_9^{-1} S^{\frac{1}{2}}$
separately, we find that in either case, there exists some constants $\delta_{11} > 0$ such that $\Psi^* \le \delta_{11} |\varphi|^{2(1-k)}$. Thus, the inequality (4.7) becomes

$$\frac{d\Psi}{dt} + \delta_7 S \le \delta_{12} S^k \rho^{2(1-k)} S^{(1-k)} \Psi(t) + \delta_{10} D_1 \ell \varepsilon^2$$

where $\delta_{12} \ge 2\delta_9 \delta_{11}$. Using (4.1) on Ψ, we get

$$\frac{d\Psi}{dt} + ((\delta_{13} - \delta_{14}) \rho^m(t)) \Psi(t) \le \delta_{10} D_1 \ell \varepsilon^2 \qquad (4.8)$$

where δ_{13}, δ_{14} as positive constants.
On integrating (4.8) from t_o to $t (t \ge t_o)$, we obtain

$$\Psi(t) \le \delta_{15} \Psi(t_o) \exp\{-\delta_{13}(t - t_o)\} + \delta_{14} \int_{t_o}^{t} \rho^m(s) d(s)\}$$
$$+\delta_{16} \ell \varepsilon^2, \qquad (4.9)$$

where $\delta_{15} = \frac{\delta_9}{\delta_7}$ and $\delta_{16} = \frac{\delta_{15} \delta_{10} D_1}{\delta_{13}}$.
If

$$\int_{t_o}^{t} \rho^m(s) d(s) < \delta_{13} \delta_{14}^{-1} (t - t_o),$$

then, the exponential index remains negative for all $(t-t_o)\geq 0$. As $t=(t-t_o)\rightarrow\infty$ and that $\Psi(t_o)$ is finite in (4.9), we have that
$\Psi(t)\leq \delta_{16}\ell\varepsilon^2 \quad for \quad any \quad t.$
Since $\Psi(t)$ satisfies (4.1),
$\Psi(t)\leq D_4^{-1}\delta_{16}\ell\varepsilon^2.$
Also by definition of $\Psi(t)$ in (4.1), we have that

$$|x(t+\zeta)-x(t)|+|y(t+\zeta)-y(t)|+|z(t+\zeta)-z(t)|\leq (\frac{3\ell\delta_{16}}{D_4})^{\frac{1}{2}}\varepsilon. \qquad (4.10)$$

choose $\ell=\frac{D_4}{3\delta_{16}}$, inequality (4.10) implies

$$|x(t+\zeta)-x(t)|+|y(t+\zeta)-y(t)|+|z(t+\zeta)-z(t)|\leq \varepsilon, \qquad (4.11)$$

where ζ is chosen to satisfy (4.4) is relatively dense and hence (4.11) implies that the solutions $(x(t),y(t),z(t))$ or equivalently $x(t),x'(t),x''(t)$ of (1.1) are uniformly almost periodic in t.
To show that the solutions are also periodic, we assume that
$q(t+\omega)=q(t)$
$r(t+\omega,x(t),y(t),z(t)+q(t))=r(t,x(t),y(t),z(t)+q(t)),$
for $(x^2+y^2+z^2)^{\frac{1}{2}}\leq D_1$, for some constants $D_1>0$.
Since the perturbation $r(t,x,y,z)$ has period ω in t, we replace ζ in the definition of $\Psi(t)$ with ω. The terms in the left hand side of (4.4) is identically zero, thus we may have inequality (4.11) as
$|x(t+\omega)-x(t)|+|y(t+\omega)-y(t)|+|z(t+\omega)-z(t)|\leq 0.$
Thus,
$|x(t+\omega)-x(t)|+|y(t+\omega)-y(t)|+|z(t+\omega)-z(t)|=0.$
which implies that
$x(t+\omega)=x(t), \quad y(t+\omega)=y(t) \quad and \quad z(t+\omega)=z(t).$
That is, $x(t),y(t),z(t)$ or equivalently $x(t),x'(t),x''(t)$ of (1.1) are periodic in t with period ω.
Example 1 *We consider third-order nonlinear delay differential equation*

$$x'''+2x''+[3x(t-\tau(t))+12x'^2(t-\tau(t))]=\frac{1}{1+t^2+x^2+x'^2+x''^2} \qquad (4.12)$$

with equivalent system of (4.12) as:

$$x'=y$$
$$y'=z-(2+\frac{4}{|sint|+1})$$
$$z'=-2z-(3x+12y^2)-2(2+\frac{4}{|sint|+1})+\frac{1}{1+t^2+x^2+y^2+z^2}$$
$$+\int_{t-\tau(t)}^{t}3ds+\int_{t-\tau(t)}^{t}24y(s)ds, \qquad (4.13)$$

