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# ANALYTICAL SOLUTIONS OF SCHRÖDINGER TIME INDEPENDENT WAVE EQUATION WITH VARSHNI POTENTIAL

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***Abstract***

***In this study, we have solved the Schrödinger time independent wave equation for the Varshni potential using the generalized Pekeris approximation model for the centrifugal term. We have obtained normalized wave functions and energy eigenvalues in closed forms. We have studied the variation of energy eigenvalues, normalization constants with energy determining parameter for three quantum states viz. 3s, 3p and 3d states, the results we obtained suggests the existence of a critical value of the energy determining parameter at which the energy of the system is optimized.***

**Keywords**: Varshni potential, Schrödinger equation, energy determining parameter, wave functions

## 1.0 Introduction

Extensive literature review has shown that in quantum mechanics, prominence is placed on the solution of wave equation owing to the fact that the quantum system under investigation is well defined if the wave functions and the associated energy eigenvalues derivable from the wave equation are known [1-3]. Physical variables of the system such as momentum, frequency of vibration, speed and wavelength can readily be determined from knowledge of the wave functions [1-2]. In order to determine the wave functions and energy eigenvalues of the system, the idea is to solve the Schrödinger equation for a given potential energy function of the system. Solving the Schrödinger equation usually poses a greater challenge because only few potential models [4] give exact solution for all quantum states nℓ (n being the principal quantum number and ℓ is the principal angular momentum quantum number) and only a handful of potential models have exact solution for the case of s-wave (ℓ = 0 state) [5-8], on the other hand, most potential models have no exact solutions for all quantum states [9-14]. In situations like this, approximate solutions (numerical or analytical) can be used. In the numerical approach, the solutions are not closed form solutions but the approach addresses issues of convergence of the solution, while in the analytical method, the solutions are closed forms but issues of convergence of solutions are not addressed. Therefore, the two methods of solutions are considered as complementary in the sense that the validity of analytical method of solution can be confirmed by the numerical method. Several approximations to the centrifugal term potential [15-17] have been proposed and used to solve the Schrödinger equation, most of these proposed approximations are limited to screening parameters of the potential and restricted to exponential-type potential models. Some couple of years ago, the approximation proposed by Pekeris [15] was viewed from another perspective [9] in a new approximation model termed the generalized Pekeris approximation [10], the approximation model gave quite excellent results when used to solve the Schrödinger equation for various potential models [1-2, 9-10]. A number of techniques have been used to solve the Schrödinger equation, among which include: exact and proper quantization rules [18-19], factorization method [20], generalized pseudospectral [21] method, parametric Nikiforov-Uvarov method [22].

## 2.0 Theoretical Approach

### 2.1 The Varshni Potential

The Varshni potential has useful applications in many areas of physics such as molecular, solid state and condensed matter physics. It has been used in molecular and condensed matter modelling [23]. So much works has been reported on the Varshni potential. The formula method was used [24] to obtain approximate solution of Schrödinger equation with the Varshni potential. The relations between the Varshni and Morse potentials [25] was established for covalent bond stretching energy. In another work, pseudospin, spin and symmetric solutions of Dirac equation for three potentials *viz*: Hellmann, Wei, Hua and Varshni potentials were determined [23]. The Varshni potential [23, 24] can be expressed as:

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$$V(r)=a-\frac{\nu b e^{-\delta(r-r_e)}}{r}. \quad (1)$$

where a and bare the potential strengths, δ is the screening parameter, ν is an adjustable parameter, r is the internuclear separation and $r_e$ is the equilibrium internuclear separation.

**2.2 Analytical Solution of Schrödinger Equation with Varshni Potential**

**2.2.1 The Radial Schrödinger Equation**

The radial Schrödinger equation [21] can be expressed as:

$$\frac{d^2\psi_{n\ell}}{dr^2}+\frac{2\mu}{\hbar^2}\left\{E_{n\ell}-V(r)-\frac{\ell(\ell+1)\hbar^2}{2\mu r_e^2}\left(\frac{r_e}{r}\right)^2\right\}\psi_{n\ell}=0. \quad (2)$$

where μ is the reduced mass of the molecule, $E_{n\ell}$ is the energy eigenvalue and $\psi_{n\ell}$ is the radial wave function. If we substitute Eq. (1) into Eq. (2), we get:

$$\frac{d^2\psi_{n\ell}}{dr^2}+\frac{2\mu}{\hbar^2}\left\{E_{n\ell}-a+\frac{\nu b e^{-\delta(r-r_e)}}{r}-\frac{\ell(\ell+1)\hbar^2}{2\mu r_e^2}\left(\frac{r_e}{r}\right)^2\right\}\psi_{n\ell}=0. \quad (3)$$

