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ON ROW-PRODUCTS OF FUZZY SOFT MATRICES

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Abstract

In this paper, we introduce the concept of row-products of fuzzy soft matrices. We define new identities and complements of fuzzy soft matrices and construct examples. We also investigate basic properties of these concepts.

Keywords: Soft matrices, fuzzy soft matrices Row-products .

Introduction

Most mathematical tools usually employed in modelling and computational methods need certain and precise data to test the validity of these tools in order to better understand the system of interest.This is quite challenging since uncertainty and imprecision of data are common in practical problems within the fields such as medical science, social science, engineering, environment. In the last decades, theories of fuzzy sets [1], vague sets [2], and interval mathematics [3] were developed to handle the difficulties associated with the imprecision of data. In spite of many research findings being discovered, difficulties with uncertain and imprecise data still persist.

In 1999, D. Molodtsov pioneered the idea of soft set theory as a mathematical tool to deal with uncertainties which existing mathematical principle cannot handle. The applications of this idea as identified in [4] are in the areas of game theory and operations research, Riemann integration, stability and regularization. In addition to these, R. Roy et al [5] pioneered the study of operations of soft set in which the De Morgan laws and others were verified.

The study of algebraic soft substructures of rings, fields, and modules was done by Akin and Aslihah[6]. The notions of soft subring and soft ideal of a ring, soft subfield of a filed and soft submodule of a module were also investigated. The ideas of soft mappings were introduced by Pinaki and Samanta [7]. They further study the images and inverse images of crisp sets and soft sets under soft mappings. Fend et al [8] initiated the idea of soft semirings by using the soft set theory.

Won Keun Min in [9] gave a note on soft topological spaces that was an improvement on the work of Shabir and Naz [10] and investigated properties of soft separation axioms defined in [11]. Soft regular spaces and some of their properties were also verified. Koyuncu et al in [12] introduced initial concepts of soft rings and initial basic properties of soft ideals, soft homomorphism etc. Lee et al [13] applied the theory of soft sets to an algrebraic structure, called d-algebras.

Segzin and Atagun [14] introduced the concepts of normalistic soft group and normalistic group homomorphism and investigated the several related properties. Wen Xiang Gu et al [15] initiated the study of algebraic hyperstructures of soft sets, the concepts of soft polygroups and some related properties were also investigated. Molodtsov et al in [16] considered the stability of sets given by constraints within the context of the theory of soft sets. Won Keun Min [17] introduced the concept of similarity between soft sets which is an extension of the equality for soft theory.

Soft matrix theory and its applications in decision making was investigated by Cagman and Enginoglu in [18] which are more functional to make theoretical studies in the soft set theory.

Fuzzy soft set theory and its applications was introduced in [19] by Cagman et al. The corresponding fuzzy soft matrix and its applications was introduced by the same authors in [20].

The rest of the paper is presented as follows. The preliminaries are highlighted in section 1. The generalized product of fuzzy soft matrices are presented in section 2. The concept of row-products operation on fuzzy soft matrices are discussed with examples in section 3. In section 1, the identities, complements of fuzzy soft matrices and the investigated properties of the concepts introduced in the paper are presented.

1 Preliminaries

Definition 1 ([18]) Let U be the set of alternatives, E the set of all parameters and $A \subseteq E$. A soft set F_A over U is defined by the set of ordered pairs

$F_A = \{(e, f_A(e)) : e \in E, f_A(e) \in P(U))\}$

where $f_A : A \to P(U)$ such that $f_A(e) = \emptyset$ if $e \notin A$ and P(U) is the power set of U.

In this definition, f_A is called the approximate function of the soft set F_A, and the value $f(e)$ is a set called e-element of the soft set $\forall e \in E$.

Definition 2 ([1]) Let U be an initial universe. A fuzzy set X over U is a set defined by a function μ_x representing a mapping $\mu_x U \to [0,1]$ μ_x is called the membership function of X and the value μ_x is called the degree of membership of $u \in U$. Thus, a fuzzy set X over U can be represented as follows:

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$\left\{\left(\frac{\mu_x(u)}{u}\right): u \in U, \mu_x \in [0,1]\right\}$

The set of all fuzzy sets over U will be denoted as F(U).

Definition 3 [19] A fuzzy soft set over U is a set of ordered pairs:

$\Gamma_A = \{(x, \gamma_A(x)): e \in E, \gamma_A(x) \in F(U)\}$

where $\gamma_A E \rightarrow F(U)$ such that $\gamma_A(x) = \emptyset$ if $x \in A$. γ_A is called fuzzy approximate function of the fuzzy soft set(fs-set) Γ_A, and the value $\gamma_A(x)$ is a set called x-element of the fs-set $\forall x \in E$. The set of all fs-set over U will be denoted as FS(U).

