Journal of the Nigerian Association of Mathematical Physics
***Volume 55* (February *2020 Issue*), pp47 –52**
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ON STABILITY ANALYSIS OF CERTAIN FOURTH-ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract

By the use of the eigenvalue approach, a new result was obtained to ascertain the stability analysis of a certain fourth-order non-linear delay differential equation. The result obtained is less restrictive than those reported in the literature.

***Keywords*:** non-linear delay differential equations of fourth order; stability; eigenvalue approach.

1.0. Introduction

The stability is a very important problem in the applications and theory of differential equations. In fact, finding a solution to a differential equation may or may not be so necessary if that solution never appears in the physical model represented by the system. Thus, equilibrium solutions which correspond to configurations in which the physical system does not move, only occurs in everyday situations if they are stable. However, unstable equilibrium will not appear in practice, since slight perturbations in the system or its physical surroundings will immediately dislodge the system far away from equilibrium [7].Perhaps, the most effective method to know the stability behavior of solutions of linear and non-linear differential equations is the Liapunov's direct(or second) method. The Lyapunov theorem on stability and asymptotic stability has been used to study the existence of a continuously differentiable Lyapunov function for non-linear differential equations of different types [1-4]. The major advantage of this method is that stability in a whole can be obtained without any concrete knowledge of solutions. Today, the technique is widely recognized as a very good techniques not only in the study of differential equations but also in the theory of control systems, power system analysis, dynamical systems, systems with time lag, time varying non-linear feedback systems etc [5-8]. Its characteristic is the construction of a scalar functional, such as; the Liapunov function or functional. System of delay differential equations in the other hands occupy a place of central importance in all areas of sciences, for instance in Mathematical modeling, delay differential equations is widely used for analysis and predictions in various areas of life sciences, for examples, population dynamics, epidemiology, immunology, physiology, neutral network etc. But, finding an appropriate Liapunov functions or functional for higher order differential equations with or without delay is generally a difficult task. However, during the past few years, by the use of the Liapunov's method (or second method) many wonderful results have been obtained on the stability and boundedness of solutions of various second-order, third-order, fourth-order, fifth-order and sixth-order non-linear differential equations with delay and without delay [9-19]. However, one should be noted that there are only a few results on the stability of solutions of certain fourth order nonlinear differential equations with delay.

In 1973, [18]considered the fourth-order delay differential equation of the form:

$$x^{(4)}(t)+f\left(x''(t)\right)x'''(t)+f_2\left(x'(t),x''(t)\right)x''(t)+g\left(x'(t-\tau)\right)+h\left(x(t-\tau)\right)=0, \tag{1.1}$$

and proved the asymptotic stability of this equation. After that, in 1989, [12] obtained sufficient conditions for uniform asymptotic stability and boundedness of solutions of a fourth-order scalar delay-differential equation of the form:

$$x^{(4)}(t)+f\left(x''(t)\right)x'''(t)+\alpha_2 x''(t)+\beta_2 x''(t-h)+g\left(x'(t-h)\right)+\alpha_4 x(t)+\beta_4 x(t-h)=p(t) \tag{1.2}$$

In 1998, [19]established sufficient asymptotic conditions for the uniform asymptotic stability of the zero solution of the following fourth-order delay differential equation:

$$x^{(4)}(t)+e\left(t,x(t),x'(t),x''(t),x'''(t)\right)x'''(t)+f\left(t,x''(t-\tau)\right)+g\left(t,x'(t-\tau)\right)+h\left(x(t-\tau)\right)=0 \tag{1.3}$$

Recently, in 2006, [7]considered the fourth-order nonlinear delay differential equations of the form:

$$x^{(4)}(t)+\alpha_1\dddot{x}+\alpha_2\ddot{x}+\alpha_3\dot{x}+f\left(x(t-\tau)\right)=0 \tag{1.4}$$

and he derived sufficient conditions for the asymptotic stability of the zero-solution of this equations by constructing two new Liapunov functional. More recently the authors in[1-4,6-7,19]investigated the asymptotic stability of zero solution of the differential equations such as fourth order non-linear delay differential equations of the form:

$$x^{(4)}(t)+\varphi(\ddot{x})\dddot{x}+h(\dot{x})\ddot{x}+\phi\left(\dot{x}(t)\right)+f\left(x(t-\tau)\right)=0 \tag{1.5}$$

and

$$x^{(4)}(t)+\varphi(\ddot{x})\dddot{x}+h(\dot{x})\ddot{x}+\phi\left(\dot{x}(t-\tau)\right)+f\left(x(t)\right)=0 \tag{1.6}$$

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In this paper, we study the same problem for the following non-linear delay differential equation of the form;

$$x^{(4)}(t)+\alpha_1\dddot{x}+\alpha_2\ddot{x}+\alpha_3\dot{x}+f\left(x(t-\tau)\right)=0 \quad (1.7)$$

where, τ is a positive constant, $f\left(x(t-\tau)\right)$ and $f(x)$ are continuous functions, $f(0)=0$. The derivatives $\frac{dx}{dt}=\dot{x}(t)$ exists and also continuous.