Let

$$z=e^{-\delta(r-r_e)}. \quad (4)$$

using Eq. (4), Eq. (3) transforms to:

$$z^2\psi''_{n\ell}(z)+z\psi'_{n\ell}(z)+\left\{\frac{2\mu}{\delta^2\hbar^2}(E_{n\ell}-a)+\frac{2\mu\nu b}{\delta^2\hbar^2 r_e}\left(\frac{r_e}{r}\right)z-\frac{\ell(\ell+1)}{\delta^2 r_e^2}\left(\frac{r_e}{r}\right)^2\right\}\psi_{n\ell}=0. \quad (5)$$

where prime denotes derivatives with respect to z. The factors $r/r_e$ and $(r/r_e)^2$ can be approximated by terms of a Taylor series expansion. In this paper we have used:

$$\frac{r_e}{r}\approx c_0+c_1(\tau-\alpha_{n\ell}). \quad (6)$$

where τ and its inverse $\tau^{-1}$ are appropriately chosen functions, and $\alpha_{n\ell}$ is an element in the domain of$\tau^{-1}$. The coefficients, $c_N$ (N = 1, 2, 3) are given by:

$$c_N=\left.\frac{d^N F}{d\tau^N}\right|_{\tau=\alpha_{n\ell}}. \quad (7)$$

with the function F given by [9, 10]

$$F(\tau)\equiv\left(\frac{r_e}{r}\right)^m=\left(1-\frac{\tau^{-1}}{\alpha_{n\ell}r_e}\right)^{-m}. \quad (8)$$

In the present work, we have chosen:

$$\tau=e^{\alpha_{n\ell}r_e}z. \quad (9)$$

and

$$\tau^{-1}=\log_e\tau. \quad (10)$$

By taking m = 1, Eq. (7) and Eq. (8) gives:

$$c_0=\frac{\delta r_e}{\delta r_e-\log_e\alpha_{n\ell}}. \quad (11)$$

and

$$c_1=\frac{\delta r_e}{(\delta r_e-\log_e\alpha_{n\ell})^2}. \quad (12)$$

Similarly, $(r/r_e)^2$ can be approximated by:

$$\left(\frac{r_e}{r}\right)^2\approx d_0+d_1(\tau-\alpha_{n\ell})+\tfrac{1}{2}d_2(\tau-\alpha_{n\ell})^2. \quad (13)$$

By taking m = 2, $c_N\equiv d_N$ in Eq. (7), Eq.( 8) gives

$$d_0=\frac{\delta^2 r_e^2}{(\delta r_e-\log_e\alpha_{n\ell})^2}. \quad (14)$$

$$d_1=\frac{2\delta^2 r_e^2}{\alpha_{n\ell}(\delta r_e-\log_e\alpha_{n\ell})^3}. \quad (15)$$

$$d_2=\frac{2\delta^2 r_e^2(3-\delta r_e+\log_e\alpha_{n\ell})}{\alpha_{n\ell}^2(\delta r_e-\log_e\alpha_{n\ell})^4}. \quad (16)$$

where $\alpha_{n\ell}\in[1,\infty)$

If we substitute Eq. (6) and Eq. (13) in Eq. (5) and using Eq. (9) get:

$$z^2\psi''_{n\ell}(z)+z\psi'_{n\ell}(z)+\left\{\begin{array}{l}\dfrac{2\mu}{\delta^2\hbar^2}\left(E_{n\ell}-a\right)+\dfrac{2\mu\nu b}{\delta^2\hbar^2 r_e}\left[\left(c_0-\alpha_{n\ell}c_1\right)z+c_1 e^{\alpha_{n\ell}r_e}z^2\right]\\ -\dfrac{\ell(\ell+1)}{\delta^2 r_e^2}\left[d_0-\alpha_{n\ell}d_1+\frac{1}{2}\alpha_{n\ell}^2 d_2+\left(d_1-\alpha_{n\ell}d_2\right)e^{\alpha_{n\ell}r_e}z+\frac{1}{2}d_2 e^{2\alpha_{n\ell}r_e}z^2\right]\end{array}\right\}\psi_{n\ell}=0 \tag{17}$$

