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HANKEL AND TEOPLITZ DETERMINANT FOR SUBCLASSES OF BAZILEVIC FUNCTION DEFINED BY SALAGEAN AND RUSCHEWEYH OPERATORS

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Abstract

In the present work, the authors are focusing on the best possible upper bound to the second Hankel determinants $|a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)|$ ***and Teoplitz*** $T_2(2)$***,*** $T_2(3)$ ***and*** $T_3(2)$ ***for the functions belonging to class of Bazilevic functions defined by Salagean and Ruscheweyh operators normalized in the open unit disk.***

1. Introduction

Bieberbach [1] conjecture asserts that if f defined by

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1}$$

is analytic and univalent in the unit disk and normalized with $f(0)=0$ and $f'(0)=1$, then $|a_n|\le n$. This conjecture is renowned in geometric function theory and much investigation has been devoted to establish it's validity. This problem posed an open challenge for many researchers in the field of study such as starlike function and convex function defined by

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0 \tag{2}$$

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \tag{3}$$

respectively.

In the univalent theory, many researchers has been devoted time to find estimate on bounds of Hankel matrices because of it is usefulness and application in different branches [2]. Nooman and Thomas [3] introduced and studied the qth Hankel determinant is defined as

$$\begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & .. & a_{k+q} \\ .. & .. & .. & .. \\ a_{k+q-1} & a_{k+q} & .. & a_{k+2(q-1)} \end{vmatrix} \qquad (k,q\in N=1,2,3...)\,. \tag{4}$$

This determinant was studied by many authors among are: Babalola [4], Ehrenborg [5], Hamzat [6] , Hayami and Owa [7] and many others. The sharp bounds on $H_2(2)$ were obtained by several authors for detail see [8,9,10,11,12]. Its observed that the well-known Fekete and Szego functional is $|H_2(1)|=|a_3-a_2^2|$. Fekete and Szego then further generalized the estimate $|a_3-\mu a_2^2|$ where μ is real.

The closer relation from the Hankel determinants are the Toeplitz determinants. A Teoplitz determinant can be thought as an "upside-down" Hankel determinant [14], in that Hankel determinant have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. Application of Toeplitz determinants to wider areas of pure and Applied Mathematics can be found in [14]. The symmetric Toeplitz determinants $T_q(k)$

$$T_q(k)=\begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & .. & a_{k+q} \\ .. & .. & .. & .. \\ a_{k+q-1} & a_{k+q} & .. & a_{k+2(q-1)} \end{vmatrix}$$

and in particular

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$$T_2(2)=\begin{vmatrix}a_2 & a_3\\ a_3 & a_2\end{vmatrix},\ T_2(3)=\begin{vmatrix}a_3 & a_4\\ a_4 & a_3\end{vmatrix},\quad T_3(2)=\begin{vmatrix}a_2 & a_3 & a_4\\ a_3 & a_2 & a_3\\ a_4 & a_3 & a_2\end{vmatrix}. \tag{5}$$

There is a long standing history in line with problem of finding best possible bounds $\left\| a_{k+1}\right| - \left| a_k \right\|$ for some function $f \in S$ as in [15].

From (1) we have that

$$f(z)^{\alpha} = \left(z + \sum_{k=2}^{\infty} a_k z^k \right)^{\alpha}. \tag{6}$$

Expand binomially we have

$$f(z)^{\alpha} = z^{\alpha} + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}. \tag{7}$$

Applying Salagean differential operator on (7) yields

$$D^n f(z)^{\alpha} = \alpha^n z^{\alpha} + \sum_{k=2}^{\infty} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1} \tag{8}$$

Remark 1: If $\alpha = 1$ in (8) we obtain Salagean differential operator [16].
Again, applying Ruscheweyh operator [17] on (8) yields

$$R^n f(z)^{\alpha} = \frac{(n+\alpha-1)!}{n!(\alpha-1)!} z^{\alpha} + \sum_{k=2}^{\infty} \frac{(n+\alpha+k-2)!}{n!(\alpha+k-2)!} b_k(\alpha) z^{\alpha+k-1} \tag{9}$$

