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ON THE EXISTENCE OF EXPLICIT SURJECTIVE L_∞ -QUASI – ISOMORPHISM

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Abstract

Smaller L_∞- model for the cover of the classifying space up to homotopy is imminent. This model retains all the information of the larger models found in literature. We give immediate consequences of L_∞ - structure transferred interacting with one of the grading and study structural properties of koszul spaces deciding the model. The induce morphisms of homology groups are isomorphic for all n.

***Keywords*:** Classifying spaces for groups and H-spaces, Homology, Homotopy.

1. Introduction

Let V denote a graded vector space. It comes with L_∞-structure corresponding precisely to a square zero coderivation of degree – 1 on the cofree cocommutative coalgebra $\wedge(s\vee)$ completely determined by its corestriction$\pi\delta: \wedge(s\vee) \to s\vee$. Write$\delta = \sum_{r\geq 0}\delta_r$,where δ_r lowers word length by r, i.e. for any $n \geq 0$ we have restriction $\delta_r: \wedge^n(s\vee) \to \wedge^{n-r}(s\vee)$. In particular, $\delta_r: \wedge(s\vee)^{r+1} \to s\vee$. If δ has a degree-1, then the family of maps δ_r correspond to the operation l_r for an L_∞-algebra, by setting

$$Sl_r(v_1, \dots, v_r) = (-1)\sum_i i|x_{r-i}| + r\,\delta_{r-1}(sv_1 \wedge \dots v_1 sv_r)$$

The condition $\delta^2 = 0$ corresponds to the generalized Jacobi identities. Taking the graded dual determines dg algebra, and if V is of a finite type and concentrated in positive degrees, then the opposite is true. A differential d on the free graded commutative algebra $\wedge((s\vee)^n)$ determines an L_∞-structure on$V$. The differential is determined by restriction to $(s\vee)^v$. Similarly, we write$d_n: (s\vee)^v \to \wedge^n((s\vee)^v)$ for the restriction, and the nn-array operation can be read from this. Furthermore, an L_∞-morphism is just a dg algebra morphism. This gives a convenient way of packaging the data of an L_∞-algebra with easy access to structural properties. For example, a minimal L_∞-structure on a positively graded vector space of finite type is given by a free graded commutative algebra equipped with a differential with nonlinear part.

Next, let (W, d_w) and (V, d_v) be chain complex, and

$$h \circlearrowleft W \underset{g}{\overset{f}{\rightleftarrows}} V \qquad \text{(a contraction)} \qquad (1)$$

If W is an L_∞-algebra, then there is an induced L_∞-structure on V. This is the Homotopy Transfer Theorem for L_∞-algebras stated in [1] without proof was later shown in [2]. A version with explicit formulae for resulting structure appeared in [3 and 4]. The latter also contains details on how to extend the maps occurring to L_∞-morphisms.

Example 1.1: Given a contraction (1) and L_∞-structure $\{l_n\}$ on W. Consider a rooted trees with each leaf labeled by g, each vertex by l_n where $n+1$ is the valence of the vertex, each internal edge by h, and the root by f. Such a tree with n leaves may be taken as recipe for building a map $V^{\otimes n} \longrightarrow V$, by using from leaves to root, each leaf taking an input from one of the n copies of V in the source. We can form a signed sum over all such rooted trees with n leaves and labels as described, to get a map $l_n: V^{\otimes n} \longrightarrow V$, which may depict as

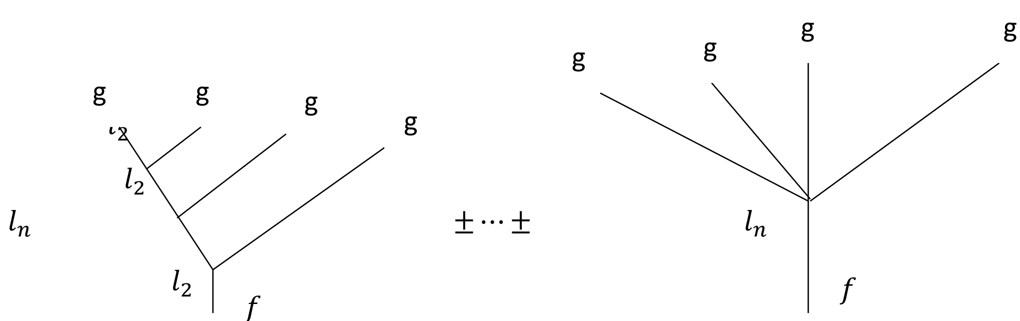

Figure 1.1 (a) Composed R...... Tree (b) Rooted Tree after contraction

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If we denote the root of each tree by h instead of f, we get a recipe for building maps
$V^{\otimes n} \longrightarrow W$, and forming the signed sum over all rooted trees with n leafs and this decoration,
we get a map
$g_n: V^{\otimes n} \longrightarrow W$

The bundle theory, foliation theory, and delooping theory, classifying spaces of topological groups and groupoids were the major focus of research in the 1960s – 1980s. Since then, Many different constructions of classifying spaces of topological groups and groupoids have been introduced, for example, the construction in [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] etc. Some of them have even been generalized to any internal category of topological spaces. For topological groups, most of the constructions give rise to homotopy equivalence spaces. The homotopy theory of mapping space and in particular, spaces of self-equivalences is well developed in [12, 15, and 22]. Thus, the model in [15, 22] does not address $\pi_0(\text{auto } X)$.
It is well known the maps
$\pi_0(autX_{\mathbb{Q}}) \longrightarrow autH^*(X:\mathbb{Q})$
$\pi_0(autX_{\mathbb{Q}}) \longrightarrow aut\pi_*\Omega(X)\otimes\mathbb{Q}$
given by sending a homotopy class to the induced map on respectively cohomotopy and homotopy structure. The surjectivity of the first of these maps has been studied in [14] for a formal space, and the second map in [21] for a conformal space, where $\pi_0(autoX_{\mathbb{Q}})$ is a linear algebraic group if X is either a finite CW – complex or has finite Postnikov tower (see. [23]). There exist general models for mapping spaces expressed in term of the so called Maurr-Cartan elements of simplicial dg Lie algebra (see, [21 and 24]). The first of these is particularly useful to investigates $autX_{\mathbb{Q}}$, while the Lie model of derivations given in [15 and 22] is finite for theoretical purpose but it is very large.
Let ℓ denote Quillen model for a relatively small formal space, let $A = H^*(X;\mathbb{Q})$ implies Quillen construction on the cohomology. For a space which is not formal, the Quillen model is quasi-isomorphic to the homotopy Lie algebra L of the space, $f: \ell \to L$. One can now extend this using contractive mapping to obtain more information in a distinct way.

1.2 Proof Plan

Homotopy Lie algebra of a simply connected space X, is the graded abelian group $\pi_*(\Omega X) \otimes \mathbb{Q}$, equipped with the samelson bracket. To obtain more information is by Homotopy Transferred Theorem for $L_\infty -$ algebras, we have the following steps.

STEP 1. The dg Lie structure on $Der\ell$ is transfer along the contraction to $SA \otimes_k L$ and further to homology $H_*(sA \otimes_k L)$. With this transferred structure, the homology computes $\pi_*(\text{aut X}, 1_x) \otimes \mathbb{Q}$ not only as a graded abelian group, but as a graded Lie algebra, the homotopy Lie algebra of the 1-connected space B, $autX(1)$. Furthermore, the $L_\infty -$algebra $H_*(SA \otimes_k L)$ completely determines the rational homotopy type B, aut X(1). This was noticed in [5 and 25].