which can be written as:

$$z^2\psi''_{n\ell}(z)+z\psi'_{n\ell}(z)+\left(-\eta z^2+\kappa z-\lambda\right)\psi_{n\ell}(z)=0\,. \tag{18}$$

where

$$\eta=-\frac{2\mu\nu b e^{\alpha_{n\ell}}}{\delta^2\hbar^2 r_e}c_1+\frac{\ell(\ell+1)e^{2\alpha_{n\ell}}}{2\delta^2 r_e^2}d_2\,. \tag{19}$$

$$\kappa=\frac{2\mu\nu b}{\delta^2\hbar^2 r_e}\left(c_0-\alpha_{n\ell}c_1\right)-\frac{\ell(\ell+1)e^{\alpha_{n\ell}r_e}}{\delta^2 r_e^2}\left(d_1-\alpha_{n\ell}d_2\right). \tag{20}$$

$$\lambda=-\frac{2\mu}{\delta^2\hbar^2}\left(E_{n\ell}-a\right)+\frac{\ell(\ell+1)}{\delta^2 r_e^2}\left(d_0-\alpha_{n\ell}d_1+\frac{1}{2}\alpha_{n\ell}^2 d_2\right). \tag{21}$$

To solve Eq. (18), consider the ansatz [1, 15]:

$$\psi_{n\ell}(z)=N_{n\ell}e^{-\frac{1}{2}pz}z^{\frac{1}{2}q}\varphi_{n\ell}(z). \tag{22}$$

where $N_{n\ell}$ are the analytical normalization constants, p and q are constants to be determined when Eq. (18) is satisfied by Eq. (22). From Eq. (22), we find:

$$\psi'_{n\ell}(z)=\left\{\frac{\varphi'_{n\ell}(z)}{\varphi_{n\ell}(z)}-\frac{p}{2}+\frac{q}{2z}\right\}\psi_{n\ell}(z). \tag{23}$$

and

$$\psi''_{n\ell}(z)=\left\{\frac{\varphi''_{n\ell}(z)}{\varphi_{n\ell}(z)}+\left(-p+\frac{q}{2}\right)\frac{\varphi'_{n\ell}(z)}{\varphi_{n\ell}(z)}+\frac{p^2}{4}-\frac{pq}{2z}-\frac{q}{2z^2}+\frac{q^2}{4z^2}\right\}\psi_{n\ell}(z). \tag{24}$$

Substitute Eq. (23) and Eq. (24) in Eq. (18) and simplifying, get:

$$z\varphi''_{n\ell}(z)+(q+1-pz)\varphi'_{n\ell}(z)+\left\{\kappa-\tfrac{1}{2}pq-\tfrac{1}{2}p+\left(\tfrac{1}{4}p^2-\eta\right)z+\frac{\frac{1}{4}q^2-\lambda}{z}\right\}\varphi_{n\ell}(z)=0 \tag{25}$$

Eq. (25) reduces to the Laguerre hypergeometric differential equation if the last-two terms of the coefficient of $\varphi_{n\ell}$ (z)separately varnish, this is true provided:

$$p=2\eta^{\frac{1}{2}}. \tag{26}$$

and

$$q=2\lambda^{\frac{1}{2}}. \tag{27}$$

Therefore, by putting Eq. (26) and Eq. (27) in Eq. (25), the hypergeometric equation is:

$$u\varphi''_{n\ell}(u)+\left(2\lambda^{\frac{1}{2}}+1-u\right)\varphi'_{n\ell}(u)+\left(\frac{\kappa}{2\eta^{\frac{1}{2}}}-\lambda^{\frac{1}{2}}-\tfrac{1}{2}\right)\varphi_{n\ell}(u)=0. \tag{28}$$

where

$$u=2\eta^{\frac{1}{2}}z\,. \tag{29}$$

Eq. (28) has the hypergeometric function as solution given by:

$$\varphi_{n\ell}(u)={}_1F_1\left(-n,2\lambda^{\frac{1}{2}}+1;u\right). \tag{30}$$