Remark 2: If $\alpha = 1$ in (9) we get Ruscheweyh operator [17].
Denote by $DR^n : A \to A$ the operator given by the Hadamard product (the convolution product) of the Salagean operator $D^n f^{\alpha}$ and the Ruscheweyh operator $R^n f^{\alpha}$:

$$DR^n f(z)^{\alpha} = (D^n * R^n) f(z)^{\alpha}$$

For any $z \in U$ and each non-negative integer n we obtain

$$DR^n f(z)^{\alpha} = \frac{\alpha^n (n+\alpha-1)!}{n!(\alpha-1)!} z^{\alpha} + \sum_{k=2}^{\infty} \frac{(\alpha+k-1)^n (n+\alpha+k-2)!}{n!(\alpha+k-2)!} a_k(\alpha) b_k(\alpha) z^{\alpha+k-1} \tag{10}$$

Remark 3: If $\alpha = 1$ in (10) we have operator defined by Andrei [18].

Definition 1: Let $f^{\alpha} \in T_n(\alpha)$, then

$$\operatorname{Re}\left(\frac{n!(\alpha-1)!\, DR^n f(z)^{\alpha}}{\alpha^n (n+\alpha-1)!\, z^{\alpha}} \right) > 0 \tag{11}$$

$n \in N, \alpha \geq 1.$

In the first part of our main results we will be dealing with Hankel coefficient estimates for the functions in the earlier defined class. The second part dealing with Teoplitz determinant.

2. Lemmas

Lemma 1: [19] If $p \in P$ then $|c_k| \leq 2$ for each k.

Lemma 2: [19] Let the function $p \in P$ be given by the power series $p(z) = 1 + c_1 z + c_2 z^2 + ...$ then

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

for some x, $|x| \leq 1$ and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1 x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z$$

for some x, $|z| \leq 1$

3 Results

Theorem 1: Let $f^{\alpha} \in T_n(\alpha)$. Then

$$\left| a_2(\alpha) a_4(\alpha) - a_3^2(\alpha) \right| \leq \lambda t^2 + \gamma t + \frac{B}{4} \tag{12}$$

where $t = \dfrac{4B-3A}{A-B}$, $\lambda = \dfrac{A-B}{8}$, $\gamma = \dfrac{3A-4B}{4}$,

$$A = \frac{\alpha^{2n} \alpha! (\alpha+2)! (n+\alpha-1)!^2}{(\alpha+1)^n (\alpha+3)^n (\alpha-1)!^2 (n+\alpha)! (n+\alpha+2)!},\ B = \frac{\alpha^{2n} (\alpha+1)!^2 (n+\alpha-1)!^2}{(\alpha+2)^{2n} (\alpha-1)!^2 (n+\alpha+1)!}. \tag{13}$$

Proof: Let $f^{\alpha} \in T_n(\alpha)$, then

$$\frac{n!(\alpha-1)!\,DR^{n}f(z)^{\alpha}}{\alpha^{n}(n+\alpha-1)!\,z^{\alpha}}=p(z). \quad (14)$$

Where

$$p(z)=1+c_1z+c_2z^2+c_3z^3+.... \quad (15)$$

Equating the like terms in (14), we have

$$a_2(\alpha)=\frac{\alpha^{n}\alpha!(n+\alpha-1)!\,c_1}{(\alpha+1)^{n}(\alpha-1)!(n+\alpha)!\,b_2(\alpha)} \quad (16)$$

$$a_3(\alpha)=\frac{\alpha^{n}(\alpha+1)!(n+\alpha-1)!\,c_2}{(\alpha+2)^{n}(\alpha-1)!(n+\alpha+1)!\,b_3(\alpha)} \quad (17)$$

$$a_4(\alpha)=\frac{\alpha^{n}(\alpha+2)!(n+\alpha-1)!\,c_3}{(\alpha+3)^{n}(\alpha-1)!(n+\alpha+2)!\,b_4(\alpha)}. \quad (18)$$

