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APPLICATION OF FOURIER TRANSFORM TO SINGULAR INTEGRAL EQUATION OF THE FIRST KIND.

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Abstract

Many physical problems that are usually solved by differential equation methods can be solved more effectively by integral equation method. In this paper, we explored Leibniz's method for Singular Integral Equation of the first kind to obtain a boundary value problem, which only gives a trivial solution at $t \approx 0$ for some integer n, we further assume the problem to be a non-homogeneous which on applying the Fourier Transform yield a one half of the two-sided decaying exponential function.

***Keywords*:** Singular Integral Equation of the first kind, Leibniz's rule, Fourier Transform.

1. Introduction

An integral equation is an equation in which the unknown function to be determined appears under the integral sign. The integral equation is said to be linear if the unknown function under the integral sign occurs linearly otherwise it is nonlinear. The most frequently used linear integral equations falls under two classes namely; Fredholm and Volterra integral equations. However, the integral equation is said to be singular if the integration is improper. This usually occurs if the interval of integration is infinite, or if the kernel becomes unbounded at one or more points of the interval of consideration. In the theory of integral equations, the convolution type integral equations and singular integral equations are two important classes of equations, which had been studied by many mathematical researchers and there were already rather complete theoretical systems [4, 5]. These theories have been widely used in practical applications, such as engineering mechanics, fracture mechanics, and elastic mechanics [8, 9]Many researchers have applied the singular integral equation in different areas of Mathematics, the solution of this integration can be obtained both analytically and numerically. [1,2]in their paper gave the exact solutions of a singular integral equation with logarithmic singularities in two classes of functions and construct formulae for the approximate solutions. [3], in their paper investigated the numerical solution of various cases of Cauchy type singular integral equations using reproducing kernel Hilbert space method. Other researchers [7-12] have applied the singular integral equation of the first kind in different areas of Mathematics, for instance, [7] presented a new numerical technique to discover a new solution of singular nonlinear Volterra Integral Equations. The technique is delineated with two numerical cases to illustrate the benefit of the techniques used.

2. Preliminaries

Definition 2.1: (Leibniz's rule); let $f(t)=\int_{\alpha}^{\beta} G(t,x)\,dx$, where $\alpha=\alpha(t)$ and $\beta=\beta(t)$, then

$$\frac{df(t)}{dt}=\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} G(t,x)\,dx=\int_{\alpha(t)}^{\beta(t)}\frac{\partial G(t,x)}{\partial t}dx+G(t,\beta(t))\frac{d\beta(t)}{dt}-G(t,\alpha(t))\frac{d\alpha(t)}{dt} \tag{2.1}$$

Definition 2.2 (Singular integral equation of the first kind): The integral equation is said to be singular if the lower limit, the upper limit orboth limits of integration are finite. It is said to be of the first kind if

$$f(x)=\lambda\int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)\,dt \tag{2.2}$$

Where, $\alpha(x)$ and $\beta(x)$ are the lower and upper limits respectively of the integration, $K(x,t)$ is the kernel of the integral equation.

Definition 2.3 (Fourier Transform); The Fourier Transforms of the function $y(x,t)$ with respect to x is defined as

$$\Im\{y(x,t)\}=\hat{y}(\xi,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} y(x,t)\exp(i\xi x)\,dx \tag{2.3}$$

Theorem 2.4 (Fourier Inversion Theorem); Let $\Im\{y(x,t)\}=\hat{y}(\xi,t)$be the Fourier Transform of $y(x,t)$, we defined the inverse Fourier Transform of $\hat{y}(\xi,t)$ as

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$$y(x,t)=\Im^{-1}\{\hat{y}(\xi,t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{y}(\xi,t)\exp(-\xi x)\,d\xi \tag{2.4}$$

Definition 2.5 (Fourier Transform of $\frac{\partial u(x,t)}{\partial t}$); the Fourier Transform of $\frac{\partial u(x,t)}{\partial t}$ is given as

$$\frac{\partial \bar{u}(\xi,t)}{\partial t}=\int_{-\infty}^{\infty}\frac{\partial u(x,t)}{\partial t}.\exp(-i\xi x)\,dx=\frac{\partial}{\partial t}\left\{\int_{-\infty}^{\infty}u(x,t)\exp(-i\xi x)\,dx\right\} \tag{2.5}$$

Definition 2.6 (Fourier Transform of $\frac{\partial^2 u(x,t)}{\partial x^2}$); the Fourier Transform of $\frac{\partial^2 u(x,t)}{\partial x^2}$ as

$$\frac{\partial^2 \bar{u}(\xi,t)}{\partial x^2}=\int_{-\infty}^{\infty}\frac{\partial^2 u(x,t)}{\partial x^2}.\exp(-i\xi x)\,dx=-\int_{-\infty}^{\infty}\frac{\partial u(x,t)}{\partial x}\{(-i\xi).\exp(-i\xi x)\}\,dx$$

$$=(i\xi)\int_{-\infty}^{\infty}\frac{\partial u(x,t)}{\partial x}.\exp(-i\xi x)\,dx=(i\xi)^2\int_{-\infty}^{\infty}\frac{\partial u(x,t)}{\partial x}.\exp(-i\xi x)\,dx=(i\xi)^2\,\bar{u}(\xi,t) \tag{2.6}$$