The motivation of this paper has been inspired basically by the papers of[7, 16 and 17].

2.0. Preliminaries

In order to reach our main results we will first give some important stability criteria for thegeneral autonomous delay differential system.

Definition 2.1 (Stability)

Consider the nonlinear time-invariant system

$$\dot{x}=f(x),\ f:\mathbb{R}\to\mathbb{R} \quad (2.1)$$

A point $x_e\in\mathbb{R}$, is an equilibrium point of the system if $f(x)=0$.We remark that $x_e$is an equilibrium point if and only if $x(t)=x_e$ is a trajectory.

Suppose x_e is an equilibrium point, then

i. The system is globally asymptotically stable if for every trajectory x, we have $x(t)\to x_e$ as t→ ∞.

ii. The system is locally asymptotically stable near or at x_e if there is an $\mathcal{R}>0\ \ni \|x(0)-x_e\|\le\mathcal{R}\ \Rightarrow x(t)\longrightarrow x_e\ as\ t\longrightarrow\infty$.

Definition 2.2

Consider also the linear system

$\dot{x}=Ax$ (2.2)

i. The system is said to be globally asymptotically stable with $\varkappa=0$ if and only if $\Re_e\lambda_i(A)<0, i=1,2,3,\dots n$, where $\Re_e$ $means\ the\ real\ part$.

ii. The system is said to be locally asymptotically stable (near $\varkappa=0$) if and only if $\Re_e\lambda_i(A)<0,\ i=1,2,3,\dots n.$ *Thus for linear system , Locally asymptoticaally stable* $\Leftrightarrow$ *Globally asymptotically stable*).

Definition 2.3 (Asymptotic stability)

The equilibrium point $\varkappa$ is said to be asymptotically stable, if for all $\varepsilon>0,\exists\ \delta>0\ \ni$

i. $f(t;0,\bar{x})\in\beta(\varkappa,\varepsilon)$for all$t\ge0$.

ii. $\lim_{t\to\infty}f(t,0,\bar{x})=\varkappa$

Definition 2.4 (Lyapunov function)

Consider the differential equation

$\dot{x}=f(x),\qquad f(0)=0$

Where the solutions are unique and vary continuously with the initial data. Let $V:\mathbb{R}^n\to\mathbb{R}$ be continuous together with its first partial derivatives, $\frac{\partial v}{\partial x}$ $(i=1,2,\dots)$ on some open set $\Omega\subset\mathbb{R}^n$, where $\Omega=B_r(0)$, $\Omega=\{x\colon x\epsilon\mathbb{R}^n\colon\|x\|<r, where\ r\ is\ the\ radius\ of\ the\ ball\}$

i. A Function $V:\Omega\to\mathbb{R}$ is said to be positive definite/negative definite if $v(0)=0\ and\ v$ assumes positive/negative values onΩ.

ii. A Function $V:\Omega\to\mathbb{R}$ is said to be positive/negative semi definite if $v(0)=(0)\ and\ v(x)\ge0$ or $v(x)\le0$ onΩ.

If the function assumes arbitrary values, then it is said to be indefinite.

Theorem 2.5

The critical point $0\in\mathbb{R}^n$ for the linear system $\dot{x}=A\underline{x}$ is asymptotically stable provided that all the eigenvalues of A have negative real parts otherwise it is unstable.

Theorem 2.6

Consider $\dot{x}=f(x)$, and assume that $f(0)=0$

Linearization

$\dot{x}=A(\underline{x})+g(\underline{x}),\quad \|g(x)\|=0(\|\underline{x}\|)\ as\ x\to0$

i. $\Re_e\lambda_k(A)<0$, for $k\Longrightarrow x=0\ is$ Locally Asymptotically Stable$(L.A.S)$.

ii. $\exists\ K\colon\Re_e\lambda_k(A)>0\Longrightarrow x=0\ is\ unstable.$

3.0 Main Results

We consider the fourth order non-linear delay differential equation of the form;

$$\ddddot{x}+\alpha_1\dddot{x}+\alpha_2\ddot{x}+\alpha_3\dot{x}+f\left(x(t-\tau)\right)=0 \quad (3.1)$$