From (16), (17) and (18), we have

$$\left|a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)\right|=$$

$$\left[\frac{\alpha^{n}\alpha!(n+\alpha-1)!\,c_1}{(\alpha+1)^{n}(\alpha-1)!(n+\alpha)!\,b_2(\alpha)}\right]\left[\frac{\alpha^{n}(\alpha+2)!(n+\alpha-1)!\,c_3}{(\alpha+3)^{n}(\alpha-1)!(n+\alpha+2)!\,b_4(\alpha)}\right]-\left[\frac{\alpha^{n}(\alpha+1)!(n+\alpha-1)!\,c_2}{(\alpha+2)^{n}(\alpha-1)!(n+\alpha+1)!\,b_3(\alpha)}\right]^2 \quad (19)$$

Upon simplification and using Lemma 1 that $b_2(\alpha)=b_3(\alpha)=b_4(\alpha)\le 2$ we have

$$a_2a_4-a_3^2=\frac{\alpha^{2n}\alpha!(\alpha+2)!(n+\alpha-1)!^2\,c_1c_3}{4(\alpha+1)^{n}(\alpha+3)^{n}(\alpha-1)!^2\,(n+\alpha)!(n+\alpha+2)!}-\frac{\alpha^{2n}(\alpha+1)!^2\,(n+\alpha-1)!^2\,c_2^2}{4(\alpha+2)^{2n}(\alpha-1)!^2\,(n+\alpha+1)!} \quad (20)$$

Substituting for c_2 and c_3 from Lemma 4 and letting $c_1=c$

$$a_2a_4-a_3^2=\frac{1}{4}Ac\left[\frac{c^3}{4}+\frac{c(4-c^2)x}{2}-\frac{c(4-c^2)x^2}{4}+\frac{(4-c^2)(1-|x|^2)z}{2}\right]$$

$$-\frac{B}{4}\left[\frac{c^4}{4}+\frac{xc^2(4-c^2)}{2}+\frac{x^2(4-c^2)^2}{4}\right] \quad (21)$$

where $A=\frac{\alpha^{2n}\alpha!(\alpha+2)!(n+\alpha-1)!^2}{(\alpha+1)^{n}(\alpha+3)^{n}(\alpha-1)!^2\,(n+\alpha)!(n+\alpha+2)!}$, $B=\frac{\alpha^{2n}(\alpha+1)!^2\,(n+\alpha-1)!^2}{(\alpha+2)^{2n}(\alpha-1)!^2\,(n+\alpha+1)!}$.

Since $|c|=|c_1|\le 2$ by using Lemma 2,we may assume without restriction $c\in[0,2]$. Then using the inequality, with $\rho=|x|$ we obtain

$$\left|a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)\right|\le$$

$$\frac{1}{16}(A-B)c^4+\frac{1}{8}(A-B)c^2(4-c^2)\rho+\frac{1}{16}Ac^2(4-c^2)\rho^2+\frac{1}{16}B(4-c^2)^2\rho^2+\frac{1}{8}A(4-c^2)(1-\rho^2)=F(c,\rho). \quad (22)$$

Then,

It is clear that, $\frac{\partial F}{\partial \rho}>0$ which show that $F(c,\rho)$ is an increasing function on the close interval [0,1]. This implies that maximum occurs at $\rho=1$.

Therefore max $F(c,\rho)=F(c,1)=G(c)$. Now

$$F(c,1)=G(c)=\frac{1}{16}(A-B)c^4+\frac{1}{8}(A-B)c^2(4-c^2)+\frac{1}{16}Ac^2(4-c^2)+\frac{1}{16}B(4-c^2)^2. \quad (23)$$

Thus,

$$G(c)=\lambda c^4+\gamma c^2+\frac{B}{4} \quad (24)$$

where $\lambda=\frac{A-B}{8}$, $\gamma=\frac{3A-4B}{4}$.