Definition 2.7 (Derivative property of the Fourier Transform): The Fourier Transform of the derivative of $g(t)$ is given by:

$$\Im\left\{\frac{dg(t)}{dt}\right\}=2i\pi f.G(f)$$

Definition 2.8 (Duality property of the Fourier Transform): Suppose $g(t)$ has Fourier Transform $G(t)$. Then we automatically know the Fourier Transform of the function $G(t)$:

$$\Im\{G(t)\}=g(-f)$$

3. Main Results

We consider the singular integral equation of the form

$$y(t)=\lambda\int_0^1 K(t,x)\,y(x)\,dx \tag{3.1}$$

where,

$$K(t,x)=\begin{cases} t(1-x) & \text{for } t\le x\le 1 \\ x(1-t) & \text{for } 0\le x\le t\end{cases}$$

Equation (1) can be written as

$$y(t)=\lambda\int_0^t x(1-t)\,y(x)\,dx+\lambda\int_t^1 t(1-x)\,y(x)\,dx \tag{3.2}$$

Applying the Leibniz's rule

$$\frac{df(t)}{dt}=\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)}G(t,x)\,dx=\int_{\alpha(t)}^{\beta(t)}\frac{\partial G(t,x)}{\partial t}dx+G(t,\beta(t))\frac{d\beta(t)}{dt}-G(t,\alpha(t))\frac{d\alpha(t)}{dt} \tag{3.3}$$

If we differentiate (2) with respect to t, we get

$$\frac{dy(t)}{dt}=\lambda\frac{d}{dt}\int_0^t x(1-t)\,y(x)\,dx+\lambda\frac{d}{dt}\int_t^1 t(1-x)\,y(x)\,dx \tag{3.4}$$

Now, for $\lambda\frac{d}{dt}\int_0^t x(1-t)\,y(x)\,dx$

Here, $G(t,x)=x(1-t)\,y(x)\Rightarrow\frac{\partial G(t,x)}{\partial t}=-xy(x),\ \alpha(t)=0\Rightarrow\frac{d\alpha(t)}{dt}=0,$

$\beta(t)=t\Rightarrow\frac{d\beta(t)}{dt}=1,\ G(t,\beta(t))=G(t,t)=t(1-t)\,y(t),\ G(t,\alpha(t))=G(t,0)=0$

Substituting this into (3.3), gives

$$\lambda\frac{d}{dt}\int_0^t x(1-t)\,y(x)\,dx=\lambda\int_0^t -xy(x)\,dx+\lambda t(1-t)\,y(t) \tag{3.5}$$

For $\lambda\frac{d}{dt}\int_t^1 t(1-x)\,y(x)\,dx$

Here, $G(t,x)=t(1-x)\,y(x)\Rightarrow\frac{\partial G(t,x)}{\partial t}=(1-x)\,y(x),\ \alpha(t)=t\Rightarrow\frac{d\alpha(t)}{dt}=1,$

$\beta(t)=1\Rightarrow\frac{d\beta(t)}{dt}=0,\ G(t,\beta(t))=G(t,1)=0,\ G(t,\alpha(t))=G(t,t)=t(1-t)\,y(t)$

Substituting this into (3.3), gives

$$\lambda\frac{d}{dt}\int_t^1 t(1-x)\,y(x)\,dx=\lambda\int_t^1(1-x)\,y(x)\,dx-\lambda t(1-t)\,y(t) \tag{3.6}$$

Combining (3.5) and (3.6), gives

$$\frac{dy(t)}{dt}=\lambda\int_0^t -xy(x)dx+\lambda\int_t^1(1-x)y(x)dx \qquad (3.7)$$

If we further differentiate again, we get

$$\frac{d^2y(t)}{dt^2}=\lambda\frac{d}{dt}\int_0^t -xy(x)dx+\lambda\frac{d}{dt}\int_t^1(1-x)y(x)dx \qquad (3.8)$$

For $\lambda\frac{d}{dt}\int_0^t -xy(x)dx$

Here,

$$G(t,x)=-xy(x)\Rightarrow\frac{\partial G(t,x)}{\partial t}=0,\ \alpha(t)=0\Rightarrow\frac{d\alpha(t)}{dt}=0,\ \beta(t)=t\Rightarrow\frac{d\beta(t)}{dt}=1,\ G(t,\beta(t))=G(t,t)=-ty(t),\ G(t,\alpha(t))=G(t,0)=0$$

Substituting this into (3.3), gives

$$\lambda\frac{d}{dt}\int_0^t -xy(x)dx=-\lambda ty(t) \qquad (3.9)$$

For $\lambda\frac{d}{dt}\int_t^1(1-x)y(x)dx$

Here,

$$G(t,x)=(1-x)y(x)\Rightarrow\frac{\partial G(t,x)}{\partial t}=0,\ \alpha(t)=t\Rightarrow\frac{d\alpha(t)}{dt}=1,\ \beta(t)=1\Rightarrow\frac{d\beta(t)}{dt}=0,\ G(t,\beta(t))=G(t,1)=0,\ G(t,\alpha(t))=G(t,t)=(1-t)y(t)$$