STEP 2. Since Quillen models are too large, taking the derivation of the Lie algebra of the model would not help. In this case, we apply several properties of Koszul spaces to produce a much smaller $L_\infty -$ algebra, retaining all the information of the larger model.

1.3 $L_\infty -$ algebras: Contraction is one of the key ingredient in studying the structure of $L_\infty -$ algebras. We found the correct notion in [1 and 22], and a modern treatment in [3]. We follow the sign conventions from the latter.

Definition 1.2: Let V be a graded vector space. An $L_\infty -$ structure on V is a family maps $\ell_n: V^{\otimes n} \to V_1$, $n \geq 1$ of degree$n-2$, satisfying anti-symmetry
$\ell_n(\dots, x, y, \dots) = -(-1)^{|x||y|}\ell_n(\dots, x, y, \dots), \forall n \geq 1,$
The generalized Jacobi identities,

$$\sum_{p=1}^{n}\sum_{\sigma}^{n} sgn(\sigma)(-1)\ell_{n+1-p}\left(\ell_p\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, x_{\sigma(p+1)}, \dots, x_{\sigma(n)}\right)\right) = 0,$$

where we sum over all $(p, n-p)$ – un-shuffled, i.e.
$\sigma^{-1}(1) < \cdots < \sigma^{-1}(p)$and $\sigma^{-1}(p+1) < \cdots < \sigma^{-1}(n)$
The sign is given by

$$\epsilon = p(n-p) + \sum_{i<j} |x_i||x_j|, \sigma^{-1}(i) > \sigma^{-1}(j)$$

The generalized Jacobi identities for $n \leq 3$ are $\ell_1^2(x) = 0$

$$\ell_2(\ell_1(x), y) + (-1)^{|x|}\ell_2(x, \ell_1(y))$$
$$= \ell_1(\ell_2(x, y)), \ell_2(\ell_2(x, y), z)$$
$$+(-1)^{|y||z|+1}\ell_2(\ell_2(x, z), y) - \ell_2(x, \ell_2(y, z))$$
$$= -(\ell_1\ell_3 + \ell_3\ell_1)(x \otimes y \otimes z) \qquad (2)$$

From (2) we see that ℓ_1 is a differential, and a derivation with respect to ℓ_2. For our $L_\infty -$ structure on a graded vector space$V$, the chain complex (V, ℓ_1) is called the underlying complex. We see from (2) that if either ℓ_1 or ℓ_2 are zero, then ℓ_2 is a Lie bracket, but in general (2) just states that Jacobi relation holds up to chain homotopy in $V^{\otimes 3}$ given by - ℓ_3(note that we have abused notation slightly so the differential on $V^{\otimes 3}$ induced by ℓ_1, is also denoted by ℓ_1). We shall say that ℓ_n for$n > 2$, is a higher operation. An $L_\infty -$ structure with trivial higher operations is just a dg Lie structure.
Definition 1.3: An $L_\infty - morphism$ $g: (V, \ell) \to (W, \ell)$ is a family of graded alternating maps $\{g_n: V^{\otimes 3} \to (W, \ell)\}_n$ of degree -1 , such that for every $n \geq 1$, g_n satisfies

$$\sum_{p=1}^{n}\sum_{\sigma}^{n} sgn(\sigma)(-1)^{\epsilon} g_{n+1-p}\left(\ell_p\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, x_{\sigma(p+1)}, \dots, x_{\sigma(n)}\right)\right)$$
$$= \sum_{k\geq 1}^{n}\sum_{r}^{n} sgn(\tau)(-1)^{n}\ell_k\left(g_{i_1}\left(x_{r(1)}, x_{r(2)}, \dots, x_{r(i_1)}\right), \dots, g_{i_k}\left(x_{r(i_{k-1}+1)}, , \dots, x_{r(i_k)}\right)\right),$$

where σ is a $(p, n-p)$ - unshuffles as above, and τ is an $(i_1, \dots, i_k)$ – unshuffles, i.e.
$\tau^{-1}(i_j+1) < \dots < \tau^{-1}(i_{j+1}), \forall j \in \{0, \dots, k-1\}$
satisfying the extra condition that
$\tau^{-1}(1) < \tau^{-1}(i_j+1) < \dots < \tau^{-1}(i_1 + i_1 \dots + i_{k+1} + 1)$
The signs are given by

$$\epsilon = p(n-p) + \sum_{i<j} |x_i||x_j|, \sigma^{-1}(i) > \sigma^{-1}(j)$$

$$\eta = \sum_{j=1}^{k}(k-j)(i_j-1) + \sum_{i<j}|x_i||x_j| + \sum_{j=2}^{k}(i_j-1)\sum_{j=2}^{k}\left|x_{r(m)}\right|, \tau^{-1}(i) > \sigma^{-1}(j)$$

For $n = 1$, the condition implies that g_1 is a chain map. For $n = 2$, it is
$-g_2(\ell_1(x_1), x_2) - (-1)^{|x_1||x_2|+1} g_2(\ell_1(x_2), x_1) + g_1(\ell_2(x_1, x_2))$
$= \ell_2(g_1(x_1), g_1(x_2)) + (-1)^{|x_1||x_2|+1}\ell_2(g_1(x_2), g_1(x_1)) + \ell_2(g_2(x_1, x_2)),$

On re-arranging, we have
$g_2(\ell_1(x_1), x_2) + (-1)^{|x_1|} g_2(x_1, \ell_1(x_2)) + \ell_1(g_2(x_1, x_2))$
$= g_2(\ell_1(x_1, x_2)) - \ell_2(g_1(x_1), g_1(x_2))$

It is obvious that g_2 is a chain homotopy between $g_1\ell_2$ and $\ell_2(g_1 \otimes g_1)$, so g_1 respects the binary operations up to homotopy. Similarly the higher maps $\{g_n\}_{n\geq 3}$ can be thought of as homotopies between homotopies, and so on.

Definition 1.4: L_∞ quasi-isomorphism is an L_∞-morphism $\{g_n\}_n$, such that g_1 is a quasi-isomorphism of chain complexes. There is an equivalent definition of $L_\infty -$ algebras, which the following theorem expresses.
Theorem 1.5 (Homotopy Transfer Theorem [4]): Let $(L, \{\ell_n\})$ be an L_∞-algebras. Let (V, d_v) be a chain complex. Given a contraction

$$h \circlearrowleft L \underset{g}{\overset{f}{\rightleftarrows}} V$$

where the collection of maps (ℓ_n) defines an$L_\infty -$ structure on V, the collection of $\{g_n\}$ defines an extension of g to an L_∞ quasi-isomorphism$(V, \{\ell_n\}) \to (L, \{\ell_n\})$. There is an extension of f to an L_∞ quasi-isomorphism in [4].