Example 1: Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be an initial universe and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters and $A = \{e_2, e_3, e_4, e_5\} \subseteq E$. Let $\gamma_A(e_2) = \left\{\frac{0.7}{u_1}, \frac{0.2}{u_6}\right\}$, $\gamma_A(e_3) = \left\{\frac{0.0}{u_4}, \frac{0.8}{u_5}\right\}$, $\gamma_A(e_4) = \emptyset$,

$\gamma_A(e_5) = U$, then the fuzzy soft set is given by:

$\Gamma_A = \left\{\left(e_2, \left\{\frac{0.7}{u_1}, \frac{0.2}{u_6}\right\}\right), \left(e_3, \left\{\frac{0.0}{u_4}, \frac{0.8}{u_5}\right\}\right), (e_4, \emptyset), (e_5, U)\right\}$

Example 2 Let $U = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be initial universe. Here, $\alpha_i, i = 1, \ldots, 4$ are symptoms of diseases, where α_1 = high fever, α_2 = nausea, α_3 = fatigue, α_4 = catarrh. Also let $E = \{\sigma_1, \sigma_2, \sigma_3\}$, where σ_1 = Malaria, σ_2 = Yellow fever, σ_3 = pneumonia. The set of each disease is constructed below:

$\gamma_E(\alpha_1) = \left\{\frac{0.8}{\sigma_1}, \frac{0.7}{\sigma_2}, \frac{0.8}{\sigma_3}, \frac{0.4}{\sigma_4}\right\}$, $\gamma_E(\alpha_2) = \left\{\frac{0.9}{\sigma_1}, \frac{0.8}{\sigma_2}, \frac{0.8}{\sigma_3}, \frac{0.3}{\sigma_4}\right\}$, $\gamma_E(\alpha_3) = \left\{\frac{0.4}{\sigma_1}, \frac{0.2}{\sigma_2}, \frac{0.8}{\sigma_3}, \frac{0.6}{\sigma_4}\right\}$

The corresponding fuzzy soft set:

$$\Gamma_A = \left\{\left(\alpha_1, \left\{\frac{0.8}{\sigma_1}, \frac{0.7}{\sigma_2}, \frac{0.8}{\sigma_3}, \frac{0.4}{\sigma_4}\right\}\right), \left(\alpha_2, \left\{\frac{0.9}{\sigma_1}, \frac{0.8}{\sigma_2}, \frac{0.8}{\sigma_3}, \frac{0.3}{\sigma_4}\right\}\right), \left(\alpha_3, \left\{\frac{0.4}{\sigma_1}, \frac{0.2}{\sigma_2}, \frac{0.8}{\sigma_3}, \frac{0.6}{\sigma_4}\right\}\right)\right\}$$

Throughout this work, the corresponding fuzzy soft matrix of any fuzzy soft set, Γ_A, will be denoted by $[\Gamma_A]$. Thus, the corresponding fuzzy soft matrix(fs-matrix) of the above fs-set is given by:

$$[\Gamma_A] = \begin{bmatrix} 0.8 & 0.9 & 0.4 \\ 0.7 & 0.8 & 0.2 \\ 0.8 & 0.8 & 0.8 \\ 0.4 & 0.2 & 0.6 \end{bmatrix}$$

Definition 4: Let $[\Gamma_A] = [a_{iq}] \in FSM_{n\times m}$. Then $[a_{iq}]$ is called

1. a full fs-matrix, denoted by [1], if $a_{iq} = 1 \forall i$ and q.
2. a null fs-matrix, denoted by [0], if $[a_{iq}] = 0 \forall i$ and q.

Definition 5 ([20]) Let $[a_{iq}], [b_{jq}] \in FSM_{n\times m}$. Then,

1. $[a_{iq}]$ is a fs-submatrix of $[b_{jq}]$, denoted by $[a_{iq}] \tilde{\subseteq} [b_{iq}]$, if $a_{iq} \leq b_{iq} \forall i$ and q.
2. $[a_{iq}]$ is a proper fs-submatrix of $[b_{jq}]$, denoted by $[a_{iq}] \tilde{\subset} [b_{iq}]$, if $a_{iq} \leq b_{iq} \forall i$ and q and for at least one term $a_{iq} < b_{iq}$
3. $[a_{iq}] \tilde{\cup} [b_{iq}]$ is called the union of $[a_{iq}]$ and $[b_{iq}]$, if $c_{iq} = max\{a_{iq}, b_{jq}\} \forall i$ and j.
4. $[a_{iq}] \tilde{\cap} [b_{iq}]$ is called the intersection of $[a_{iq}]$ and $[b_{iq}]$, if$c_{iq} = min\{a_{iq}, b_{jq}\} \forall i$ and j.

2 Generalized Products of Fuzzy Soft Matrices

Let $[\Gamma_A] = [a_{iq}] \in FSM_{n\times m_1}$, $[\Gamma_B] = [b_{ir}] \in \in FSM_{n\times m_2}$ be fs-matrices of the fuzzy soft sets Γ_A and Γ_B over common initial universe U respectively.