**2.2.2 The Normalization constant**

Normalization of wave functions [26, 27] requires that:

$$\int_0^{\infty}\left|\psi_{n\ell}(r)\right|^2 dr=1. \tag{31}$$

Putting Eq. (4), Eq. (29) and Eq. (22) in Eq. (31) get:

$$N_{n\ell}^2\int_0^{u_0}e^{-u}u^{q-1}\left|\varphi_{n\ell}(u)\right|^2 du=\delta p^q. \tag{32}$$

where

$$u_0=2\eta^{\frac{1}{2}}e^{\delta r_e}\,. \tag{33}$$

Eq. (32) gives the normalization constant as:

$$N_{n\ell}=\left\{\frac{\delta\, p^{q}}{\int_{0}^{u_0} e^{-u}\,u^{q-1}\left|{}_1F_1\left(-n,2\lambda^{\frac{1}{2}}+1;u\right)\right|^2 du}\right\}^{\frac{1}{2}}. \tag{34}$$

when Eq. (30) is used in Eq. (32).

**2.2.3Energy Eigenvalues**

The quantum condition [1, 15] requirement for a polynomial solution of Eq. (28) is given by:

$$\frac{\kappa}{2\eta^{\frac{1}{2}}}-\lambda^{\frac{1}{2}}-\tfrac{1}{2}=n. \tag{35}$$

Thus, we find:

$$\lambda=\left(n+\tfrac{1}{2}-\frac{\kappa}{2\eta^{\frac{1}{2}}}\right)^2. \tag{36}$$

Inserting Eq. (36) in Eq. (21), we find:

$$E_{n\ell}=a-\frac{\delta^2\hbar^2}{2\mu}\left(n+\tfrac{1}{2}-\frac{\kappa}{2\eta^{\frac{1}{2}}}\right)^2+\frac{\ell(\ell+1)\hbar^2}{2\mu r_e^2}\left(d_0-\alpha_{n\ell}\,d_1+\tfrac{1}{2}\alpha_{n\ell}^2\,d_2\right). \tag{37}$$

Substituting Eq. (19) and Eq. (20) in Eq. (37), we have:

$$E_{n\ell}=a-\frac{\delta^2\hbar^2}{2\mu}\left\{n+\tfrac{1}{2}-\frac{\frac{2\mu v b}{\delta^2\hbar^2 r_e}\left(c_0-\alpha_{n\ell}c_1\right)-\frac{\ell(\ell+1)e^{\alpha_{n\ell}r_e}}{\delta^2 r_e^2}\left(d_1-\alpha_{n\ell}\,d_2\right)}{2\left[-\frac{2\mu v b e^{\alpha_{n\ell}}}{\delta^2\hbar^2 r_e}c_1+\frac{\ell(\ell+1)e^{2\alpha_{n\ell}}}{2\delta^2 r_e^2}d_2\right]^{\frac{1}{2}}}\right\}^2 + \frac{\ell(\ell+1)\hbar^2}{2\mu r_e^2}\left(d_0-\alpha_{n\ell}\,d_1+\tfrac{1}{2}\alpha_{n\ell}^2\,d_2\right) \tag{38}$$

## 3.0 Result and Discussion

The data in Tables 1-3 shows computed normalization constants and energy eigenvalues as $\alpha_{n\ell}$ varies for three different values of screening parameters ($\delta$ = 0.001, 0.005 and 0.010**)**, as can be seen from the table, for a given value of $\delta$**,** $E_{n\ell}$ increases steeply with $\alpha$ up to a certain critical value after which it begins to increase slowly. As $\alpha_{n\ell}$ is further increased, $E_{n\ell}$ begins to decrease. The normalization constant follows an opposite trend, $N_{n\ell}$ decreases with increase in$\alpha_{n\ell}$, reaching a minimum value and later increases with increase in $\alpha_{n\ell}$. On the other hand, for a given value of $\alpha_{n\ell}$, $E_{n\ell}$ increases across a quantum state with $\ell$. For the special case of s-wave, ($\ell$ = 0) Eq. (38) gives:

$$E_{n0}=a-\frac{\delta^2\hbar^2}{2\mu}\left\{n+\tfrac{1}{2}-\frac{\frac{2\mu v b}{\delta^2\hbar^2 r_e}\left(c_0-\alpha_{n\ell}c_1\right)}{2\left[-\frac{2\mu v b e^{\alpha_{n\ell}}}{\delta^2\hbar^2 r_e}c_1\right]^{\frac{1}{2}}}\right\}^2. \tag{39}$$