We obtain the first order systems from the scalar differential equation by letting

$$\left.\begin{aligned}\dot{x}&=y\\ \dot{y}&=z\\ \dot{z}&=w\\ \dot{w}&=\varphi\end{aligned}\right\} \quad (3.2)$$

Equation (3.1) becomes

$$\dddot{y}+\alpha_1\ddot{y}+\alpha_2\dot{y}+\alpha_3 y+f\left(x(t-\tau)\right)=0$$
$$\ddot{z}+\alpha_1\dot{z}+\alpha_2 z+\alpha_3 y+f\left(x(t-\tau)\right)=0$$
$$\dot{w}+\alpha_1 w+\alpha_2 z+\alpha_3 y+f\left(x(t-\tau)\right)=0$$
$$\Rightarrow \dot{w}=-\alpha_3 y-\alpha_2 z-\alpha_1 w-f\left(x(t-\tau)\right) \quad (3.3)$$

The equivalent system is now

$$\left.\begin{aligned}\dot{x}&=y\\ \dot{y}&=z\\ \dot{z}&=w\\ \dot{w}&=-\alpha_3 y-\alpha_2 z-\alpha_1 w-f\left(x(t-\tau)\right)\end{aligned}\right\} \quad (3.4)$$

The equation (3.4) is the equivalent system obtained directly from the scalar equation, which can be written in matrix form as;

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \dot{w}\end{pmatrix}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&1&0\\0&-\alpha_3&-\alpha_2&-\alpha_1\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}+\begin{pmatrix}0\\0\\0\\-f\left(x(t-\tau)\right)\end{pmatrix}\Rightarrow \dot{\gamma}=A\gamma+g(\gamma)$$

The equation (3.1) can be written as

$$\ddddot{x}+\alpha_1\dddot{x}+\alpha_2\ddot{x}+\alpha_3\dot{x}+f\left(x(t-\tau)\right)\equiv\frac{d}{dt}\left(\dddot{x}+\alpha_1\ddot{x}+\alpha_2\dot{x}\right)+\alpha_3\dot{x}+f\left(x(t-\tau)\right)=0 \quad (3.5)$$

Let

$$\dddot{x}+\alpha_1\ddot{x}+\alpha_2\dot{x}=w$$
$$\ddot{y}+\alpha_1\dot{y}+\alpha_2 y=w$$
$$\dot{z}+\alpha_1 z+\alpha_2 y=w\Rightarrow \dot{z}=-\alpha_2 y-\alpha_1 z+w$$

Also, equation (3.5) can be written as;

$$\left.\begin{aligned}\frac{d}{dt}\left(\dddot{x}+\alpha_1\ddot{x}+\alpha_2\dot{x}\right)+\alpha_3\dot{x}+f\left(x(t-\tau)\right)&\equiv\frac{d}{dt}(w)+\alpha_3\dot{x}+f\left(x(t-\tau)\right)=0\\ &=\dot{w}+\alpha_3 y+f\left(x(t-\tau)\right)=0\\ &\Rightarrow \dot{w}=-\alpha_3 y-f\left(x(t-\tau)\right)\end{aligned}\right\} \quad (3.6)$$

The equivalent system is now

$$\left.\begin{aligned}\dot{x}&=y\\ \dot{y}&=z\\ \dot{z}&=-\alpha_1 y-\alpha_2 z+w\\ \dot{w}&=-\alpha_3 y-f\left(x(t-\tau)\right)\end{aligned}\right\} \quad (3.7)$$

The equation (3.7) is the second of the fourth equivalent first order systems which can be written in matrix form as;

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \dot{w}\end{pmatrix}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&-\alpha_1&-\alpha_2&1\\0&-\alpha_3&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}+\begin{pmatrix}0\\0\\0\\-f\left(x(t-\tau)\right)\end{pmatrix}\Rightarrow \dot{\gamma}=A\gamma+g(\gamma) \quad (3.8)$$

The system in (3.1) above can be expressed as;

$$\ddddot{x}+\alpha_1\dddot{x}+\alpha_2\ddot{x}+\alpha_3\dot{x}+f\left(x(t-\tau)\right)\equiv\ddddot{x}+\frac{d}{dt}\left(\alpha_1\ddot{x}+\alpha_2\dot{x}\right)+\alpha_3\dot{x}+f\left(x(t-\tau)\right) \quad (3.9)$$

Let $\alpha_1\ddot{x}+\alpha_2\dot{x}=z$

$$\alpha_1\dot{y}+\alpha_2 y=z\Rightarrow \dot{y}=-\frac{\alpha_2}{\alpha_1}y+\frac{1}{\alpha_1}z$$