Let denote ℓ_2 by $[-1-]$ for now and let $x, y \in V$. For binary and ternary transferred operations, we get
$\ell_2(x, y) = f([g(x), g(y)])$
$\ell_3(x, y, z) = fo(-[h[g(x), g(y)], g(z)])$
$+(-1)^{|x|}[g(x), h[g(y), g(z)]]$
$+(-1)^{|y||z|}[h[g(x), g(z), g(y)]]$
$+\ell_3[g(x), g(y), g(z)]$ (3)

1.3 Preliminary Results
We shall need the following,

Proposition 1.6 ([17]). There is always a minimal Sullivan model for a simply connected space. The minimal is unique up to isomorphism.
Proposition 1.7 ([17]). There is always a minimal quillen model for a simply connected space. The minimal model is unique up to isomorphism.
Lemma 1.8 ([3], section 10.4). Let L and L' be dg Lie algebras, and consider them as $L_\infty -$ algebra with trivial higher operations. Then there exists $L_\infty -$ quasi-isomorphism $L \xrightarrow{\sim} L'$ if and only if L and L' are quasi isomorphism as dg Lie algebras.
Remark 1.9. From Theorem 1.8, it is clear, a dg Lie algebras L is formal if and only if there exist $L_\infty -$ quasi-isomorphism $H_*(L) \xrightarrow{\sim} L'$, where L and $H_*(L)$ are considered $L_\infty -$algebras with trivial higher operations. Now, we may always choose contraction

$$k \circlearrowleft L \underset{i}{\overset{q}{\rightleftarrows}} H_*(L) \quad (4)$$

onto homology in accordance with the following result.
Lemma 1.10 For any chain complex V (over a field) we may choose a contraction

$$h \circlearrowleft V \underset{i}{\overset{p}{\rightleftarrows}} H_*(V)$$

where we consider the graded vector space $H_*(V)$ is a chain complex with zero differential.

Proof. Consider the short exact sequences

$$0 \longrightarrow Z_n \underset{}{\overset{j}{\rightleftarrows}} V_n \overset{d}{\rightleftarrows} B_{n-1} \longrightarrow 0$$

$$0 \longrightarrow B_n \underset{j}{\rightleftarrows} Z_n \underset{d}{\rightleftarrows} H_n(V) \longrightarrow 0$$

Since we are working over a field, these are split exact, and we may choose splitting as already indicated. It is easy to check that the data

$$\sigma\rho r \circlearrowleft V \underset{jw}{\overset{qr}{\rightleftarrows}} H_*(V)$$

is a contraction of V onto homology.
Therefore, theorem 1.5 for L_∞ −algebras then produces a minimal L_∞ − structure on $H_*(L)$ with L_2 − operation the standard bracket induced on the homotopy, and L_∞-Quasi-isomorphism

$$H_*(V) \overset{\sim}{\to} L$$

Lemma 1.11 ([5]). Let X be a simple connected space such that $H^*(X;\mathbb{Q})$ is of finite type. The following are equivalent:

I. X is both formal and coformal
II. X is formal and the cohomology algebra $H^*(X;\mathbb{Q})$ is a Koszul graded commutative algebra.
III. X is coformal and the homotopy Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ is a Koszul graded Lie algebra

Lemma 1.12 ([5]). Let X be a Koszul space with homotopy Lie algebra L and cohomology A. The Koszul dual graded commutative algebra L' is isomorphic to A, and the Koszul dual graded algebra $A'$is isomorphic to L.

Corollary 1.13 ([5]). Let X be a Koszul space with homotopy Lie algebra L and cohomology algebra A. Then

I. $\ell(A^v)$ is the minimal Quillen model for X
II. There is a surjective quasi-isomorphism$\ell(A^v) \overset{\sim}{\to} L$ corresponding to a twisting morphism $k: A^v \to L$

The existence of this explicit surjective quasi-isomophism is the special feature of Koszul spaces upon which this paper is build.

Lemma 1.14 ([15], Corollary VII. 4(4)). Let X be a simply connected space homotopy equivalent to a finite CW-complex. If ℓ is a Quillen model for X, then 1-connected cover of the map induced by inclusion of monoids $B\ aut_*(X)\langle 1\rangle \longrightarrow B\ aut\ (X)\langle 1\rangle$ is modeled by the map of dg Lie algebras $(Der\ \ell)_+ \longrightarrow (Der\ \ell\ //\ ad\ell)_+$, given by inclusion of the derivation.

The rest of this paper is organized as follows: In Section 2, we set up contractions and isomorphisms needed. In Section 3, we specialize to the case of interest; koszul algebras. In Section 4, we record how several gradings interact with the maps set-up, and give immediate consequences of L_∞ − structure transferred with one of the gradings. Finally, we study in Section 5, the induced L_∞ − structure on $H(SA \otimes_k L)_*$ for the configuration spaces.

2. L_∞ −Transferred Structure

We are now more determined to reduce the study of $Der\ \ell(c)$ to the study of a twisted version of the complex $A^v \otimes L$ by basic perturbation theory, and application of a standard isomorphism. Little novelty exists. Therefore, we skipped some of the proofs.

Lemma 2.1 Let A be a dg commutative algebra. Let L be a dg Lie algebra. Let the chain complex $A \otimes L$ be equipped with a graded Lie bracket making a dg Lie algebra. Then the bracket is given by $[a\otimes x, b\otimes y] = (-1)^{|b||x|}ab \otimes [x,y]$ for $a,b \in A$ and $x,y \in L$

For any Maurer-Cartan element in this dg Lie algebra $\tau \in MC(A\otimes L)$, we have the twisted differential $d^\tau_{A\otimes L} = d_{A\otimes L} + \mathrm{a}d_\tau$, we write $A \otimes_\tau L$ for the resulting dg Lie algebra (in this way $A \otimes L$ equals $A\otimes_\tau L$ – the dg Lie algebra twisted by the Maurer-Cartan element 0). Let the following be a contraction of dg Lie algebras

$$h \circlearrowleft M \underset{g}{\overset{f}{\rightleftarrows}} L \qquad (5)$$

where L and M are dg Lie algebras, f is a quasi-isomorphism of dg Lie algebras, g is a chain map and h a chain homotopy such that $fg = I_L$ and $dh + hd = gf - I_L$. Imposing a strong assumption $hg = 0, h^2 = 0$ and $fg = 0$, one can now see that g and h are general not Lie maps.

Theorem 2.2 Given a graded commutative algebra A, a contraction (5) of a dg Lie algebras and Maurer-Cartan element $\tau \in MC(A\otimes L)$ such that A is nilpotent. Then, there is an induced contraction of chain complexes

$$h' \circlearrowleft A\otimes_\tau M \underset{g'}{\overset{1\otimes f}{\rightleftarrows}} A \otimes_{(1\otimes f)(\tau)} L$$

where

I. $1\otimes f$ is a quasi-isomorphism of chain complexes
II. g' is given by the recursive formula $g' = 1\otimes g + (1\otimes h)ad_\tau g'$

III h' is given by the recursive formula $h' = 1 \otimes h + h' ad_\tau (1 \otimes h)$

Proof. The contraction (5) induces a contraction

$$\circlearrowleft A \otimes M \underset{1 \otimes g}{\overset{1 \otimes f}{\rightleftarrows}} (A \otimes L, 1 \otimes d_\tau + \ell'),$$

where the maps are defined by the recursive formulae
$f' = 1 \otimes f + f' ad_\tau (1 \otimes h)$
$g' = 1 \otimes g + (1 \otimes h) ad_\tau g'$
$h' = 1 \otimes h + h' ad_\tau (1 \otimes h)$
$t' = f' ad_\tau (1 \otimes g)$
since f is a morphism of Lie algebras, we have for any $a \otimes m \in A \otimes M$
$(1 \otimes f) ad_\tau (1 \otimes h)(a \otimes m) = ad_{\tau (1 \otimes f)_\tau} a \otimes fh(x)$
$fh = 0$, we $\zeta' = 1 \otimes f$
Furthermore, for any $a \otimes \ell \in A \otimes L$, we get
$t'(a \otimes \ell) = (1 \otimes f) ad_\tau (1 \otimes g)(a \otimes \ell)$
$= ad_{(1 \otimes f)\tau} (a \otimes \ell),$
Hence
$t' = ad_{(1 \otimes f)\tau}$

These formulae converges because A is nilpotent. For most of our applications, A will be a finite dimensional Koszul algebra and thus nilpotent, see [1] for a discussion of weaker assumptions which may be adapted to our situation.