Clearly, Eq. (39) is dependent of the energy determining parameter. The variation of $E_{n\ell}$ with $\alpha_{n\ell}$ are shown in Figures 1-3 for three states *viz*. 3s, 3p and 3d

## 4.0 Conclusion

In this research work, we have solved the Schrödinger time independent wave equation analytically for the Varshni potential and obtained the normalized wave functions and energy eigenvalues in closed forms and also studied the variation of the energy eigenvalues with energy determining parameter, our result suggested that there exists a critical value of the energy determining parameter at which the energy of the system is optimized, this work will be extended to cover other potential models to verify the relationship between the energy determining parameter and the optimized energy of the system

**Table 1.Energy Determining Parameter, Normalization Constants and Energy Eigenvalues in atomic unitsfor δ = 0.001, a = ν = -2, b = 1**

| $\alpha_{n\ell}$ | 33 | | 3p | | 3d | |
|---|---|---|---|---|---|---|
| | $N_{n\ell}(\alpha_{n\ell})$ | $-E_{n\ell}(\alpha_{n\ell})$ | $N_{n\ell}(\alpha_{n\ell})$ | $-E_{n\ell}(\alpha_{n\ell})$ | $N_{n\ell}(\alpha_{n\ell})$ | $-E_{n\ell}(\alpha_{n\ell})$ |
| 1.25 | 1.12E+35 | 2.04669959 | NaN | 2.01741102 | … | 1.99197878 |
| 1.50 | 2.37E+19 | 2.01430476 | 1.27E+25 | 2.00972965 | 4.34E+33 | 2.00338873 |
| 1.75 | 1.12E+14 | 2.00725779 | 3.79E+16 | 2.00557460 | 6.56E+20 | 2.00284898 |
| 2.00 | 1.67E+11 | 2.00434148 | 5.69E+12 | 2.00344522 | 2.67E+15 | 2.00189459 |
| 2.25 | 2.31E+09 | 2.00277509 | 3.07E+10 | 2.00219330 | 2.86E+12 | 2.00115324 |
| 2.50 | 8.08E+07 | 2.00181713 | 7.35E+08 | 2.00139124 | 3.23E+10 | 2.00061645 |
| 2.75 | 4.08E+06 | 2.00119025 | 3.05E+07 | 2.00085277 | 1.03E+09 | 2.00023304 |
| 3.00 | 3.23E+05 | 2.00076631 | 1.87E+06 | 2.00048362 | 4.76E+07 | 1.99996221 |
| 3.25 | 4.52E+04 | 2.00047662 | 2.03E+05 | 2.00023001 | 3.54E+06 | 1.99977462 |
| 3.50 | 1.44E+04 | 2.00028013 | 4.36E+04 | 2.00005825 | 5.00E+05 | 1.99964907 |
| 3.75 | 6.18E+03 | 2.00015027 | 5.60E+04 | 1.99994578 | 1.56E+05 | 1.99956992 |
| 4.00 | 5.93E+02 | 2.00006884 | 3.07E+03 | 1.99987670 | 3.06E+05 | 1.99952537 |
| 4.25 | 0.00E+00 | 2.00002283 | 0.00E+00 | 1.99983945 | 0.00E+00 | 1.99950642 |
| 4.50 | 0.00E+00 | 2.00000270 | 4.89E+01 | 1.99982539 | 9.64E+03 | 1.99950614 |
| 4.75 | 0.00E+00 | 2.00000133 | 0.00E+00 | 1.99982803 | 4.18E+02 | 1.99951917 |
| 5.00 | 2.63E+01 | 2.00001336 | 0.00E+00 | 1.99984243 | 2.64E+02 | 1.99954141 |
| 5.25 | 1.84E+02 | 2.00003471 | 0.00E+00 | 1.99986480 | 0.00E+00 | 1.99956969 |
| 5.50 | 1.11E+03 | 2.00006229 | 4.09E+02 | 1.99989230 | 0.00E+00 | 1.99960161 |
| 5.75 | 5.62E+03 | 2.00009376 | 1.75E+03 | 1.99992276 | 0.00E+00 | 1.99963539 |
| 6.00 | 2.32E+04 | 2.00012736 | 8.88E+03 | 1.99995455 | 5.39E+01 | 1.99966969 |