Also the equation (3.9) can be expressed as;

$$\begin{aligned}\ddddot{x}+\frac{d}{dt}\left(\alpha_1\ddot{x}+\alpha_2\dot{x}\right)+\alpha_3\dot{x}+f\left(x(t-\tau)\right)&\equiv\ddddot{x}+\frac{d}{dt}(z)+\alpha_3\dot{x}+f\left(x(t-\tau)\right)=0\\ &=\dddot{y}+\dot{z}+\alpha_3 y+f\left(x(t-\tau)\right)=0\\ &=\ddot{z}+w+\alpha_3 y+f\left(x(t-\tau)\right)=0\\ &=\dot{w}+w+\alpha_3 y+f\left(x(t-\tau)\right)=0\\ &\Rightarrow \dot{w}=-\alpha_3 y-f\left(x(t-\tau)\right)\end{aligned} \quad (3.10)$$

The equivalent system is now

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{\alpha_2}{\alpha_1}y + \frac{1}{\alpha_1}z \\ \dot{z} &= -\alpha_1 y - \alpha_2 z + w \\ \dot{w} &= -\alpha_3 y - f\left(x\left(t-\tau\right)\right) \end{aligned}\right\} \qquad (3.11)$$

Equation (3.11) is the third of the fourth equivalent first order system. This can be written in matrix form as;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\alpha_2}{\alpha_1} & \frac{1}{\alpha_1} & 0 \\ 0 & -\alpha_1 & -\alpha_2 & 1 \\ 0 & -\alpha_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -f\left(x\left(t-\tau\right)\right) \end{pmatrix} \Rightarrow \dot{\gamma} = A\gamma + g\left(\gamma\right) \qquad (3.12)$$

The system in (3.1) above can be expressed as;

$$\ddddot{x} + \alpha_1 \dddot{x} + \alpha_2 \ddot{x} + \alpha_3 \dot{x} + f\left(x\left(t-\tau\right)\right) \equiv \ddddot{x} + \alpha_1 \dddot{x} + \frac{d}{dt}\left(\alpha_2 \dot{x} + \alpha_3 x\right) + f\left(x\left(t-\tau\right)\right) \qquad (3.13)$$

Let
$$\begin{aligned} &\alpha_2 \dot{x} + \alpha_3 x = y \\ &\alpha_2 y + \alpha_3 x = \dot{x} \\ &\Rightarrow \dot{x} = \alpha_3 x + \alpha_2 y \end{aligned}$$

Also, equation (3.13) can be expressed as

$$\begin{aligned} \ddddot{x} + \alpha_1 \dddot{x} + \frac{d}{dt}\left(\alpha_2 \dot{x} + \alpha_3 x\right) + f\left(x\left(t-\tau\right)\right) &\equiv \ddddot{x} + \alpha_1 \dddot{x} + \frac{d}{dt}\left(y\right) + f\left(x\left(t-\tau\right)\right) = 0 \\ &= \dddot{y} + \alpha_1 \ddot{y} + \dot{y} + f\left(x\left(t-\tau\right)\right) = 0 \\ &= \ddot{z} + \alpha_1 \dot{z} + z + f\left(x\left(t-\tau\right)\right) = 0 \\ &= \dot{w} + \alpha_1 w + z + f\left(x\left(t-\tau\right)\right) = 0 \\ &\Rightarrow \dot{w} = -\alpha_1 w - z - f\left(x\left(t-\tau\right)\right) \end{aligned}$$

The equivalent system is now

$$\left.\begin{aligned} \dot{x} &= \alpha_3 x + \alpha_2 y \\ \dot{y} &= -\frac{\alpha_2}{\alpha_1}y + \frac{1}{\alpha_1}z \\ \dot{z} &= -\alpha_1 y - \alpha_2 z + w \\ \dot{w} &= -\alpha_1 w - z - f\left(x\left(t-\tau\right)\right) \end{aligned}\right\} \qquad (3.14)$$

Equation (3.14) is the fourth of the fourth equivalent first order system. This can be written in matrix form as;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \alpha_3 & \alpha_2 & 0 & 0 \\ 0 & -\frac{\alpha_2}{\alpha_1} & \frac{1}{\alpha_1} & 0 \\ 0 & -\alpha_1 & -\alpha_2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -f\left(x\left(t-\tau\right)\right) \end{pmatrix} \Rightarrow \dot{\gamma} = A\gamma + g\left(\gamma\right)$$

4.0. Stability Analysis

We linearize the nonlinear parts of each of the equivalent system derived using Maclaurin series expansion i.e.