Definition 6 ([20]) The generalized And-product of $[a_{iq}]$ and $[b_{ir}]$denoted by $\wedge$ is defined as:

$\wedge: FSM_{n\times m_1} \times FSM_{n\times m_2} \rightarrow FSM_{n\times m_1\times m_2}$

$[a_{iq}], [b_{ir}] \rightarrow [a_{iq}] \wedge [b_{ir}] = [c_{ip}]$

$c_{ip} = \min\{a_{iq}, b_{ir}\}, \quad q = \alpha, \quad p = (\alpha - 1)m_2 + r$

Definition 7 ([20]) The generalized Or-product of $[a_{iq}]$ and $[b_{ir}]$ denoted by $\vee$ is defined as:

$\vee: FSM_{n\times m_1} \times FSM_{n\times m_2} \rightarrow FSM_{n\times m_1\times m_2}$

$[a_{iq}], [b_{ir}] \rightarrow [a_{iq}] \vee [b_{ir}] = [c_{ip}]$

$c_{ip} = \max\{a_{iq}, b_{ir}\}, \quad q = \alpha, \quad p = (\alpha - 1)m_2 + r$

Definition 8 ([20]) The generalized And-Not-product of $[a_{iq}]$ and $[b_{ir}]$ denoted by $\underline{\wedge}$ is defined as:

$\underline{\wedge}: FSM_{n\times m_1} \times FSM_{n\times m_2} \rightarrow FSM_{n\times m_1\times m_2}$

$[a_{iq}], [b_{ir}] \rightarrow [a_{iq}] \underline{\wedge} [b_{ir}] = [c_{ip}]$

$c_{ip} = \min\{a_{iq}, 1 - b_{ir}\}, \quad q = \alpha, \quad p = (\alpha - 1)m_2 + r$

Definition 9 ([20]) The generalized Or-Not-product of $[a_{iq}]$ and $[b_{ir}]$ denoted by $\underline{\vee}$ is defined as:

$\underline{\vee}: FSM_{n\times m_1} \times FSM_{n\times m_2} \rightarrow FSM_{n\times m_1\times m_2}$

$[a_{iq}], [b_{ir}] \rightarrow [a_{iq}] \underline{\vee} [b_{ir}] = [c_{ip}]$

$c_{ip} = \max\{a_{iq}, 1 - b_{ir}\}, \quad q = \alpha, \quad p = (\alpha - 1)m_2 + r$

3 Row-product of Fuzzy Soft Matrices

In this section, four distinguished row-product of fs-matrices are defined. Let $[\Gamma_A]=[a_{iq}]\in FSM_{n_1\times m}$, $[\Gamma_B]=[b_{jq}]\in FSM_{n_1\times m}$ be fs-matrices of the fuzzy soft sets Γ_A and Γ_B over common initial universe U respectively.

Definition 10 The And row-product of $[a_{iq}]$ and $[b_{jq}]$ denoted by $\tilde{\wedge}_r$ is defined by:

$\tilde{\wedge}_r: FSM_{n_1\times m}\times FSM_{n_2\times m}\to FSM_{n_1n_2\times m}$

$[a_{iq}],[b_{jq}]\to[a_{iq}]\,\tilde{\wedge}_r\,[b_{jq}]=[c_{vq}]$

where $c_{vq}=\min\{a_{iq},b_{jq}\}$ such that $i=\alpha$, $v=(\alpha-1)n_2+j$ and α is the least positive integer such that $v\le\alpha n_2$.

Example 3 Let $[a_{iq}]$ and $[b_{jq}]$ be fs-matrices such that $[a_{iq}]\in FSM_{3\times 4}$ and $[b_{jq}]\in FSM_{2\times 4}$ given as follows:

$$[a_{iq}]=\begin{bmatrix}0.8&0&0.4&0\\0&1&0.3&0\\0.3&0&0.2&0\end{bmatrix},\quad [b_{jq}]=\begin{bmatrix}0.1&0&0.7&0.3\\0.5&0&0.2&0\end{bmatrix}$$

To compute $[a_{iq}]\,\tilde{\wedge}_r=[c_{vq}]$, it is important to find $c_{vq}\,\forall v=1,2,\dots,6$ and $q=1,\dots,4$. Assume $\min\{a_{13},b_{23}\}$ is to be computed, it is obvious that $i=\alpha=1$, then $v=2$. Thus, $\min\{a_{13},b_{23}\}=c_{23}$ so that $c_{23}=0.2$. The other inputs of $[c_{vq}]$ can be obtained similarly. So, the matrix of $[a_{iq}]\,\tilde{\wedge}_r=[c_{vq}]$ is obtained as:

$$[c_{vq}]=\begin{bmatrix}0.1&0&0.4&0\\0.5&0&0.2&0\\0&0&0.3&0\\0&0&0.2&0\\0.1&0&0.2&0\\0.3&0&0.2&0\end{bmatrix}$$

Definition 11 The Or row-product of $[a_{iq}]$ and $[b_{jq}]$ denoted by $\tilde{\vee}_r$ is defined by:

$\tilde{\vee}_r: FSM_{n_1\times m}\times FSM_{n_2\times m}\to FSM_{n_1n_2\times m}$

$[a_{iq}],[b_{jq}]\to[a_{iq}]\,\tilde{\vee}_r\,[b_{jq}]=[c_{vq}]$

where $c_{vq}=\max\{a_{iq},b_{jq}\}$ such that $i=\alpha$, $v=(\alpha-1)n_2+j$ and α is the least positive integer such that $v\le\alpha n_2$.