Preposition 2.3 Let C be a dg coalgebra. Let L be a dg Lie algebra. Let $Hom(C, L)$ denote the convolution dg Lie algebra. Then, the map $\varphi: C^v \otimes L \to Hom(C, L)$ given by $\varphi(1 \otimes f)(C) = (-1)^{|c||x|} f(c)x$, is a map of dg Lie algebras with respect to the structure of Definition 2.1 on the left hand side, natural in C and L. If C and L are of finite type and either C^v and L are either both bounded above or both bounded below, then φ is an isomorphism (i.e. either C or L is finite, the other need just be of finite type for φ to be an isomorphism).
For any Maurer-Cantan element $\tau \in MC(C^v \otimes L)$, the same formula defines a map $\varphi: C^v \otimes_\tau L \to Hom(C, L)^{\varphi(\tau)}$ with the same properties. The converse to φ is given by sending a map $f: C \to L$ to the expression $\sum_i (-1)^{|c_i|(|f| + |c_i|)} C_i^* \otimes f(c_i)$ where we have chosen basis $\{c_i\}$ for C, and $\{C_i^*\}$ for the dual basis of C^v. Note that this is a finite sum if either C is finite dimensional or if C^v and L are both bounded above or below.

Remark 2.4 We will need Proposition 2.3 to relate $sA \otimes L \approx Hom(s^{-1}C, L)$ in which case the signs work out as follows.
For $sa \otimes x \in sA \otimes L, \varphi(sa \otimes x)(s^{-1}c) = (-1)^{|c_i|(|f| + |c_i|)} a(c)x$
For $f \in Hom(s^{-1}C, L), \varphi^{-1}(f) = \sum_i (-1)^{|f||sc_i^*| + 1} sc_i \otimes f(s^{-1}c_i)$
Theorem 2.5 Let $(L, [-,-], d_L)$ be a dg Lie algebra, and (C, D_c, d_c) a coaugmented dg coalgebra, then for any twisting morphism $\tau \epsilon TW(C, L)$, restriction to generators, give a natural isomorphism of chair complexes $c^*: Der_f(L(C), L) \xrightarrow{\cong} Hom(\bar{C}, L)$ of degree -1, where f corresponds to τ under the bijection of {32}.
Proof. It is not difficult to see that restriction gives an isomorphism of graded vector spaces. We must show that i^* is a chain map of degree -1, i.e.

$$i^*(\partial(\theta)) = \partial^\tau (i^*(\theta)) \quad (6)$$

For any f-derivation θ, observe first that i is a twisting morphism. i.e. it satisfies

$$0 = \partial(l) + \frac{1}{2}[l, l] = d_{L(c)} i + i d_c + \frac{1}{2}[l, l] \quad (7)$$

We expand the left hand side of (6) using (7), this yields the following
$i^*(\partial(\theta)) = i^*(d_L \theta - (-1)^{|\theta|} \theta d_{\ell(c)})$
$= (-1)^{|\theta| - 1} d_L \theta_l + d_{\ell(c)^i}$
$= (-1)^{|\theta| - 1} d_L \theta_l + \theta \left(-l d_c - \dfrac{1}{2}[l, l]\right)$
By definition of the bracket in the convolution Lie algebra, and the fact that θ is an f- derivation, we get
$i^*(\partial(\theta)) = (-1)^{|\theta| - 1} d_L \theta_l - \theta_l d_c - \theta \dfrac{1}{2}[-,-]_L (l \otimes l)\Delta c$
$= d_L i^*(\theta) - (-1)^{|\theta|} i^*(\theta) d_c - \dfrac{1}{2}[-,-]_L (\theta \otimes f + f \otimes \theta)(l \otimes l)\Delta c$
$= -\Delta c \partial(i^*(\theta)) - \dfrac{1}{2}\left((-1)^{|\theta|}[i^*(\theta), \tau]\right) + [\tau, i^*(\theta)]$
$= -\Delta \partial(i^*(\theta) - [\tau, i^*(\theta)])$
This is precisely, $-\partial^\tau(i^*(\theta))$, and we are done.
A map dg Lie algebras $f: L \to M$ induces a chain map$f^*: DerL \to Der_f(M, L)$. Composing with ad gives a natural chain map $DerL \to Der_f(M, L)$, and we may consider the mapping cone $Der_f(M, L)//(f^* o ad)(L)$, which we just write $Der_f(M, L)//(L)$ for short.

Corollary 2.6 Let τ be a twisting morphism in $TW(C, L)$. Restriction to generators gives a natural isomorphism of chain complexes $Der_f(M, L)//L \xrightarrow{\simeq} sHom(C, L)^{\tau}$

Proof. The isomorphism of theorem 2.5 extends to a natural isomorphism of graded vector spaces $\emptyset: Der_f(M, L)//L \xrightarrow{\simeq} sHom(C, L)^{\tau}$ where $\emptyset(sx)$ for $x \in L$, is the (suspension of the) linear map which annihilates $\bar{c}$ and on the counit is given by $\emptyset(sx)(1) = x$. We check that this extension is still a chain map. On one hand

$-\partial^{\tau}\emptyset(sx)(1) = (\partial\emptyset(sx)) + [\tau, \emptyset(sx)](1)$
$= d_L\emptyset(sx)(1) - (-1)^{|\emptyset(sx)|}\emptyset(sx)d_c(1) + [\tau(1), \emptyset(sx)(1)]$
$= d_L(x)$
$\partial^{\tau}\emptyset(sx)(c) = (\partial\emptyset(sx)) + [\tau, \emptyset(sx)](c)$
$= -(-1)^{|\emptyset(sx)|}\emptyset(sx)d_c(c) + (-1)^{|c||x|}[\tau(c), \emptyset(sx)(1)]$
$= (-1)^{|c||x|}[\tau(c), x]$ for $c \in \bar{C}$.
On the other hand,
$\emptyset(\partial\emptyset(sx))(1) = \emptyset(ad_x \pm sd_L(x))(1)$
$= -d_L(x)$
$\emptyset(\partial(sx))(c) = \emptyset(ad_x f - sd_L(x))(c)$
$= \emptyset(ad_x f)(c)$
$= (-1)^{|x|}[x, f(s^{-1}c)]$
$= (-1)^{|x|+(|c|+1)|x|+1}[\tau(c), x]$
$= (-1)^{|c||x|+1}[\tau(c), x]$

The formula we have given is for a map to $Hom(C, L)^{\tau}$, and as such, has degree -1. Thus, the calculation shows that $\emptyset$ is a chain map.