$$g\left(x\right) = g\left(0\right) + xg'\left(0\right) + \frac{1}{2!}\left\|x\right\|^2 g''\left(0\right) + \cdots, \quad \left\|g\left(x\right)\right\| = 0, \text{ as } x \to 0$$

The linearized term is now

$g\left(x\right) = g\left(0\right) + xg'\left(0\right)$, but $g\left(0\right) = 0 \Rightarrow g\left(x\right) = xg'\left(0\right)$

Hence, the system $\dot{\gamma} = A\gamma + g\left(\gamma\right)$ becomes $\dot{\gamma} = A\gamma + \gamma g'\left(0\right)$

From the above, we see that $g'\left(0\right)$ is necessarily an $n \times n$ matrix since it must be compatible with matrix A. Then we have a linearized system given by $\dot{\gamma} = B\gamma$, where B is called the linearized matrix i.e. $B = A + g'\left(0\right)$

Applying Theorem (2.6), all the nonlinear terms varnishes living behind the linear terms i.e. $\dot{\gamma} = A\gamma$. Now, we want to test the stability for each of the matrices derived using the eigenvalue method i.e. $\left|A_n - \lambda I\right| = 0$, where $n = 1, 2, 3, 4.$

Where,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha_1 & -\alpha_2 & 1 \\ 0 & -\alpha_3 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\alpha_2}{\alpha_1} & \frac{1}{\alpha_1} & 0 \\ 0 & -\alpha_1 & -\alpha_2 & 1 \\ 0 & -\alpha_3 & 0 & 0 \end{pmatrix} \text{ and } A_4 = \begin{pmatrix} \alpha_3 & \alpha_2 & 0 & 0 \\ 0 & -\frac{\alpha_2}{\alpha_1} & \frac{1}{\alpha_1} & 0 \\ 0 & -\alpha_1 & -\alpha_2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and applying Theorem (2.6), but for convenience, we assumed $\alpha_1 = \alpha_2 = \alpha_3 = 1$, then we have that

$$|A_1 - \lambda I| = 0 \Rightarrow \left|\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & -1 & -1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^4 - \lambda^2 = 0 \qquad \Rightarrow \lambda = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \tag{4.1}$$

Since all the eigenvalues are not negative real parts then, (4.1) is unstable.

$$|A_2 - \lambda I| = 0 \Rightarrow \left|\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\alpha_2 & -\alpha_1 & 1 \\ 0 & -\alpha_3 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & -1 & -1-\lambda & 1 \\ 0 & -1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^4 + \lambda^3 = 0 \Rightarrow \lambda = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{4.2}$$

Since all the eigenvalues are not negative real parts then, (4.2) is unstable.

$$|A_3 - \lambda I| = 0 \Rightarrow \left|\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -1-\lambda & 1 & 0 \\ 0 & -1 & -1-\lambda & 1 \\ 0 & -1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^4 + 2\lambda^3 + 2\lambda^2 + \lambda = 0 \Rightarrow \lambda = \begin{pmatrix} 0 \\ -1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \end{pmatrix} \tag{4.3}$$

Since all the eigenvalues are not negative real parts then,(4.3) is unstable.

$$|A_4 - \lambda I| = 0 \Rightarrow \left|\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -1-\lambda & 1 & 0 \\ 0 & -1 & -1-\lambda & 1 \\ 0 & 0 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^4 + \lambda^3 - \lambda - 1 = 0 \Rightarrow \lambda = \begin{pmatrix} -1 \\ 1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \end{pmatrix} \tag{4.4}$$

Since all the eigenvalues are not negative real parts then, (4.4) is unstable.

5.0. Conclusion

The eigenvalue method is very easy to handle as verified in this paper. The need to first convert higher order differential equations to first order differential equations has been a key to easily solving higher order differential equations. Therefore, it is recommended that one can check for the stability of the system by picking only one of the equivalent first order systems derived instead of checking for each of the equivalent system. Furthermore, this method tends to be complex if higher numbers are assumed hence; the technique requires the smallest possible number to be assumed. However, since all the eigenvalues of the matrices do not all have negative real parts, we conclude that the fourth-order delay differential equation (3.1) is unstable in respect to Theorem (2.6).

6.0 Acknowledgements

First of all, I will like to express my deepest gratitude to God Almighty for giving me the strength to initiate and execute this work. Special honor goes to my M.Sc. supervisor in the person of Dr. S. O. Maliki, without whose singular effort, I could not have had the endurance and perseverance to complete my Thesis. I am deeply grateful Sir for your great inspiration, excellent guidance, deep thoughts and friendship during the course of the work. I also appreciate my other Lecturers and colleagues for their encouragements. Thank you all for your effort, I cannot do without you people.

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