Journal of the Nigerian Association of Mathematical Physics
***Volume 57* (June – July *2020 Issue*), pp19 – 20**
© J. of NAMP

FURTHER GENERALIZATIONS OF OPIAL–TYPE INEQUALITIES ON TIME SCALES

Aribike E. E. and Anthonio Y. O.

Department of Mathematics and Statistics, Lagos State, Polytechnic, Ikorodu, Nigeria

Abstract

This paper establishes some generalizations of Opial inequalities on time scales. Also, some Opial–type inequalities with weight functions were established.

***Keywords*:** Opial–type inequalities, Time scales, Hölder's inequalities, rd–continuous functions

1. Introduction

Inequality involving integrals of a function and its derivative was established by Opial in [1]. It has proved to be one of the most useful inequalities in analysis. The result is as follows:

Theorem 1.1 *If $f(x)$ is absolutely continuous on $[0,h]$ such that $f(0)=f(h)=0$ and $f(x)>0$ on $(0,h)$, then*

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{4}\int_0^h (f'(x))^2 dx. \tag{1.1}$$

$\frac{h}{4}$ is the best possible constant.

Olech [2] provided a modified version of the result in the following result:

Theorem 1.2 *If $f(x)$ is absolutely continuous on $[0,h]$ with $f(0)=0$, then*

$$\int_0^h f(x)f'(x)dx \leq \frac{h}{2}\int_0^h (f'(x))^2 dx. \tag{1.2}$$

Time scale calculus was initiated in [3] in order to create a theory that can unify discrete and continuous analysis. A *time scale* is an arbitrary non-empty closed subset of the real numbers. The three most popular examples of time scale calculus are differential calculus, difference calculus and quantum calculus, that is $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$, $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$, where $q>1$, referenced in [4]. Delta derivative f^{Δ} for a function f defined on $\mathbb{T}$ as:

(i) $f^{\Delta}=f'$ is the usual derivative if $\mathbb{T}=\mathbb{R}$; and

(ii) $f^{\Delta}=\Delta f$ is the usual forward difference operator if $\mathbb{T}=\mathbb{Z}$.

The summary of time scale calculus and its applications could be sourced from [5–9] and the references therein.

Table 1: NOTATIONS

SYMBOLS	NAMES
$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers
$\mathbb{T}$	Time scales
inf	Infimum
sup	Supremum
C_{rd}	rd-Continuous
Σ	Forward jump operato

The aim of this work is to generalize some inequalities of Opial-type by using Hölder's inequality for convex functions.

2. Opial–type inequalities

Theorem 2.1 *Let $\mathbb{T}$ be a time scale with $\alpha,\beta\in\mathbb{T}$ and $r\in C_{rd}([\alpha,\beta]_{\mathbb{T}},\mathbb{R}^+)$ be such that $r(t)$ is nonincreasing on $[\alpha,\beta]_{\mathbb{T}}$ and $\kappa>0$ and $\eta>1$. If $y:[\alpha,\beta]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable with $\gamma(\alpha)=0$,. Then*

$$\int_\alpha^\beta r(t)|\gamma(t)|^{\kappa}|\gamma^{\Delta}(t)|^{\eta}\Delta t \leq \frac{\eta(\beta-\alpha)^{\kappa}}{\kappa+\eta}\int_\alpha^\beta r(t)|\gamma^{\Delta}(t)|^{\kappa+\eta}\Delta t. \tag{2.1}$$

Proof:

Suppose that the function $f(t)$ is defined by

$$f(t)=\int_\alpha^t r^{\frac{\eta}{\kappa+\eta}}(s)|\gamma^{\Delta}(s)|^{\eta}\Delta s, \tag{2.2}$$

therefore

$$f(\alpha)=0 \quad and f^{\Delta}(t)=r^{\frac{\eta}{\kappa+\eta}}(s)|\gamma^{\Delta}(s)|^{\eta}>0. \tag{2.3}$$

When $\eta>1$, using indices η and $\eta/\eta-1$ and by Hölder's inequality,

$$\begin{aligned}|\gamma(t)| &\leq \int_\alpha^t |\gamma^{\Delta}(s)|\Delta s = \int_\alpha^t r^{\frac{-1}{\kappa+\eta}}(s) r^{\frac{1}{\kappa+\eta}}(s)|\gamma^{\Delta}(s)|\Delta s \\ &\leq \left(\int_\alpha^t \left(r^{\frac{-1}{\kappa+\eta}}(s)\right)^{\frac{\eta}{\eta-1}}\Delta s\right)^{\frac{\eta-1}{\eta}} \left(\int_\alpha^t r^{\frac{\eta}{\kappa+\eta}}(s)|\gamma^{\Delta}(s)|^{\eta}\right)^{\frac{1}{\eta}}\Delta s \\ &\leq r^{\frac{-1}{\kappa+\eta}}(t)(t-\alpha)^{\frac{\eta-1}{\eta}} f^{\frac{1}{\eta}}(t),\end{aligned} \tag{2.4}$$

yields

$$r^{\frac{-1}{\kappa+\eta}}(t)|\gamma(t)|^{\kappa} \leq (t-\alpha)^{\frac{\kappa(\eta-1)}{\eta}} f^{\frac{\kappa}{\eta}}(t). \tag{2.5}$$