Illustration I. Let C be a coaugumented dg co-comummutative coalgebra and let L be a dg Lie algebra, such that C or L is finite, or C^v and L are both bounded above or both bounded below. Write $A := C^v$. We observe that

I. $Der\,\ell(C) = Der_{id}(\ell(C), \ell(C))$

II. $Der\,\ell(C) \overset{r}{\rightarrow} sHom(\bar{C}, \ell(C))^{l} \rightarrow s\bar{A} \otimes_{\varphi^{-1}} \ell(C)$

III. $\dfrac{Der\frac{\ell(C)}{}}{\ell(C) \overset{r}{\rightarrow} sHom(\bar{C}, \ell(C))^{l} \overset{\varphi^{-1}}{\rightarrow} s\bar{A} \otimes_{\varphi^{-1}(l)} \ell(C)}$

IV.
$$\begin{array}{ccc} s\bar{A}\otimes_{\varphi^{-1}(l)}\ell(c) & \xrightarrow{|\otimes f} & s\bar{A}\otimes_{\varphi^{-1}(\tau)}\ell \\ \simeq\downarrow & & \downarrow\simeq \\ Der\ell(c) & \underset{f_*}{\rightarrow} & Der_f(\ell(c), L) \end{array}$$

V.
$$\begin{array}{ccc} s\bar{A}\otimes_{\varphi^{-1}(l)}\ell(c) & \xrightarrow{|\otimes f} & s\bar{A}\otimes_{\varphi^{-1}(\tau)}\ell \\ \simeq\downarrow & & \downarrow\simeq \\ Der\ell(c)//\ell(c) & \underset{f_*}{\rightarrow} & Der_f(\ell(c), L)//L \end{array}$$

From I, the identity on $L(C)$ correspond to $i \in Tw\,(C, L(C))$ the universal twisting isomorphism (see [11], Appendix B). By Proposition 2.3, 2.5 and 2.6 respectively, we get natural isomorphisms stated in II and III combining these natural isomorphisms with the maps of theorem 2.2, we get the commutative diagram of complexes in IV and V which naturally ensures that $(|\otimes f)(\varphi^{-1}(l)) = \varphi^{-1}(\tau)$. We will suppress the natural isomorphism μ in the notation from now on. Under the correspondence expressed by these diagrams (IV and V), the contraction produced in Theorem 2.2 is the same as [26]. In particular, the positive parts of the diagram IV and V above provide contractions of the schlessinger-stasheff classifying dg Lie algebra for dg Lie algebra fibrations with kernels is quasi-isomorphic to $\ell(C)$. There is no need to treat the case $\bar{A}$ (modeling B Aut $(X)\langle 1\rangle$, (see [15], Corollary VII 4 (4)) separately from the case with A. We may think of $s\bar{A}\otimes_k L$ as a sub-complex of $s\bar{A}\otimes_k L$, at least until we reach homology. Thus we proceed with the only one case.

Remark 2.7 Theorem 2.2 illustrates the need for A to be nilpotent, and this assumption on A is carried throughout this paper unless otherwise we state a completely different techniques. Our application of theorem 2.2 will be for $M = L(C)$ and the corresponding koszul morphism in the [3, 5, 10, and 19]. Nilpotency of A may be replaced by nilpotency of L as follows. It was shown in [15 and 16] that Der M_x is a Lie model for B Aut $X\langle 1\rangle$ simply connected to a space X, when M_x is the minimal Sullivan model for X. For a koszul space X, dualizing the bar construction $\ell(L)^v$ give the minimal Sullivan model for X. The injective quasi-isomorphism $L' \rightarrow \ell(L)$ (see [10]) gives rise to a contraction.

$$h \circlearrowleft \ell(L)^v \underset{g}{\overset{f}{\rightleftarrows}} A.$$

If L is nilpotent, then for any Maurer-Cartan element $\tau \in Mc(\ell(L)^v \otimes L)$, we get an induced contraction

$$h \circlearrowleft \ell(L)^v \otimes_{\tau} L \rightleftarrows A \otimes_{(1\otimes f)(\tau)} L \qquad (8)$$

The formulae in Theorem 2.2, converges because L is nilpotent. As analogous to the above, we have any isomorphism of dg Lie algebras $Der\ell(L)^v \simeq \ell(L)^v \otimes_\tau L$. The L_∞- structure transferred to $sA \otimes_{(1\otimes f)(\tau)} L$ along (8) will be L_∞-isomorphic to the one we produce below, because there are quasi-isomorphism such that

$Der\ell(c)$ $\overset{\sim}{\to}$ $Der\ell(L)^v$

$\sim\searrow$ $\swarrow\sim$

$sA\otimes_k L$

Commutes (see [22 and 27]).Thus, everything goes through the case where L and not A is nilpotent.

3. Contraction for Koszul Algebras

We now specialised the results of Section 2 to the case of koszul algebras, our primary interest

Theorem 3.1 Let C be a koszul algebra of finite type, with koszul dual graded Lie algebra L_i. Then, the surjective quasi-isomorphism $f: \ell(c) \to L$ associated to the twisting morphism k, give rise to a contraction

$$h \circlearrowleft \ell(L)^v \underset{g}{\overset{f}{\rightleftarrows}} A$$

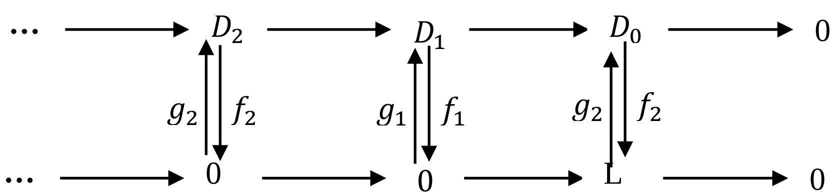
where we may choose h and g to have the following properties

I. The image of g is contained in the diagonal $D\ell = \{x \in \ell(c) | \ell_w(x) = \ell_b(x)\}$
II. h pressure the total weight
III. the contraction satisfies the annihilation conditions: $fh = 0, hg = 0$ and $h^2 = 0$

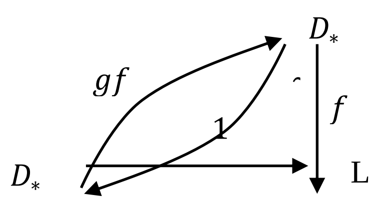
Proof. For I, we note that ℓ is given by projection to D, as the quotient map, is trivially a chain map. It is a section of f, and it has the desired property. For II, consider the bounded below chain complex

$$\cdots \longrightarrow D_2 \longrightarrow D_1 \longrightarrow D_0 \longrightarrow 0$$

The maps f and g then give rise to chain maps



which we also denote f and g. We now have a chain complex of vector spaces (projective modules) D_* and a diagram.



where both gf and 1 are lifts of f along the surjection quasi-isomorphism f. Thus, we have the standard proof of the fact that gf and 1 is homotopy. We proceed by constructing a homotopy.

First, note that $gf - 1$ factors through $kerf$, so we may construct a homotopy $h: D_* \longrightarrow kerf \cap D_*$. That is, a family of maps $h_i: D_i \longrightarrow kerf \cap D_{i+1}$ such that

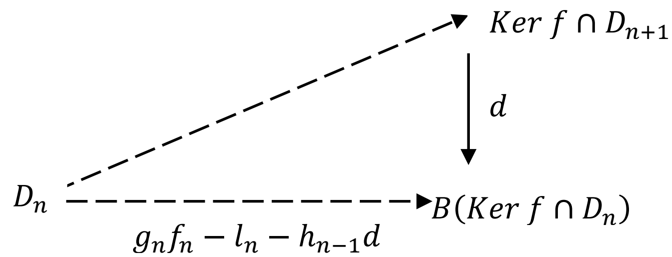
$$dh_i + h_{i-1}d = g_i f_i - 1_i \quad (9)$$

It suffices to show that h_i preserves the total weight and the annihilation conditions are satisfied.