Substituting this into (3.3), gives

$$\lambda\frac{d}{dt}\int_t^1(1-x)y(x)dx=\lambda(1-t)y(t) \qquad (3.10)$$

Combining (9) and (10), gives

$$\frac{d^2y(t)}{dt^2}=-\lambda ty(t)-\lambda(1-t)y(t)=-\lambda ty(t)-\lambda y(t)+\lambda ty(t)=-\lambda y(t) \qquad (3.11)$$

Hence, the Singular integral equation is equivalent to the boundary value problem

$$\begin{cases}\frac{d^2y(t)}{dt^2}+\lambda y(t)=0\\ y(0)=y(1)=0\end{cases} \qquad (3.12)$$

The solution of the homogeneous problem, for $\lambda>0$ is given as

$$y(t)=\alpha_1\sin\sqrt{\lambda t}+\alpha_2\cos\sqrt{\lambda t} \qquad (3.13)$$

$y(0)=0\Rightarrow\alpha_2=0$ and $y(1)=0\Rightarrow\alpha_1\sin\lambda=0$, so either $\alpha_2=0$ which only gives the trivial solution $y\approx 0$ or $\sqrt{\lambda}=n\pi,$ for some integer n and $\lambda=n^2\pi^2$.

Hence, the eigenvalues are $\lambda_n=n^2\pi^2$ and the corresponding Eigen-functions are $y_n(t)=\sin(n\pi t)$

For the non-homogeneous, we assume the right hand side to be $-g(t)$ and $\lambda=1$, so that the equation (12) becomes

$$\begin{cases}\frac{d^2y(t)}{dt^2}+y(t)=-g(t)\\ y(0)=y(1)=0\end{cases} \qquad (3.14)$$

We are looking for the function $y(t)$ that will satisfy equation (14). Since the Fourier Transform is a linear operation, the time domain will produce an equation where each term corresponds to the term in the frequency domain.

If we take the Fourier Transform of (14), we get

$$\begin{cases}\Im\left\{\frac{d^2y(t)}{dt^2}\right\}+\Im\{y(t)\}=\Im\{-g(t)\}\\ y(0)=y(1)=0\end{cases} \qquad (3.15)$$

Equation (15) can be written as

$$\begin{cases}\Im\left\{\frac{d^2y(t)}{dt^2}\right\}+K(f)=-G(f)\\ y(0)=y(1)=0\end{cases} \qquad (3.16)$$

If we recall the differential property of the Fourier Transform, we observe that the derivatives in time become simple multiplication in the frequency domain i.e.

$$\Im\left\{\frac{dy(t)}{dt}\right\}=(2i\pi f)K(f),\ \Im\left\{\frac{d^2y(t)}{dt^2}\right\}=(2i\pi f)^2K(f),...,\Im\left\{\frac{d^ny(t)}{dt^n}\right\}=(2i\pi f)^nK(f) \qquad (3.17)$$

Equation (16) becomes

$$(2i\pi f)^2K(f)+K(f)=-G(f) \qquad (3.18)$$

$$\Rightarrow\left\{(2i\pi f)^2+1\right\}K(f)=-G(f)$$

$$\Rightarrow K(f)=-\frac{G(f)}{(2i\pi f)^2+1}=-\frac{G(f)}{1-4\pi^2f^2} \qquad (3.19)$$

In general, the solution is the inverse Fourier Transforms of the result in (19).

We observe that the multiplication of the two functions in the time domain results to a convolution in the Fourier domain. In other way, the multiplication of two functions in the Fourier domain will give the convolution in the time domain. Therefore, we have

$$y(t)=\Im^{-1}\left\{K(f)\right\}=\Im^{-1}\left\{-\frac{1}{1-4\pi^2f^2}\times G(f)\right\}=\Im^{-1}\left\{-\frac{1}{1-4\pi^2f^2}\right\}*\Im^{-1}\left\{G(f)\right\}$$

$$=\frac{e^{-|t|}}{2}*g(t)=\frac{1}{2}\int_{-\infty}^{\infty}e^{-|t-\xi|}g(\xi)d\xi \qquad (20)$$

Observe that the solution to the non-homogeneous side also satisfied the boundary condition.

Hence, the general solution is the sum of the solution we obtained i.e.

$$y(t)=\alpha_1\sin\sqrt{\lambda t}+\alpha_2\cos\sqrt{\lambda t}+\frac{1}{2}\int_{-\infty}^{\infty}e^{-|t-\xi|}g(\xi)d\xi \qquad (21)$$

4. Conclusion

It is shown in this work that singular integral equation of the first kind can be converted to boundary value problem. We further show that the Boundary value problem obtained when solved gives a trivial solution, also when we assume the problem to be a non-homogeneous and applying the Fourier Transform we obtained a one half of the two-sided decaying exponential function.

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