Definition 12 The And-Not row-product of $[a_{iq}]$ and $[b_{jq}]$ denoted by $\underline{\tilde{\wedge}}_r$ is defined by:

$\underline{\tilde{\wedge}}_r: FSM_{n_1\times m}\times FSM_{n_2\times m}\to FSM_{n_1n_2\times m}$

$[a_{iq}],[b_{jq}]\to[a_{iq}]\,\underline{\tilde{\wedge}}_r\,[b_{jq}]=[c_{vq}]$

where $c_{vq}=\min\{a_{iq},1-b_{jq}\}$ such that $i=\alpha$, $v=(\alpha-1)n_2+j$ and α is the least positive integer such that $v\le\alpha n_2$.

Definition 13 The Or-Not row-product of $[a_{iq}]$ and $[b_{jq}]$ denoted by $\underline{\tilde{\vee}}_r$ is defined as:

$\underline{\tilde{\vee}}_r: FSM_{n_1\times m}\times FSM_{n_2\times m}\to FSM_{n_1n_2\times m}$

$[a_{iq}],[b_{jq}]\to[a_{iq}]\,\underline{\tilde{\vee}}_r\,[b_{jq}]=[c_{vq}]$

where $c_{vq}=\max\{a_{iq},1-b_{jq}\}$ such that $i=\alpha$, $v=(\alpha-1)n_2+j$ and α is the least positive integer such that $v\le\alpha n_2$.

Example 4 Let $[a_{iq}]$ and $[b_{jq}]$ be fs-matrices such that $[a_{iq}]\in FSM_{3\times 4}$ and $[b_{jq}]\in FSM_{2\times 4}$ as given in example 3. The other row-products are computed below.

$$[a_{iq}]\,\tilde{\vee}_r\,[b_{jq}]=\begin{bmatrix}0.8&0&0.7&0.3\\0.8&0&0.4&0\\0.1&1&0.7&0.3\\0.5&1&0.3&0\\0.3&0&0.7&0.3\\0.5&0&0.2&0\end{bmatrix},\quad [a_{iq}]\,\underline{\tilde{\wedge}}_r\,[b_{jq}]=\begin{bmatrix}0.8&0&0.3&0\\0.5&0&0.4&0\\0&1&0.3&0\\0&1&0.3&0\\0.3&0&0.2&0\\0.3&0&0.2&0\end{bmatrix}$$

$$[a_{iq}]\,\underline{\tilde{\vee}}_r\,[b_{jq}]=\begin{bmatrix}0.9&1&0.4&0.7\\0.8&1&0.8&1\\0.9&1&0.3&0.7\\0.5&1&0.8&1\\0.9&1&0.3&0.7\\0.5&1&0.8&1\end{bmatrix}$$

Remark 1 It is worthy to note that the commutative property does not hold under the row-product operations.

4 Identities and Complements of Fuzzy Soft Matrices

4.1 Identities of Fuzzy Soft Matrices

In this section, three types of identities for fs-matrices are presented. Let $[a_{iq}]\in FSM_{1\times m}$ and $[b_{jq}]\in FSM_{n\times m}$.

1. $[a_{iq}]\,\tilde{\vee}_r\,[b_{jq}]=[b_{jq}]$ if $\forall$ i, j and q, $a_{iq}\le b_{jq}$. In this case, $[a_{iq}]$ is called a row-identity fs-matrix of $[b_{jq}]$ with respect to And row-product operation.

For instance, let

$$[a_{iq}]=[0.4\quad 0.1\quad 0.3]\text{ and }[b_{jq}]=\begin{bmatrix}1&0.5&0.7\\0.4&0.2&0.4\\1&0.3&0.8\end{bmatrix}.$$

So,

$$[a_{iq}]\,\widetilde{\vee}_r\,[b_{jq}] = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.4 \\ 1 & 0.3 & 0.8 \end{bmatrix}$$

2. $[a_{iq}]\,\widetilde{\wedge}_r\,[b_{jq}] = [b_{jq}]$ if ∀ i, j and q, $a_{iq} \geq b_{jq}$. In this case, $[a_{iq}]$ is a called a row-identity fs-matrix of $[b_{jq}]$ with respect to the Or row-product operation. The following illustrates the definition clearly. Let $[a_{iq}] = [0.4 \quad 0.7 \quad 1]$ and $[b_{jq}] = \begin{bmatrix} 0.3 & 0.2 & 1 \\ 0.4 & 0.3 & 0.5 \\ 0.1 & 0 & 0 \end{bmatrix}$.

$$[a_{iq}]\,\widetilde{\wedge}_r\,[b_{jq}] = \begin{bmatrix} 0.3 & 0.2 & 1 \\ 0.4 & 0.3 & 0.5 \\ 0.1 & 0 & 0 \end{bmatrix}$$