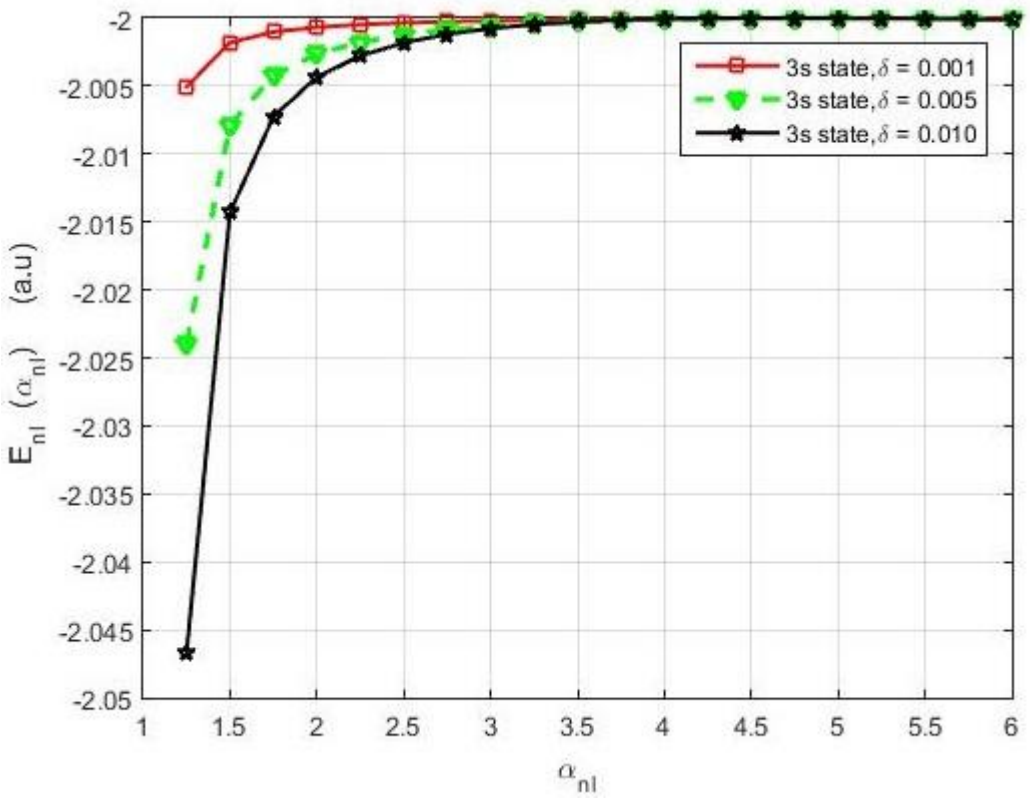


Figure 1. Plot of Energy Eigenvalues vs. Energy Determining parameter at δ = 0.001

**Table 2.Energy Determining Parameter, Normalization Constants and Energy Eigenvalues in atomic units for δ = 0.005, a = ν = -2, b = 1**