where $q(t)=(2+\frac{4}{|sint|+1})$ is the perturbation.
Comparing (1.3) with (4.13), it is easy to see that
$(2+\frac{4}{|sint|+1})=q(t)\leq d_1=4$
The function $f(x,y)=(3x+12y^2)$, it is clear from the equation that
$\frac{f(x,y)}{y}\geq 12=b>0, \quad y\neq 0$
Also,
$\frac{f(x,y)}{x}\geq 3=\nu>0, \quad x\neq 0$
$f'_x(x,0)\leq 3=c, \quad c>0$
$|f'_x(x,y)|\leq 3=L$
$|f'_y(x,y)|\leq 24=M$
$\frac{1}{2}>\delta>\frac{12}{3}, \quad we \ choose, \delta=1$
Since
$0<\beta<1, \quad we \ choose, \beta=\frac{1}{2},$
we have
$\alpha<min\{0.84,24,2,48\}$
thus, we take $\alpha=0.64$.
It follows that
$\gamma<min\{0.018,0.046,0.003\}$
hence, we can choose $\tau(t)=0.001$.
Thus, all the conditions of Theorem 1 hold. That is, the solutions of (4.12) having the properties that are almost periodic and periodic in t.
The plot of $x(t)$, $y(t)$ and $z(t)$ of equation (4.13) which are almost periodic solutions characterizing the system (4.13) is shown in Fig. 1, Fig.2 and Fig.3 respectively below while in Fig. 4, Fig. 5 and Fig. 6 shows the periodic behaviour of solutions of $x(t)$, $y(t)$ and $z(t)$ of equation (4.13) with period $\omega=0.80401$ in t.

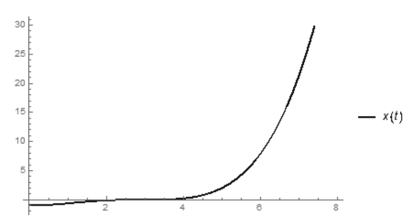

Fig. 1 The almost periodic graph of $x(t)$ after a force function with $\tau(t) = 0.001$

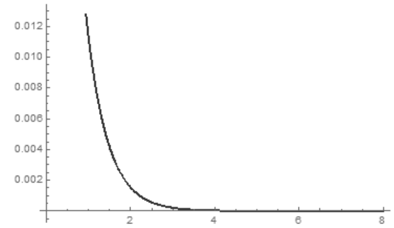

Fig. 2 The almost periodic graph of $y(t)$ after a force function with $\tau(t) = 0.001$

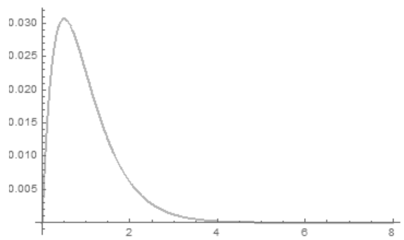

Fig. 3 The almost periodic graph of $z(t)$ after a force function with $\tau(t) = 0.001$

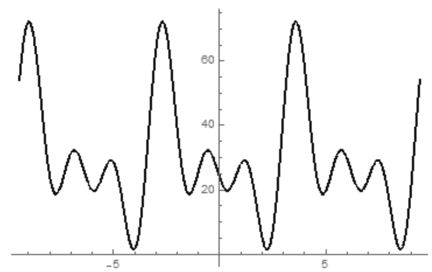

Fig. 4 The periodic graph of $x(t)$ after a force function with $\omega = 0.80401$ and $\tau(t) = 0.001$

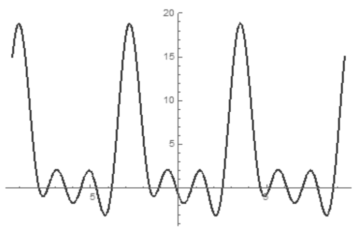

Fig. 5 The periodic graph of $y(t)$ after a force function with $\omega = 0.80401$ and $\tau(t) = 0.001$

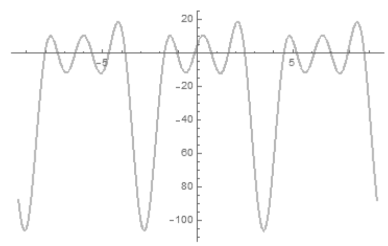

Fig. 6 The periodic graph of $z(t)$ after a force function with $\omega = 0.80401$ and $\tau(t) = 0.001$

Conclusion

Analysis of nonlinear delay diffrential equations show that Lyapunov's theory in periodic properties of solutions is rarely scarce. The Lyapunov's method allow us to predict and describe the periodic behaviour of solutions of nonlinear delay differential equations when the forcing term p is periodic or almost periodic in t. The solutions of the third order nonlinear delay differential equation (1.1) are periodic or almost periodic uniformly in in x, x' and x'' according to Lyapunov's theory if the conditions of Theorm 1 hold.

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