Thus

$$G'(c)=4\lambda c^3+2\gamma c \quad (25)$$

and

$$G''(c)=12\lambda c^2+2\gamma<0. \quad (26)$$

For optimum value of $G(c)$, consider $G'(c)=0$ from (25), we get

$$c^2=\frac{4B-3A}{A-B}=t. \quad (27)$$

Substituting the value of c^2 from (27) in (26), it can be shown that

$$G''(c)=\frac{3}{2}[A-B]+\frac{1}{2}[3A-4B]. \qquad (28)$$

Therefore, by second derivative test $G(c)$ has the maximum value at c, where c^2 is given by (27). Substituting the obtained value of c^2 in the expression (19), which gives the maximum value of $G(c)$ as

$$\left|a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)\right|\le \lambda t^2+\gamma t+\frac{B}{4}$$

which complete the proof.

4. Teoplitz Determinant

Theorem 2: Let $f^{\alpha}\in T_n(\alpha)$. Then

$$\left|a_3^2(\alpha)-a_2^2(\alpha)\right|\le \frac{1}{8}\left[A_1^2\beta^2+\lambda_1\beta+\lambda_1\right] \qquad (29)$$

where $A_1=\frac{\alpha^n(\alpha+1)!(n+\alpha-1)!}{(\alpha+2)^n(\alpha-1)!(n+\alpha+1)!}$, $B_1=\frac{\alpha^n\alpha!(n+\alpha-1)!}{(\alpha+1)^n(\alpha-1)!(n+\alpha)!}$, $\lambda_1=2B_1^2-5A_1^2$

$$\beta=\frac{5}{2}-\left[\frac{\alpha!(\alpha+2)^n(n+\alpha+1)!}{(\alpha+1)^n(n+\alpha)!(\alpha+1)!}\right]^2. \qquad (30)$$

Proof: By (16), (17) and (18), we noticed that

$$a_3^2(\alpha)-a_2^2(\alpha)=\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2c_2^2}{(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2b_3^2(\alpha)}-\frac{\alpha^{2n}\alpha!^2(n+\alpha-1)!^2c_1^2}{(\alpha+1)^{2n}(\alpha-1)!^2(n+\alpha)!^2b_2^2(\alpha)} \qquad (31)$$

For $b_2(\alpha)\le 2$ and $b_3(\alpha)\le 2$ we have

$$a_3^2(\alpha)-a_2^2(\alpha)=\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2c_2^2}{4(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}-\frac{\alpha^{2n}\alpha!^2(n+\alpha-1)!^2c_1^2}{4(\alpha+1)^{2n}(\alpha-1)!^2(n+\alpha)!^2} \qquad (32)$$

Substituting for c_2 from Lemma 4 and setting $c_1=c$ yields

$$a_3^2(\alpha)-a_2^2(\alpha)=\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{16(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}c^4+\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{8(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}c^2(4-c^2)x+$$
$$\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{16(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}x^2(4-c^2)-\frac{\alpha^{2n}\alpha!^2(n+\alpha-1)!^2}{4(\alpha+1)^{2n}(\alpha-1)!^2(n+\alpha)!^2}c^2. \qquad (33)$$

Since $|c|=|c_1|\le 2$ by using Lemma 2, we may assume without restriction $c\in[0,2]$. Then using the inequality, with $\rho=|x|$ we obtain

$$\left|a_3^2(\alpha)-a_2^2(\alpha)\right|\le\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{16(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}c^4+\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{8(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}c^2(4-c^2)\rho+$$
$$\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{16(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}\rho^2(4-c^2)+\frac{\alpha^{2n}\alpha!^2(n+\alpha-1)!^2}{4(\alpha+1)^{2n}(\alpha-1)!^2(n+\alpha)!^2}c^2. \qquad (34)$$

Then,

$$\frac{\partial F}{\partial\rho}=\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{8(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}c^2(4-c^2)+\frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{8(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}\rho(4-c^2). \qquad (35)$$

It is clear that, $\frac{\partial F}{\partial\rho}>0$ which show that $F(c,\rho)$ is an increasing function on the close interval [0,1]. This implies that maximum occurs at $\rho=1$.