3. If $[a_{iq}] = [0]$, then $[a_{iq}]\,\widetilde{\vee}_r\,[b_{jq}] = [b_{jq}]$ ∀ i, j and q. In this case, $[a_{iq}]$ is called null identity fs-matrix of $[b_{jq}]$ with respect to the Or row-product operation. Let $[a_{iq}] = [0 \quad 0 \quad 0]$ and $[b_{jq}]$ as in definition 2 above, then

$$[a_{iq}]\,\widetilde{\vee}_r\,[b_{jq}] = \begin{bmatrix} 0.3 & 0.2 & 1 \\ 0.4 & 0.3 & 0.5 \\ 0.1 & 0 & 0 \end{bmatrix}$$

4. If $[a_{iq}] = [1]$, then $[a_{iq}]\,\widetilde{\wedge}_r\,[b_{jq}] = [b_{jq}]$ ∀ i, j and q. In this case, $[a_{iq}]$ is called a full identity fs-matrix of $[b_{jq}]$ with respect to the And row-product operation. Let $[a_{iq}] = [1 \quad 1 \quad 1]$ and $[b_{jq}]$ as in definition 2 above, then

$$[a_{iq}]\,\widetilde{\wedge}_r\,[b_{jq}] = \begin{bmatrix} 0.3 & 0.2 & 1 \\ 0.4 & 0.3 & 0.5 \\ 0.1 & 0 & 0 \end{bmatrix}$$

4.2 Complements of Fuzzy Soft Matrices

Let $[\Gamma_A] = [a_{ij}] \in FSM_{n\times m}$ be a fuzzy soft matrix of the corresponding fuzzy soft set, Γ_A.

Definition 14 The fuzzy soft set $\Gamma_A^o = (\gamma_A, E)^o = (\gamma_A^o, E)$ is a complement of Γ_A, where

$\gamma_A^o: E \to F(U)$

is a function such that $\gamma_A^o(x) = U - \gamma_A(x) \forall x \in E$.

Definition 15 Let $\Gamma_A = (\gamma_A, E)$ be a fuzzy soft set over U. The fuzzy soft set $\Gamma_A^{oA} = (\gamma_A^{oA}, E)$ is called A-complement of Γ_A, where

$\gamma_A^{oA}: E \to F(U)$

is a function such that $\gamma_A^{oA}(x) = U - \gamma_A(x) \forall x \in A$. Thus, $x \notin A$ implies that $\gamma_A^{oA}(x) = \emptyset$.

Definition 16 Let $A \subseteq E = \{e_i: 1 \leq i \leq n\}$, $I_A = \{i: e_i \in A\}$ and $[\Gamma_a] = [a_{ij}] \in FSM_{n\times m}$. The fuzzy soft matrix $[\Gamma_A]^{oA} = [c_{ij}]$ is said to be the A-complement of the fuzzy soft matrix $[\Gamma_A]$ if

$$c_{ij} = \begin{cases} 1 - a_{ij}, & \text{if } j \in I_A ; \\ 0, & \text{if } j \notin I_A . \end{cases}$$

Example 5 Let the set of alternatives $U = \{h_1, h_2, h_3, h_4\}$, the set of the parameters be $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $A = \{e_1, e_3, e_4, e_6\}$.

The fuzzy soft set is given as:

$$\Gamma_A = \left\{\left(e_1,\left\{\frac{h_1}{0.5},\frac{h_2}{0.7},\frac{h_3}{0.8},\frac{h_4}{0.4}\right\}\right),\left(e_3,\left\{\frac{h_1}{0.0},\frac{h_2}{0.7},\frac{h_3}{0.5},\frac{h_4}{0.8}\right\}\right),\left(e_4,\left\{\frac{h_1}{0.3},\frac{h_2}{0.5},\frac{h_3}{0.7},\frac{h_4}{0.1}\right\}\right),\left(e_6,\left\{\frac{h_1}{0.3},\frac{h_2}{1},\frac{h_3}{0.8},\frac{h_4}{0.6}\right\}\right)\right\}$$

The complement of the fuzzy soft set Γ_A:

$$\Gamma_A^o = \left\{\left(e_1,\left\{\frac{h_1}{0.5},\frac{h_2}{0.3},\frac{h_3}{0.2},\frac{h_4}{0.6}\right\}\right),(e_2,U),\left(e_3,\left\{\frac{h_1}{1},\frac{h_2}{0.3},\frac{h_3}{0.5},\frac{h_4}{0.2}\right\}\right),\left(e_4,\left\{\frac{h_1}{0.7},\frac{h_2}{0.5},\frac{h_3}{0.3},\frac{h_4}{0.9}\right\}\right),(e_5,U),\left(e_6,\left\{\frac{h_1}{0.7},\frac{h_2}{0.0},\frac{h_3}{0.2},\frac{h_4}{0.4}\right\}\right)\right\}$$