When $\eta=1$, we have

$$\begin{aligned}|\gamma(t)| &\leq \int_\alpha^t |\gamma^{\Delta}(s)|\Delta s = \int_\alpha^t r^{\frac{-1}{\kappa+1}}(s) r^{\frac{1}{\kappa+1}}(s)|\gamma^{\Delta}(s)|\Delta s \\ &\leq r^{\frac{-1}{\kappa+1}}(t)\int_a^t r^{\frac{1}{\kappa+1}}(s)|\gamma^{\Delta}(s)|\Delta s = r^{\frac{-1}{\kappa+1}}(t) f(t),\end{aligned} \tag{2.6}$$

Corresponding Author**:** Aribike E.E., Email: aribike.e@mylaspotech.edu.ng, Tel: +2348034012186

Also, (2.5) holds when $\eta = 1$.

Combining (2.3) and (2.5) yields

$$\begin{aligned}\int_\alpha^\beta r(s)|\gamma(s)|^\kappa|^\Delta(s)|^\eta\Delta s &= \int_\alpha^\beta r^{\frac{\kappa}{\kappa+\eta}}(s)|\gamma(s)|^\kappa r^{\frac{\eta}{\kappa+\eta}}(s)|\gamma^\Delta(s)|^\eta\Delta s\\ &\le \int_\alpha^\beta (s-\alpha)^{\frac{\kappa(\eta-1)}{\eta}} f^{\frac{\kappa}{\eta}}(s) f^\Delta(s)\Delta s \\ &\le (\beta-\alpha)^{\frac{\kappa(\eta-1)}{\eta}}\int_\alpha^\beta f^{\frac{\kappa}{\eta}}(s)f^\Delta(s)\Delta s.\end{aligned} \tag{2.7}$$

Bohner and Peterson [5] states that:

$$f^{\frac{\kappa}{\eta}}(s)f^\Delta(s) \le \frac{\eta}{\kappa+\eta}\left(f^{\frac{\kappa+\eta}{\eta}}(s)\right)^\Delta. \tag{2.8}$$

Since $f(a) = 0$

$$\begin{aligned}\int_\alpha^\beta r(s)|\gamma(s)|^\kappa|\gamma^\Delta(s)|^\eta\Delta s &\le \frac{\eta}{\kappa+\eta}(\beta-\alpha)^{\frac{\kappa(\eta-1)}{\eta}}\int_\alpha^\beta\left(f^{\frac{\kappa+\eta}{\eta}}(s)\right)^\Delta\Delta s\\ &= \frac{\eta}{\kappa+\eta}(\beta-\alpha)^{\frac{\kappa(\eta-1)}{\eta}}\left(f^{\frac{\kappa+\eta}{n}}(\beta)\right).\end{aligned} \tag{2.9}$$

Using Hölder's inequality with indices $(\kappa+\eta)/\kappa$ and $\eta/(\kappa+\eta)$,

$$\begin{aligned}f(\beta) = &\int_\alpha^\beta r^{\frac{\eta}{\kappa+\eta}}(s)|\gamma^\Delta(s)|^\eta\Delta s\\ &\le \left(\int_\alpha^\beta 1\Delta s\right)^{\frac{\kappa}{\kappa+\eta}}\left(\int_\alpha^\beta\left(r^{\frac{\eta}{\kappa+\eta}}(s)|\gamma^\Delta(s)|^\eta\right)^{\frac{\kappa+\eta}{\eta}}\Delta s\right)^{\frac{\eta}{\kappa+\eta}}\\ &= (\beta-\alpha)^{\frac{\kappa}{\kappa+\eta}}\left(\int_\alpha^\beta\left(r^{\frac{\eta}{\kappa+\eta}}(s)|\gamma^\Delta(s)|^\eta\right)^{\frac{\kappa+\eta}{\eta}}\Delta s\right)^{\frac{\eta}{\kappa+\eta}}.\end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10) implies,

$$\int_\alpha^\beta r(s)|\gamma(s)|^\kappa|\gamma^\Delta(s)|^\eta\Delta s \le \frac{\eta(\beta-\alpha)^\kappa}{\kappa+\eta}\int_\alpha^\beta r(s)|\gamma^\Delta(s)|^{\kappa+\eta}\Delta s.$$

Hence, proof is complete.