Therefore max $F(c,\rho)=F(c,1)=G(c)$. Now

Thus

$$\left|a_3^2(\alpha)-a_2^2(\alpha)\right|\le\frac{1}{8}A_1^2c^4+\frac{1}{8}\left[2B_1^2-5A_1^2\right]c^2+\frac{1}{8}\left[2B_1^2-5A_1^2\right]=G(c). \qquad (36)$$

Thus

$$G'(c)=\frac{1}{2}A_1^2c^3+\frac{1}{4}\left[2B_1^2-5A_1^2\right]c \qquad (37)$$

and

$$G''(c)=\frac{3}{2}A_1^2c^2+\frac{1}{4}\left[2B_1^2-5A_1^2\right]. \qquad (38)$$

For optimum value of $G(c)$, consider $G'(c)=0$ from (37), we get

$$c^2=\frac{5}{2}-\left[\frac{\alpha!(\alpha+2)^n(n+\alpha+1)!}{(\alpha+1)^n(n+\alpha)!(\alpha+1)!}\right]^2=\beta \qquad (39)$$

Substituting the value of c^2 from (39) in (38), it can be shown that

$$G''(c) = \frac{3}{2}A_1^2\beta + \frac{1}{4}\left[2B_1^2 - 5A_1^2\right]. \tag{40}$$

Therefore, by second derivative test $G(c)$ has the maximum value at c, where c^2 is given by (22). Substituting the obtained value of c^2 in the expression (19), which gives the maximum value of $G(c)$ as

$$\left|a_3^2(\alpha) - a_2^2(\alpha)\right| \le \frac{1}{8}\left[A_1^2\beta^2 + \lambda_1\beta + \lambda_1\right]$$

which complete the proof.

Theorem 3: Let $f^\alpha \in T_n(\alpha)$. Then

$$\left|a_4^2(\alpha) - a_3^2(\alpha)\right| \le M_1t^3 + M_2t^2 + M_3t + 4B_1 \tag{41}$$

Where $M_1 = \frac{-5}{6}A_1$, $M_2 = \frac{7A_1 - B_1}{4}$, $M_3 = 4A_1 + B_1$

$$A_1 = \frac{\alpha^2(n+2)!(n+\alpha-1)!}{4(\alpha+3)^{2n}(\alpha-1)!^2(n+\alpha+2)!^2} \quad B_1 = \frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{4(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}. \tag{42}$$

Proof: Following the method of proof Theorem 2 and using Lemma 2.2 to express c_2 and c_3 in terms of c_1, the desired shall be obtained.

Theorem 4: Let $f^\alpha \in T_n(\alpha)$. Then

$$\left|1 + 2a_2^2(\alpha)(a_3(\alpha) - 1) - a_3^2(\alpha)\right| \le 1 + \theta_1\eta^2 + \theta_2\eta + 16\lambda_4 \tag{43}$$

where $\eta = \frac{\lambda_1 - \lambda_2 - \lambda_4}{2(\lambda_2 + \lambda_3 - 8\lambda_4)}$, $\theta_1 = \lambda_1 + \lambda_4 - \lambda_2$, $\theta_2 = 4\lambda_2 + \lambda_3 - 8\lambda_4$

$$\lambda_1 = \frac{\alpha^{3n}\alpha!^2(n+\alpha-1)!^3}{16(\alpha+1)^{2n}(\alpha+2)^2(\alpha-1)!^3(n+\alpha)!^2(n+\alpha+1)!}$$

$$\lambda_1 = \frac{\alpha^{3n}\alpha!^2(n+\alpha-1)!^3}{16(\alpha+1)^{2n}(\alpha+2)^2(\alpha-1)!^3(n+\alpha)!^2(n+\alpha+1)!} - \frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{2(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}$$

$$\lambda_3 = \frac{\alpha^{2n}(\alpha)!^2(n+\alpha-1)!^2}{4(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha)!^2}$$

$$\lambda_3 = \frac{\alpha^{2n}(\alpha+1)!^2(n+\alpha-1)!^2}{4(\alpha+2)^{2n}(\alpha-1)!^2(n+\alpha+1)!^2}.$$