The computed A-complement:

$$\Gamma_A^{oA} = \left\{\left(e_1,\left\{\frac{h_1}{0.5},\frac{h_2}{0.3},\frac{h_3}{0.2},\frac{h_4}{0.6}\right\}\right),\left(e_3,\left\{\frac{h_1}{1},\frac{h_2}{0.3},\frac{h_3}{0.5},\frac{h_4}{0.2}\right\}\right),\left(e_4,\left\{\frac{h_1}{0.7},\frac{h_2}{0.5},\frac{h_3}{0.3},\frac{h_4}{0.9}\right\}\right),\left(e_6,\left\{\frac{h_1}{0.7},\frac{h_2}{0.0},\frac{h_3}{0.2},\frac{h_4}{0.4}\right\}\right)\right\}$$

In matrix form:

$$[\Gamma_A] = \begin{bmatrix} 0.5 & 0 & 0 & 0.3 & 0 & 0.3 \\ 0.7 & 0 & 0.7 & 0.5 & 0 & 1 \\ 0.8 & 0 & 0.5 & 0.7 & 0 & 0.8 \\ 0.4 & 0 & 0.7 & 0.1 & 0 & 0.6 \end{bmatrix}$$

$$[\Gamma_A]^o = \begin{bmatrix} 0.5 & 1 & 1 & 0.7 & 1 & 0.7 \\ 0.3 & 1 & 0.3 & 0.5 & 1 & 0 \\ 0.2 & 1 & 0.5 & 0.3 & 1 & 0.7 \\ 0.6 & 1 & 0.2 & 0.9 & 1 & 0.4 \end{bmatrix}$$

$$[\Gamma_A]^{oA} = \begin{bmatrix} 0.5 & 0 & 1 & 0.7 & 0 & 0.7 \\ 0.3 & 0 & 0.3 & 0.5 & 0 & 0 \\ 0.2 & 0 & 0.5 & 0.3 & 0 & 0.7 \\ 0.6 & 0 & 0.2 & 0.9 & 0 & 0.4 \end{bmatrix}$$

Proposition 1 Let $[\Gamma_A], [\Gamma_B], [\Gamma_C] \in FSM_{n\times m}$, then:
1. $[\Gamma_A]\,\widetilde{\nabla}_r\,([\Gamma_B] \cup [\Gamma_C]) = ([\Gamma_A]\,\widetilde{\nabla}_r\,\Gamma_B]) \cup [\Gamma_C]$
2. $[\Gamma_A]\,\widetilde{\nabla}_r\,([\Gamma_B] \cap [\Gamma_C]) = ([\Gamma_A]\,\widetilde{\nabla}_r\,\Gamma_B]) \cap [\Gamma_C]$
3. $[\Gamma_A]\,\widetilde{\wedge}_r\,([\Gamma_B] \cup [\Gamma_C]) = ([\Gamma_A]\,\widetilde{\wedge}_r\,\Gamma_B]) \cup [\Gamma_C]$
4. $[\Gamma_A]\,\widetilde{\wedge}_r\,([\Gamma_B] \cap [\Gamma_C]) = ([\Gamma_A]\,\widetilde{\wedge}_r\,\Gamma_B]) \cap [\Gamma_C]$

Proof. Let $[\Gamma_A] = [a_{ij}], [\Gamma_B] = [b_{ij}], [\Gamma_C] = [c_{ij}]$ such that $[a_{ij}], [b_{ij}], [c_{ij}] \in FSM_{n\times m}$. Therefore, $[b_{ij}] \cup [c_{ij}] = [d_{ij}]$ so that $[a_{ij}]\,\widetilde{\nabla}_r\,[d_{ij}] = [e_{vj}]$
where $e_{vj} = max\{a_{ij}, d_{ij}\}$, $i = \alpha$, $v = (\alpha - 1)n + j$ such that α is the least positive integer which satisfies $v \le \alpha n$.
Likewise, let $[a_{ij}\,\widetilde{\nabla}_r\,[b_{ij}] = [f_{wj}]$, where $i = \alpha_2$, $w = (\alpha_2 - 1)n + j$ such that α_2 is the least positive integer which satisfies $w \le \alpha_2 n$. Also, $[a_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}] = [g_{uj}]$, where $i = \alpha_3$, $u = (\alpha_3 - 1)n + j$ such that α_3 is the least positive integer which satisfies $u \le \alpha_3 n$. It is obvious that $[f_{wj}]$ and $[g_{uj}]$ have the same order. So, $[f_{wj}] \cup [g_{uj}] = [h_{yj}]$, with $i = \alpha_4$, $y = (\alpha_4 - 1)n + j$, implies that

$$\begin{aligned} y &= (\alpha_4 - 1)n + j \\ &= (\alpha_3 - 1)n + j \\ &= (\alpha_2 - 1)n + j \\ &= (\alpha_1 - 1)n + j \\ &= (\alpha - 1)n + j \end{aligned}$$

Since $y = w = u = v$. Thus $[e_{vj}] = [h_{yj}]$, such that
$max\{a_{ij}, max\{b_{ij}, c_{ij}\}\} = max\{max\{a_{ij}, b_{ij}\}, c_{ij}\}$
The proofs of 2 - 4 are similar, hence left undone.