Now, set $h_{-2} = h_{-1} = 0$, since $D_i = 0$ for $i < 0$. Clearly, these maps preserve the total weight and satisfy the annihilation condition. Now for $n \geq 0$, suppose that we have constructed h_i with the desired property for all $i < n$. Consider the map $g_n f_n - 1_n - h_{n-1}d : kerf \cap D_*$. If we apply the differential and use the fact $gf - 1$ is a chain map together with the equation (8), we get

$d\,(g_n f_n - 1_n - h_{n-1}d) = (g_n f_n - 1_n)d - dh_{n-1}d$
$= (g_n f_n - 1_n)d - (g_n f_n - l - h_{n-2}d)d$
$= 0.$

Thus, $g_n f_n - 1_n - h_{n-1}d$ factors through the cycles $Z(kerf \cap D_n)$ which is exactly the boundaries $B(kerf \cap D_n)$. Since f is a quasi-isomorphism. Then we get a diagram



with a lift as indicated because D_n is vector space (thus projective) and the differential is surjective from $kerf$ onto boundaries. Such a lift is just a choice of the pre-images $d^{-1}(d\ (g_n f_n\ -1_n - h_{n-1}d)(x_j)$ for a linear basis $\{x_j\}$ of D_n. The maps $d\ (g_n f_n\ -1_n - h_{n-1}d$ preserve the total weight by part I and the assumption on h_{n-1}. Since d preserves the total weight, we can always choose pre-images such that h_n preserves total weight. Clearly, we have
$f_{n+1}h_n\ = 0$ and $h_n g_n$ is zero for $n > 0$. Now $h_0 g_0$ is a lift of $(g_0 f_0\ -1_0 - h_{n-1}d)\ g_0\ =\ (g_0 f_0\ -1_0)\ g_0 = 0$ along the differential. We may choose h_0 to vanish on $I_m\ g_0$ without violating the condition that h_0 preserves total weight. Similarly $h_{n+1}\ h_n$ is a lift of
$(g_{n+1} f_{n+1}\ -1_{n+1} - h_n d)\ h_n\ =\ -h_n - h_n d h_n$
$=\ -h_n - h_n\ (g_n f_n - l_n - h_{n-1}d)$
$=\ h_n h_{n-1} d$
along the differential. Inductively this is zero, and again we may choose h_{n+1} to be zero on $I_m h_n$ without violating the condition that h_{n+1} preserves total weight. Finally, h_n then satisfies (8) for $i = n$ and by construction, the resulting homotopy h has the properties II and III. End of proof.

It is clear, the dg lie structure on $Der\ell(C)//\ell(C)$ induces one on $sA\otimes_\tau \ell(C)$ by the neutral isomorphism in Proportion 2.3 and corollary 2.6. This is not the same as the one from definition 2.1. By $[-,-]_{Der}$, we mean induced bracket that comes from the derivations. Suppose now that A is nilpotent, by Theorem 2.2 there are contractions along which we may transfer the structure using Theorem 1.5 for $L_\infty - algebras$ to get $L_\infty - structures\ \{\ell_n\}$ and $\{l_n\}$ as below:

$$\begin{array}{ccccc} sA\otimes_\tau \ell(C) & \leftrightarrows & sA\otimes_k L & \leftrightarrows & H_*(sA\otimes_\tau L) \\ \vdots & & \vdots & & \vdots \\ [-,-]_{Der} & & \{\ell_n\} & & \{l_n\} \end{array}$$

It is easy to check that if we transfer the structure further along the contraction to the homology.

$$\circlearrowleft sA\otimes_k L \underset{i}{\overset{g}{\leftrightarrows}} H_*(sA\otimes_k L)$$

We get the same result as we do by transferring the original one along the composed contraction, and the resulting structure can be obtained by composing maps similar to example 1.1. The composed tree is an easy exercises for our readers, hence the omission.

4. Grading's and Transferred Operations

In this section, we refer to elements of bracket length q by $\ell(q)$, and elements in the subspace D_i of bracket length q by $D_i(q)$. The koszul algebra C has a weight grading and the dual algebra $A = C^v$ has an induced weight grading $A(P) = C(P)^v$, preserved by the induced multiplication on A. we still assume that C or L is finite or that $A = C^v$ and L are both bounded in the same direction.

Lemma 4.1 (Weight lemma): The given maps $f$and i, and the maps g and h chosen in Theorem 3.1, interacting with the following weight gradings of A, L and ℓ. Then, for $p \geq 0$, $q \geq 1$ and $i \geq 0$., the following maps are in order.

$f: \ell(q)\ \rightarrow\ L(q)$
$g: L(q)\ \rightarrow\ \ell(q)$
$h: D_i(q)\ \rightarrow\ D_{i+1}(q-1)$
$i: sA(p)\otimes D_i(q)\ \rightarrow\ \bigoplus_{m\geq 1} sA(p+M)\otimes D_{i+m-1}(q+1)$

It should be noted that the notation i is a short hand for the map ad_i using the structure from Definition 2.1.

Proof. By construction of $C^i = L$, the bracket length in L corresponds to that of $\ell\big(C(1)\big) = D$. Since h preserves total weight and raises the offset index for the diagonals, it lowers bracket length by 1. The map i splits as a sum $l = \sum_{m\geq 1} l_m$ with one summand for each weight m is a linear basis for A. The weight is raise by m in A, bracket length by 1 in $\ell(C)$, and raises total weight by m in $\ell(C)$.

As immediate consequence of lemma 4.1, we have the following proposition.

Proposition 4.2 The maps g', h' and $i\otimes f$ interact with the weight gradings of A, L and ℓ as follow. For $p \geq 0$, $q \geq 1$.

$i\otimes f: sA(p)\otimes_k \ell(q)\ \rightarrow\ sA(p)\otimes_k L(q)$
$g': sA(p)\otimes_k L(q)\ \rightarrow\ sA(\geq p)\otimes_k \ell(q)$
$h': sA(p)\otimes_k \ell(q)\ \rightarrow\ sA(\geq p)\otimes_k \ell(q-1)$

Proof. By Theorem 2.2 we can identify $g'\ and\ h'$ from g, h and i.

$g' =\ \sum_{i\geq 0}((1\otimes h)_i)^i\ (1\otimes g)$
$h' =\ \sum_{i\geq 0}((1\otimes h)_i)^i\ (1\otimes h)$ (9)

The Proposition 4.2 now follows by combining these formulae with Lemma 4.1.
Similar properties can be deduced for the maps in the contraction to the homology.

Proposition 4.3 There is a contraction $\circlearrowleft sA\otimes_\tau L \underset{i}{\overset{q}{\leftrightarrows}} H_*(sA\otimes_k L)$ such that q and i preserves the weight grading in L and the homotopy k decreases the weight by 1 in both A and L.
It is easy to check the splitting in Lemma 1.10, can be chosen such that the contraction produced in that Lemma has the desired properties. It suffices to show how the transferred $L_\infty - structure$ interacts with the weight gradings. Now, we write ℓ for short of $\ell(C)$, just for the sake of notational convenient. We begin by given a formula for the Lie bracket $[-,-]_{Der}$ on $A\otimes\ell$.