Proof: Similarly, following the method of proof Theorem 2 and using Lemma 2 to express c_2 and c_3 in terms c_1 the desired results shall be obtained.

References

[1] L. Bieberbach L. (1916), Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abblildung des Einheitskreises vermitteln, Semesterberichte Preuss. Akad. Wiss.38, 940-955.
[2] Ch.Pommerenke (1967), On Hankel determinants for of univalent function, Mathematika, 102-112.
[3] J.W. Noonan and D.K.Thomas (1976), On the second Hankel determinant of a really mean p-valent functions, Trans. Amer. Math. Soc., 223 (2), 337-346.
[4] K.O. Babalola (2010), On H3(1) Hankel determinants for some classes of univalent functions,Inequality Theory and Applications, vol. 6, 1.7, Nova Science Publishers, New York, NY, USA,
[5] R. Ehrenborg ((2000)), Hankel determinant of exponential polynomial. American Mathematical Monthly, 107, 557-560.
[6] J.O. Hamzat, and A.A. Oni (2017), Hankel determinant for a certain subclasses of analytic functions. Asian Research Journal of Mathematics, 5(2), 1-10.
[7] T. Hayami, S.Owa (2012), Coefficient bounds for bi-univalent functions, Pan Amer. Math. J. 22, no. 4, 15(26).
[8] A. Janteng, S. A.Halim, and M.Darus 2007, Hankel determinant for starlike and convex functions, International Journal of Mathematical Analysis, vol. 1, no. 13, pp. 619.625,
[9] K.R. Laxmi, R.B. Sharma (2017), Second Hankel determinant for some subclasses of bi-univalent functions associated with pseudo-starlike functions, J. Complex Anal., Article ID 6476391, 9 pp.
[10] I. T. Awolere (2020), Hankel determinant for bi-Bazelevic function involving error and Sigmoid function defined by derivative calculus via Chebyshev polynomials, Journal of Fractional Calculus and Applications Vol.11(2) July ,pp.208-217.
[11] I. T. Awolere and A.T.Oladipo (2019), Determinantfor bi-univalent of pseudo starlikeness for certain class of analytic univalent functions, Libertas Mathematica (new series), Volume 39, No. 2, 27-43.
[12] A. Janteng, S.A. Halim and M. Darus (2007), Hankel determinant for starlike and convex functions, Int. J. of Math. Anal., 1(13), 19-625.

[13] J.W. Layman (2001), The Hankel transform and some of its properties, J. of Integer, Sequences, 4, 1-11.
[14] Ye.Ke and Lek. Henglim (2016), Every matrix is a product of Teoplitz, Fundation of Computational Mathematics 16,577-598.
[15] P.L.Duren (1995), Univalent functions (Springer-Verlag, 1983)Mat.5b37(79), 471-476.
[16] G. St. Salagean(1983) , Subclasses of univalent functions, Lecture Notes in Math.,Springer Verlag, Berlin, 1013, 362-372.
[17] St. Ruscheweyh, New criteria for univalent functions, Proc. Amet. Math. Soc.,49(1975), 109-115.
[18] L. Andrei (2014), Differential subordination results using a generalized Salagen operator and Ruscheweyh operator, Acta Universitatis Apulensis, No.37, 45-59.
[19] C. Ramachandran, D. Kavitha (2016), Toeplitz determinant for some subclasses of analytic functions, lobal Journal of Pure and Applied Mathematics, vol.13 No.2783-793.