| $\alpha_{n\ell}$ | 33 | | 3p | | 3d | |
|---|---|---|---|---|---|---|
| | $N_{n\ell}(\alpha_{n\ell})$ | $-E_{n\ell}(\alpha_{n\ell})$ | $N_{n\ell}(\alpha_{n\ell})$ | | $N_{n\ell}(\alpha_{n\ell})$ | $-E_{n\ell}(\alpha_{n\ell})$ |
| 1.25 | 2.87E+49 | 2.02393652 | … | 2.01640220 | … | 2.00843802 |
| 1.50 | 1.17E+28 | 2.00788262 | 1.38E+32 | 2.00686448 | 8.30E+38 | 2.00525944 |
| 1.75 | 5.00E+20 | 2.00421378 | 3.09E+22 | 2.00385895 | 5.11E+25 | 2.00323424 |
| 2.00 | 6.31E+16 | 2.00264609 | 7.43E+17 | 2.00246227 | 7.39E+19 | 2.00212410 |
| 2.25 | 1.95E+14 | 2.00177972 | 1.14E+15 | 2.00166233 | 3.15E+16 | 2.00144176 |
| 2.50 | 2.75E+12 | 2.00123418 | 1.16E+13 | 2.00114916 | 1.74E+14 | 2.00098762 |
| 2.75 | 8.57E+10 | 2.00086535 | 3.05E+11 | 2.00079849 | 3.28E+12 | 2.00067069 |
| 3.00 | 3.43E+09 | 2.00060609 | 1.18E+10 | 2.00055041 | 1.17E+11 | 2.00044365 |
| 3.25 | 1.64E+08 | 2.00042018 | 5.19E+08 | 2.00037183 | 4.69E+09 | 2.00027902 |
| 3.50 | 1.29E+07 | 2.00028588 | 3.70E+07 | 2.00024255 | 2.80E+08 | 2.00015941 |
| 3.75 | 1.92E+06 | 2.00018908 | 4.94E+06 | 2.00014927 | 3.13E+07 | 2.00007303 |
| 4.00 | 9.61E+05 | 2.00012005 | 1.71E+06 | 2.00008275 | 7.58E+06 | 2.00001151 |
| 4.25 | 0.00E+00 | 2.00007181 | 0.00E+00 | 2.00003629 | 0.00E+00 | 1.99996871 |
| 4.50 | 5.09E+04 | 2.00003922 | 8.75E+04 | 2.00000494 | 5.74E+05 | 1.99994004 |
| 4.75 | 1.54E+03 | 2.00001839 | 8.45E+03 | 1.99998493 | 3.14E+05 | 1.99992197 |
| 5.00 | 5.51E+02 | 2.00000637 | 1.34E+03 | 1.99997339 | 1.61E+04 | 1.99991180 |
| 5.25 | 0.00E+00 | 2.00000089 | 4.98E+01 | 1.99996813 | 4.89E+03 | 1.99990745 |
| 5.50 | 0.00E+00 | 2.00000022 | 0.00E+00 | 1.99996743 | 1.41E+02 | 1.99990732 |
| 5.75 | 0.00E+00 | 2.00000300 | 0.00E+00 | 1.99996998 | 0.00E+00 | 1.99991015 |
| 6.00 | 3.29E+02 | 2.00000820 | 0.00E+00 | 1.99997477 | 0.00E+00 | 1.99991501 |