Definition 4.4 For any $a \in \bar{A}$ and $x \in \ell$, denote by $x\frac{\partial}{\partial a}$, the unique derivations on $\ell(C)$ extending the linear map $\varphi(sa \otimes x)$: $s^{-1}C \to \ell(C)$ given by
$\varphi(sa \otimes x)(s^{-1}C) = (-1)^{|c||x|+|a|}a(c)x$ is in order (see., Proposition 2.3).
Theorem 4.5 The Lie bracket on $sA\otimes_k\ell$ induced by the isomorphism $Der||\ell \simeq sA\otimes_l\ell$ is given by

$$[sa\otimes x, sb\otimes y]_{Der} = \begin{cases} (-1)^\alpha sb\otimes x\frac{\partial}{\partial a}y - (-1)^\beta sa\otimes y\frac{\partial}{\partial b}x & a,b \in \bar{A} \\ (-1)^{|a|+|x|+|}s\,1\otimes x\frac{\partial}{\partial a}y & a\in\bar{A}, b\in A(0) \\ 0 & a,b\in A(0)\end{cases} \quad (10)$$

where $x, y \in \ell(C)$ have signs given by
$\alpha = (|x| + |a|)(|b|+1)+1$
$\beta = |x|(|y| + |b| + 1) + |a|$

Proof. Let $\{C_i\}_i$ be a basis for $\bar{C}$. The bracket $[-,-]_{Der}$ is by definition, the composition is $\varphi^{-1}o[-,-]o(\varphi\otimes\varphi)$. For $x, y \in \ell$ and basis elements $, b \in \bar{A}$, we get

$$[sa\otimes x, sb\otimes y]_{Der} = \sum_i (-1)^c sc_i^*\otimes[x\frac{\partial}{\partial a}, y\frac{\partial}{\partial b}](s^{-1}c_i)$$

where the sign is given by

$$\in = \left(\left|x\frac{\partial}{\partial a}\right| + \left|y\frac{\partial}{\partial b}\right|\right)|sc_i^*| + 1$$
$$= (|x| + |a| + |y| + |b|)\,|sc_i^*| + 1$$

According to the Remark 2.4, we have the following evaluation

$$\left[x\frac{\partial}{\partial a}, y\frac{\partial}{\partial b}\right](sc_i^*) = \left(x\frac{\partial}{\partial a}o\,y\frac{\partial}{\partial b} - (-1)^{(|x|+|a|+1)(|y|+|b|+1)}y\frac{\partial}{\partial b}o\,x\frac{\partial}{\partial a}\right)(sc_i^*).$$

Clearly, the first term is non-zero only if $c_i^* = b$, and second term is non-zero only if $c_i^* = a$. Thus the sum reduces to

$$(-1)^{(|b|+1)(|x|+|y|+|a|)+1}sb\otimes x\frac{\partial}{\partial a}o\,y\frac{\partial}{\partial b}(s^{-1}b^*)$$
$$-(-1)^{(|x|+|a|+1)(|y|+|b|+1)+(|a|+1)(|x|+|y|+|b|)+1}sa\otimes y\frac{\partial}{\partial b}o\,x\frac{\partial}{\partial a}(s^{-1}a^*)$$
$$= (-1)^{(|x|+|a|)(|b|+1)+1}sb\otimes x\frac{\partial}{\partial a}y - (-1)^{|x|(|y|+|b|+!)+|a|}sa\otimes y\frac{\partial}{\partial b}x$$

Next, let $x, y \in \ell(C)$ and let $b = y \in A(0)$. Let $\varphi(sL\otimes y)$ be a linear map $s^{-1}c \to L$ which is non-zero only on $c(0) \simeq Q$, and $\varphi(s\,1 \otimes y)(1) = sy$. Thus,

$$[-,-]o(\varphi\otimes\varphi)(sa\otimes x\otimes s\,1\otimes y) = \left[x\frac{\partial}{\partial a}, sy\right] = (-1)^{|a|+|x|+|}sx\frac{\partial}{\partial a}y.$$

By the definition of the bracket restricted to $Der\,L\otimes sL$. The inverse φ^{-1} on $s\ell$ is given by $sx \to s\,1\otimes x$, so we get

$$[sa\otimes x, st\otimes y]_{Der} = (-1)^{|a|+|x|+|}s\,1\otimes x\frac{\partial}{\partial a}y$$

Finally, for a = b = 1 ∈ A(0), we have $[sx, sy] = 0$, and thus $[sa\otimes x, sb\otimes y]_{Der} = 0$.
Lemma 4.6 The bracket $[-,-]_{Der}$ interacts with the weight gradings of A and L as follows. For $P_2 \geq 0$ and $q_1q_2 \geq 1$

$$sA(P_1)\otimes_l\ell(q_1)\otimes sA(P_2)\otimes_l\ell(q_2)$$
$$\downarrow$$
$$sA(P_1)\otimes_l L(q_1 + q_1 - 1)\oplus sA(P_2)\otimes_l L(q_1 + q_2 - 1)$$

Proof. The composition $y\frac{\delta}{\delta b}o\,x\frac{\delta}{\delta a}$ is given by $y\frac{\delta}{\delta b}o\,x\frac{\delta}{\delta a}(s^{-1}c) = (-1)^{|c||x|+|a|}a(c)y\frac{\delta}{\delta b}(x)$ and the recursive formula.

$$y\frac{\partial}{\partial b}(x) = \{(-1)^{|x||y|+|b|}\ b(x)y, x \in \ell(1)$$
$$= s^{-1}c\ \left(\left[y\frac{\partial}{\partial b}(x_1), x_2\right] + (-1)^{\left|y\frac{\partial}{\partial b}\right||x_1|}\left[x_1, y\frac{\partial}{\partial b}(x_2)\right]\right), x = [x_1, x_2] \quad (12)$$

Clearly, the weight of $y\frac{\partial}{\partial b}\circ x\frac{\partial}{\partial a}$ (c) is the sum of weights of x and y minus 1. Furthermore, all terms of (10) for which $a(c_i) = 0$ vanish. Therefore all the sc_i^* appearing in the resulting sum will have the same weight as sa. The same holds mutatis mutandis and $x\frac{\partial}{\partial a}\circ y\frac{\partial}{\partial b}$.

Example 4.7 Let $\{a_i\}$ be a basis for A, and let $\{c_i\}$ be the dual basis for c. by calculating the first term of the bracket $\left[sa_1 \otimes [c_1, c_2], sa_2 \otimes [c_1, [c_2, c_3]]\right]_{Der}$ (but pave out the signs), we have

$$sa_1 \otimes [c_1, c_2] \frac{\partial}{\partial sa_2} [c_1, [c_2, c_3]]$$
$$= sa_1 \otimes \left([[c_1, c_2] \frac{\partial}{\partial sa_2} (c_1), [c_2, c_3] \ + \left[c_1, [c_2, c_3] \frac{\partial}{\partial sa_2} [c_2, c_3]\right]\right)$$
$$= sa_1 \otimes \left(\left[c_1, \left[[c_1, c_2] \frac{\partial}{\partial sa_2} (c_2), c_3\right]\right] + \left[c_1, \left[c_{2,}[c_1, c_2] \frac{\partial}{\partial sa_2} \ (c_3)\right]\right]\right)$$
$$= sa_1 \otimes [c_1, [[c_1, c_2] c_3]]$$

Clearly, we have scanned the word $[c_1, [c_2, c_3]]$ for occurrences of the letter $a_2^* = c_2$, and replaced it with the word $[c_1, c_2]$. It is easy to see how $sa_1$is preserved in the first term and that the bracket lengths in $\ell(c)$ goes from 2 + 3 = 5 inputs 4 = 5 - 1 in the output. The second term is computed in the same way.