Remark 2.1 *When $\mathbb{T} = \mathbb{R}$, (2.1) reduces to Yang [10]*

$$\int_a^b r(t)|\gamma(t)|^\kappa|\gamma'(t)|^\eta\Delta t \le \frac{\eta(b-a)^\kappa}{\kappa+\eta}\int_a^b r(t)|\gamma'(t)|^{\kappa+\eta}\Delta t. \tag{2.11}$$

When $r(t) = 1$, (2.11) reduces to Yang [11]

$$\int_a^b |\gamma(t)|^\kappa|\gamma'(t)|^\eta\Delta t \le \frac{\eta(b-a)^\kappa}{\kappa+\eta}\int_a^b |\gamma'(t)|^{\kappa+\eta}\Delta t. \tag{2.12}$$

Remark 2.2 *Beesack and Das [12] showed that (2.11) and (2.12) are not sharp for $\eta > 1$ but sharp for $\eta = 1$.*

Remark 2.3 *Set $\eta = 1$ in (2.12), we have Hua [13]*

$$\int_a^b |\gamma(t)|^\kappa|\gamma'(t)|\Delta t \le \frac{\eta(b-a)^\kappa}{\kappa+1}\int_a^b |\gamma'(t)|^{\kappa+1}\Delta t. \tag{2.13}$$

Some generalizations of Opial–type inequalities with weight functions were established.

Theorem 2.2 *Let $\mathbb{T}$ be a time scale with $0, \rho \in \mathbb{T}$ and $\omega(t)$ be a positive and rd–continuous function on $[0,\rho]_\mathbb{T}$ such that $\int_0^\rho \omega^{1-\eta}(t)\Delta t < \infty, \eta > 1$. For delta differentiable*

$\chi: [0,\rho]_\mathbb{T} \to \mathbb{R}$ with $\chi(0) = 0$. Then

$$\int_0^\rho |\chi(t)+\chi^\sigma(t)||\chi^\Delta(t)|\Delta t \le \left(\int_0^\rho \omega^{1-\eta}(t)\Delta t\right)^{\frac{2}{\eta}}\left(\int_0^\rho \omega(t)|\chi^{\Delta(t)}|^\kappa\Delta t\right)^{\frac{2}{\kappa}}, \tag{2.14}$$

where $\kappa > 1$and $1/\kappa + 1/\eta = 1$ and with equality when $\chi(t) = c\int_0^t \omega^{1-\eta}(s)\Delta s$. Proof:

Consider $\gamma(t) = \int_0^t |\chi^\Delta(t)|\Delta t$. Then $\gamma^\Delta(t) = |\chi^\Delta(t)|$ and $|\chi| \le \gamma$.

Using Hölder's inequality, we have

$$\begin{aligned}&\int_0^\rho |\chi(t)+\chi^\sigma(t)||\chi^\Delta(t)|\Delta t \le \int_0^\rho (|\chi(t)|+|\chi^\sigma(t)|)\chi^\Delta(t)\Delta t\\ &\le \int_0^\rho (\gamma(t)+\gamma^\sigma(t))\gamma^\Delta(t)\Delta t = \int_0^\rho (\gamma^2(t))^\Delta = \gamma^2(\rho)\\ &= \left(\int_0^\rho |\chi^\Delta(t)|\Delta t\right)^2 = \left(\omega^{\frac{-1}{\kappa}}(t)\omega^{\frac{1}{\kappa}}(t)|\chi^\Delta(t)|\right)^2\\ &\le \left(\int_0^\rho\left(\omega^{\frac{-1}{\kappa}}(t)\right)^\eta\right)^{\frac{2}{\eta}}\left(\int_0^\rho \omega|\chi^\Delta(t)|^\kappa\right)^{\frac{2}{\kappa}}\end{aligned}$$

Hence, proof is complete.

3. Conclusion

The results of this paper were some generalizations of Opial-type inequalities. The concept of Hölder's inequality on convex functions on time scales was introduced, which is an essential tool used throughout the work.

References

[1] Opial Z. (1960). Surune Int*e′*galit*e′*. Annales Polonici Mathematici, **8**, 29–32.
[2] Olech C. (1960). A Simple Proof of A Certain Result of Opial. Annales Polonici Mathematici,**8**, 61–63.
[3] Hilger S. (1988). Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, PhD thesis, Universität Würzburg.
[4] Kac V. and Cheung P. (2002). Quantum Calculus, Springer, New York, USA.
[5] Bohner M. and Peterson A. (2001). Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston, Mass, USA.
[6] Bohner M. and Peterson A. (2003). Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston, Mass, USA.
[7] Srivastava, H. M., Tseng, S.–J. & Lo, J.–C. (2010). Some weighted Opial–type inequalities on time scales, Taiwanese Journal of Mathematics, 14(1), 107–122.
[8] Rauf K. and Anthonio Y. O. (2017). Time Scales on Opial–type Inequalities. Journal of Inequalities and Special Functions, **8**(2), 1–13.
[9] Rauf K., Anthonio Y. O., Mohammed M. T. and Owolabi A. A. (2015). On Integral Inequalities of Opial–Type. Nigerian Journal of Mathematics and Applications, **24**, 53–61.
[10] Yang G. S. A Note on Some Integrodifferential Inequalities, Soochow Journal of Mathematics, **9,** 231-236 (1983).
[11] Yang G. S. (1966). On Certain Result of Z. Opial, Proceeding of Japan Academy, **42**, 78–83.
[12] Beesack P. R. and Das K. M. (1968). Extensions of Opial's Inequality. Pacific Journal of Mathematics, **26**, 215–232.
[13] Hua L. K. On an Inequality of Opial. Scientia Sinica, **14**, 789–790 (1965).