Theorem 1: Let $[\Gamma_A], [\Gamma_B], [\Gamma_C] \in FSM_{n\times m}$, then
1. $([\Gamma_A] \cup [\Gamma_B])\,\widetilde{\nabla}_r\,[\Gamma_C] = ([\Gamma_A]\,\widetilde{\nabla}_r\,[\Gamma_C]) \cup ([\Gamma_B]\,\widetilde{\nabla}_r\,[\Gamma_C])$
2. $([\Gamma_A] \cap [\Gamma_B])\,\widetilde{\nabla}_r\,[\Gamma_C] = ([\Gamma_A]\,\widetilde{\nabla}_r\,[\Gamma_C]) \cap ([\Gamma_B]\,\widetilde{\nabla}_r\,[\Gamma_C])$
3. $([\Gamma_A] \cup [\Gamma_B])\,\widetilde{\wedge}_r\,[\Gamma_C] = ([\Gamma_A]\,\widetilde{\wedge}_r\,[\Gamma_C]) \cup ([\Gamma_B]\,\widetilde{\wedge}_r\,[\Gamma_C])$
4. $([\Gamma_A] \cap [\Gamma_B])\,\widetilde{\wedge}_r\,[\Gamma_C] = ([\Gamma_A]\,\widetilde{\wedge}_r\,[\Gamma_C]) \cap ([\Gamma_B]\,\widetilde{\wedge}_r\,[\Gamma_C])$

Proof. Let $[\Gamma_A] = [a_{ij}], [\Gamma_B] = [b_{ij}], [\Gamma_C] = [c_{ij}]$. $[a_{ij}], [b_{ij}] \in FSM_{n\times m}$ implies that $[a_{ij}] \cup [b_{ij}] \in FSM_{n\times m}$. Let $[a_{ij}] \cup [b_{ij}] = [d_{ij}]$, so that $([a_{ij}] \cup [b_{ij}])\,\widetilde{\nabla}_r\,[c_{ij}] = [d_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}]$. Let $[d_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}] = [e_{vj}] \in FSM_{n\times m}$. Thus,

$$\begin{aligned} e_{vj} &= max\{max\{a_{ij}, b_{ij}\}, c_{ij}\} \forall \text{ i and j} \\ &= max\{d_{ij}, c_{ij}\} \end{aligned}$$

where $d_{ij} = a_{aij}$ or b_{ij} such that $i = \alpha$, $v = (\alpha - 1)n + j$ and α is the least positive integer such that $v \le \alpha n$ from definition. Thus, $v = (\alpha - 1)n + j$. Also, $[a_{ij}], [b_{ij}], [c_{ij}] \in FSM_{n\times m}$ implies that $[a_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}]$ and $[b_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}]$ have the same order. Let

$$[a_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}] = [d_{uj}] \quad (1)$$

and

$$[b_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}] = [f_{wj}] \quad (2)$$

such that in (1) $i = \alpha_1$, $u = (\alpha_1 - 1)n + j$ and α_1 is the least positive integer for which $w \le \alpha_1 n$. Also in (2), $i = \alpha_2$, $w = (\alpha_2 - 1)n + j$ where α_2 is the least positive integer which satisfies $w \le \alpha_2 n$.
$([a_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}]) \cup ([b_{ij}]\,\widetilde{\nabla}_r\,[c_{ij}]) = [g_{zj}] \in FSM_{nn\times m}$ where $g_{zj} = max\{d_{uj}, f_{wj}\}$. Since $[d_{uj}]$ and $[f_{wj}]$ have the same order, then $u = w = z$ so that if $u = \alpha_3$, $z = (\alpha_3 - 1)n + j$ where α_3 is the least positive integer such that $z \le \alpha_3 n$. Thus,

$$\begin{aligned} z &= (\alpha_3 - 1)n + j \\ &= (\alpha_2 - 1)n + j \\ &= (\alpha_1 - 1)n + j \\ &= (\alpha - 1)n + j \end{aligned}$$

Thus, $[e_{vj}] = [g_{zj}]$ so that,
$max\{max\{a_{ij}, b_{ij}\}, c_{ij}\} = max\{max\{a_{ij}, c_{ij}\}, max\{b_{ij}, c_{ij}\}$
The proofs of 2 - 4 are similar, hence omitted.