Definition 4.8 The complex $sA \otimes_k L$ is bigraded by weight in A and L. The shifted weight grading is the bigrading which degree (p, q) is $sA(PH) \otimes_k L(q+1)$ for $p \geq 0$.

The differential k has bi-degree $(1,1)$ in the standard weight, and so also in the shifted weight grading. This is a special case of the following theorem on the entire L_∞-structure.

Theorem 4.9 (Technical Result) Let C be a koszul graded cocummutative coalgebra with koszul dual graded Lie algebra L, such that the linear dual $A = C^v$ is nilpotent. The L_∞-structure on $sA \otimes_k L$ transferred from the derivations $Der\ell(C) // \ell(C)$ through the chosen contraction, respect the shifted weight grading in the sense that for any $r \geq 1$, the operation ℓ_r has a bidegree $2-r, 2-r$.

Proof. We now introduced yet another grading for $sA \otimes_k L \ (C)$: The mass $m(sA \otimes x)$ of an element $sA \otimes x$ is the total weight $\ell_w(x)$ of x in $\ell(C)$ minus the weight $w(a)$ of a in A. We verify that, h, i and $[-,-]_{Der}$ preserve the mass grading. It is straight forward to see that h and i preserve the mass by Lemma 4.1. From the formulae (10) and (12), we see that $[-,-]_{Der}$ preserves the mass. In general case we get

$$m([sa \otimes x, sb \otimes y]_{Der}) = \ell_w(x) - w(a) + \ell_w(y) - w(b)$$
$$= \ell_w(x) - w(b) + \ell_w(y) - w(a)$$
$$= \ell_w(y) - w(a) - \ell_w(x) - w(b)$$

where second and third line is the mass of respectively first and second term of the right hand side expression of (10) in the first case. The other cases are similar. For f and g, the total weight in $L(C)$ agrees with the weight in L since f vanish outside D_o and $I_m q$.

Thus, the interaction of $f$and g preserves the mass grading. In particular, the maps $1 \otimes f, g'$ and h' all preserve the mass grading. Now consider the operation

$$\otimes_{k=1}^{T} = \left.\begin{array}{c} sA(P_k) \otimes_k L(q_k) \\ \downarrow \\ sA \otimes_k L \end{array}\right\} \text{all mass preserve.}$$

There coincides weight grading L with the total weight grading and D_o. Then we have for any element in the source that

A-weight for $\ell_r(x)$ = L-weight of $\ell_r(x)$ – mass of $\ell_r(x)$

= L-weight of $\ell_r(x)$ – mass of x

where the mass of $x = \sum_{k=1}^{r} a_k - p_k$, and by Lemma 4.11 below, the image $\ell_r(x)$ has weight $\sum_{k=1}^{r} a_k - 2r + 3$ in L.

Thus, the image $\ell_r(x)$ has weight $\sum_{k=1}^{r} a_k - 2r + 3 - \sum_{k=1}^{r} (a_k - p_k) \ = \sum_{k=1}^{r} p_k - 2r + 3$

in A, and ℓ_r bi-degree (2-r, 2-r) in the shifted weight.

Remark 4.10 Clearly, none of the maps defining the transferred operations decrease the weight in A. Thus, the condition $\sum_{k=1}^{r} p_k \geq 2r - 3$ gives a lower bound on the weight in A for where ℓ_r is non-zero. ℓ_3 restricted to $A(1) \otimes L$ is zero. In the shifted weight grading, it is an operation from copies of weight (0,*) to (-1,*0), but then one of the maps defining ℓ_3 would have lowered the weight in $A. L_\infty - structure$ in Theorem 4.9 respect the grading. However, there is no priori connection to the homological grading in what is discussed.

Lemma 4.11 For $r \geq 1$, the operations ℓ_r on $sA \otimes_k L$, interacts with the weight grading of L as follows

$$\begin{array}{c} \otimes_{k=1}^{T} \ sA \otimes_k L(q_k) \\ \downarrow \ell_r \\ sA \otimes_k L(\sum_{k=1}^{r} a_k - 2r + 3) \end{array}$$

Proof. By Theorem 1.5, ℓ_r is given by composing along binary rooted tree with r leaves decorated maps as established earlier. For each vertex, we apply the bracket, and for each internal edge, we apply the homotopy. The bracket length decrease in ℓ by 1 and there are (r-1)t (r-2) = 2r-3 vertices and internal edges. The other maps do not change the bracket length in $\ell \ or \ L$.

If we consider only the transferred binary operation we have the following

Corollary 4.12 Consider the graded anti-commutative (non-associative) algebra $(sA\otimes L, \ell_2)$. Then, we have the following

I. $sA\otimes_k L$ (1) is a subalgebra of $sA\otimes_k L$
II. $sA\otimes \quad _k L\ (j)$ is a module over $sA\otimes \quad _k L(1)$ for $j \geq 0$
III. $\oplus_{j\geq m} sA\otimes_k L(j)$ is a subalgebra of $sA\otimes_k L, \forall\ m \geq 0$
IV. $sA(1)\otimes_k L$ is a subalgebra of $sA\otimes_k L$
V. $sA(i)\otimes_k L$ is a module over $sA(1)\otimes_k L$ for $i \geq 0$
VI. $\oplus_{i\geq m} sA(i)\otimes_k L$ is a subalgebra of $sA\otimes_k L, \forall\ m \geq 0$

5. Final Remark and Conclusion

We have produced a smaller $L_\infty - model$ for the cover of the classifying space of the homotopy automorphism, and the contraction. Explicitly,

$$h' + g'kf \circlearrowleft sA\otimes_k L \rightleftarrows H(sA\otimes_k L)_*$$

$$g' = \sum_{j\geq 0} \big((1 \otimes h)i\big)^j (1 \otimes g)$$

$$h' = \sum_{j\geq 0} \big((1 \otimes h)i\big)^j (1 \otimes h)$$

The map i splits as a sum $l = \sum_{j\geq 0} l_m, l_m$ increases the total weight in L by m and bracket length by 1. The homotopy h preserves total weight and decreases bracket length by 1. So, for any $j \geq 1$, the map $\big((1\ 0h)i\big)^j$ preserves bracket length and strictly increases total weight. In our case $A(p) = 0$ for $p \geq 3$, so only the diagonals D_0 and D_1 are non-zero, and in particular $(1 \otimes h)i\ (1 \otimes h) = 0$. In conclusion we have

$$g' = 1 \otimes g\ + (1 \otimes h)i\ (1 \otimes g)$$

$$g' = 1 \otimes h$$

The map i is given by a choice of cyclic representatives for the homology, and the map g as produced by Theorem 3.1 is given by a choice of basis for L. cycle representative may be consider our next work, while the unmixed basis for L gives a choice of g. Finally, h is inductively constructed according to the proof of Theorem 3.1. It is our hope to show these constructions in future papers. The trees in figure 1.1 are simply a composition. Before the contraction in (a) no leaf was shared, but after the contraction, it loses most of its leafs represented by g. Thus, summing all vertexes in (a) is equivalence to that of (b), and their composition gives a complete isomorphism. Nonetheless, One may investigate what can be said in general about the $L_\infty - structure$ on the homology of the model, upto degree zero. There are two ways of approaching this. The first approach relies on [6], and the second is by studying a kan complex associated to the Maurer-Cartan elements of $A\otimes L$. The following literatures [7, 20, 24, and 25] may be of help.

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