Theorem 2 *Let $[\Gamma_A], [\Gamma_B], [\Gamma_C] \in FSM_{n\times m}$, then*
1. $[\Gamma_A]\,\widetilde{\nabla}_r\,([\Gamma_B] \cup [\Gamma_C]) = ([\Gamma_A]\,\widetilde{\nabla}_r\,[\Gamma_B]) \cup ([\Gamma_A]\,\widetilde{\nabla}_r\,[\Gamma_C])$
2. $[\Gamma_A]\,\widetilde{\nabla}_r\,([\Gamma_B] \cap [\Gamma_C]) = ([\Gamma_A]\,\widetilde{\nabla}_r\,[\Gamma_B]) \cap ([\Gamma_A]\,\widetilde{\nabla}_r\,[\Gamma_C])$
3. $[\Gamma_A]\,\widetilde{\wedge}_r\,([\Gamma_B] \cup [\Gamma_C]) = ([\Gamma_A]\,\widetilde{\wedge}_r\,[\Gamma_B]) \cup ([\Gamma_A]\,\widetilde{\wedge}_r\,[\Gamma_C])$
4. $[\Gamma_A]\,\widetilde{\wedge}_r\,([\Gamma_B] \cap [\Gamma_C]) = ([\Gamma_A]\,\widetilde{\wedge}_r\,[\Gamma_B]) \cap ([\Gamma_A]\,\widetilde{\wedge}_r\,[\Gamma_C])$

Proof. The proofs are similar to that of Theorem 1 above, hence omitted.

Remark 2: Theorems 1 and 2 establish the fact that the row-products are right and left distributive over the union and intersection of fuzzy soft matrices

Theorem 3 *Let* $[\Gamma_A], [\Gamma_B] \in FSM_{n\times m}$, *then*
1. $[\Gamma_A]^{oA} \cup [0] = [\Gamma_A]^{oA}$ where $[0] \in$FSM $_{n\times m}$
2. $[\Gamma_A]^{oA} \cap [0] = [0]$
3. $[\Gamma_A]^{oA} \cap [\Gamma_A]^{oA} = [\Gamma_A]^{oA}$
4. $[\Gamma_A]^{oA} \cup [\Gamma_A]^{oA} = [\Gamma_A]^{oA}$
5. $[\Gamma_A]^{oA} \cup U = U$
6. $[\Gamma_A]^{oA} \cap U = [\Gamma_A]^{oA}$
7. $[\Gamma_A]^{oA} \cup [\Gamma_B]^{oB} = [\Gamma_A]^{oA} \cup [\Gamma_B]^{oB}$
8. $[\Gamma_A]^{oA} \cap [\Gamma_B]^{oB} = [\Gamma_A]^{oA} \cap [\Gamma_B]^{oB}$

Proof. The proofs are straight forward.

Theorem 4 *Let* $[\Gamma_A], [\Gamma_B] \in FSM_{n\times m}$, *A = B and* $[\Gamma_B] \,\tilde{\subseteq}\, [\Gamma_A]$, *then the following hold.*
1. $[\Gamma_A]^{oA} \cup [\Gamma_B]^{oB} = [\Gamma_B]^{oB}$
2. $[\Gamma_A]^{oA} \cap [\Gamma_B]^{oB} = [\Gamma_A]^{oA}$
3. $[\Gamma_A]^{o} \cup [\Gamma_A]^{oA} = [\Gamma_A]^{o}$
4. $[\Gamma_A]^{o} \cup [\Gamma_B]^{o} = [\Gamma_B]^{o}$
5. $[\Gamma_A]^{o} \cap [\Gamma_B]^{o} = [\Gamma_A]^{o}$

Proof. Let $[\Gamma_A]^{oA} = [c_{ij}]$, $[\Gamma_B]^{oB} = [d_{ij}]$ and $I_A = \{e_i : i \in A\}$, where $I_A = I_B$ since A = B. $[b_{ij}] \,\tilde{\subseteq}\, [a_{ij}]$ implies that, $\forall i$ and j, $b_{ij} \le a_{ij}$ with at least one term $b_{ij} < a_{ij}$. Now, if $b_{ij} \le a_{ij}$ then $1 - a_{ij} \le 1 - b_{ij}\, \forall i$ and j. So $\forall i$ and j,

$$c_{ij} = \begin{cases} 1 - a_{ij}, & if\ j \in I_A; \\ 0, & if\ j \notin I_A. \end{cases}$$

and

$$d_{ij} = \begin{cases} 1 - b_{ij}, & if\ j \in I_B; \\ 0, & if\ j \notin I_B. \end{cases}$$

If $[e_{ij}] = [c_{ij}] \cup [d_{ij}]$, then

$$e_{ij} = \begin{cases} max\{1 - a_{ij}, 1 - b_{ij}\}, & if\ j \in I_{A\cap B}; \\ 0, & if\ j \notin I_{A\cap B}. \end{cases}$$

$$e_{ij} = \begin{cases} 1 - b_{ij}, & if\ j \in I_{A\cap B}; \\ 0, & if\ j \notin I_{A\cap B}. \end{cases}$$

Therefore, $[e_{ij}] = [\Gamma_B]^{oB}$. The proofs of 2 - 4 follow similar arguments.

Conclusion

Many examples constructed in this work have shown how the idea of row products of soft matrices introduced by H. Kamac et al [5] is fuzzified. Some fundamental algebraic properties of identity and complements of Or row products operation and And row-product operation of fuzzy soft matrices are investigated. From the results, Or row-product is associative with the union and intersection of fuzzy soft matrices. Similarly, for the And row product as indicated by Theorem 1 of section 1. In addition to this, Theorem 1 of section 1 also indicates that union and intersection of fuzzy soft matrices are distributive over defined Or row-product and And row-product operations.

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