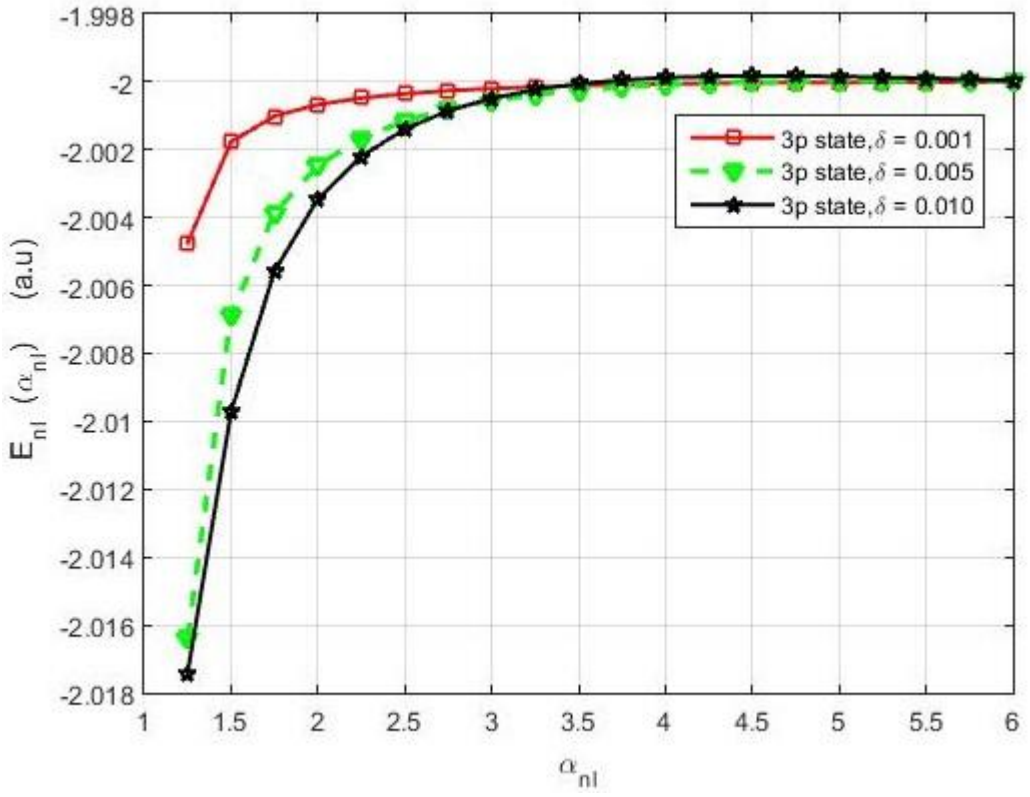


Figure 2. Plot of Energy Eigenvalues vs. Energy Determining parameter at δ = 0.005

**Table 3.Energy Determining Parameter, Normalization Constants and Energy Eigenvalues in atomic units for δ = 0.010, a = ν = -2, b = 1**

| $\alpha_{n\ell}$ | 33 | | 3p | | 3d | |
|---|---|---|---|---|---|---|
| | $N_{n\ell}(\alpha_{n\ell})$ | $-E_{n\ell}(\alpha_{n\ell})$ | $N_{n\ell}(\alpha_{n\ell})$ | | $N_{n\ell}(\alpha_{n\ell})$ | $-E_{n\ell}(\alpha_{n\ell})$ |
| 1.25 | 1.12E+35 | 2.04669959 | … | 2.01741102 | … | 1.99197878 |
| 1.50 | 2.37E+19 | 2.01430476 | 1.27E+25 | 2.00972965 | 4.34E+33 | 2.00338873 |
| 1.75 | 1.12E+14 | 2.00725779 | 3.79E+16 | 2.00557460 | 6.56E+20 | 2.00284898 |
| 2.00 | 1.67E+11 | 2.00434148 | 5.69E+12 | 2.00344522 | 2.67E+15 | 2.00189459 |
| 2.25 | 2.31E+09 | 2.00277509 | 3.07E+10 | 2.00219330 | 2.86E+12 | 2.00115324 |
| 2.50 | 8.08E+07 | 2.00181713 | 7.35E+08 | 2.00139124 | 3.23E+10 | 2.00061645 |
| 2.75 | 4.08E+06 | 2.00119025 | 3.05E+07 | 2.00085277 | 1.03E+09 | 2.00023304 |
| 3.00 | 3.23E+05 | 2.00076631 | 1.87E+06 | 2.00048362 | 4.76E+07 | 1.99996221 |
| 3.25 | 4.52E+04 | 2.00047662 | 2.03E+05 | 2.00023001 | 3.54E+06 | 1.99977462 |
| 3.50 | 1.44E+04 | 2.00028013 | 4.36E+04 | 2.00005825 | 5.00E+05 | 1.99964907 |
| 3.75 | 6.18E+03 | 2.00015027 | 5.60E+04 | 1.99994578 | 1.56E+05 | 1.99956992 |
| 4.00 | 5.93E+02 | 2.00006884 | 3.07E+03 | 1.99987670 | 3.06E+05 | 1.99952537 |
| 4.25 | 0.00E+00 | 2.00002283 | 0.00E+00 | 1.99983945 | 0.00E+00 | 1.99950642 |
| 4.50 | 0.00E+00 | 2.00000270 | 4.89E+01 | 1.99982539 | 9.64E+03 | 1.99950614 |
| 4.75 | 0.00E+00 | 2.00000133 | 0.00E+00 | 1.99982803 | 4.18E+02 | 1.99951917 |
| 5.00 | 2.63E+01 | 2.00001336 | 0.00E+00 | 1.99984243 | 2.64E+02 | 1.99954141 |
| 5.25 | 1.84E+02 | 2.00003471 | 0.00E+00 | 1.99986480 | 0.00E+00 | 1.99956969 |
| 5.50 | 1.11E+03 | 2.00006229 | 4.09E+02 | 1.99989230 | 0.00E+00 | 1.99960161 |
| 5.75 | 5.62E+03 | 2.00009376 | 1.75E+03 | 1.99992276 | 0.00E+00 | 1.99963539 |
| 6.00 | 2.32E+04 | 2.00012736 | 8.88E+03 | 1.99995455 | 5.39E+01 | 1.99966969 |

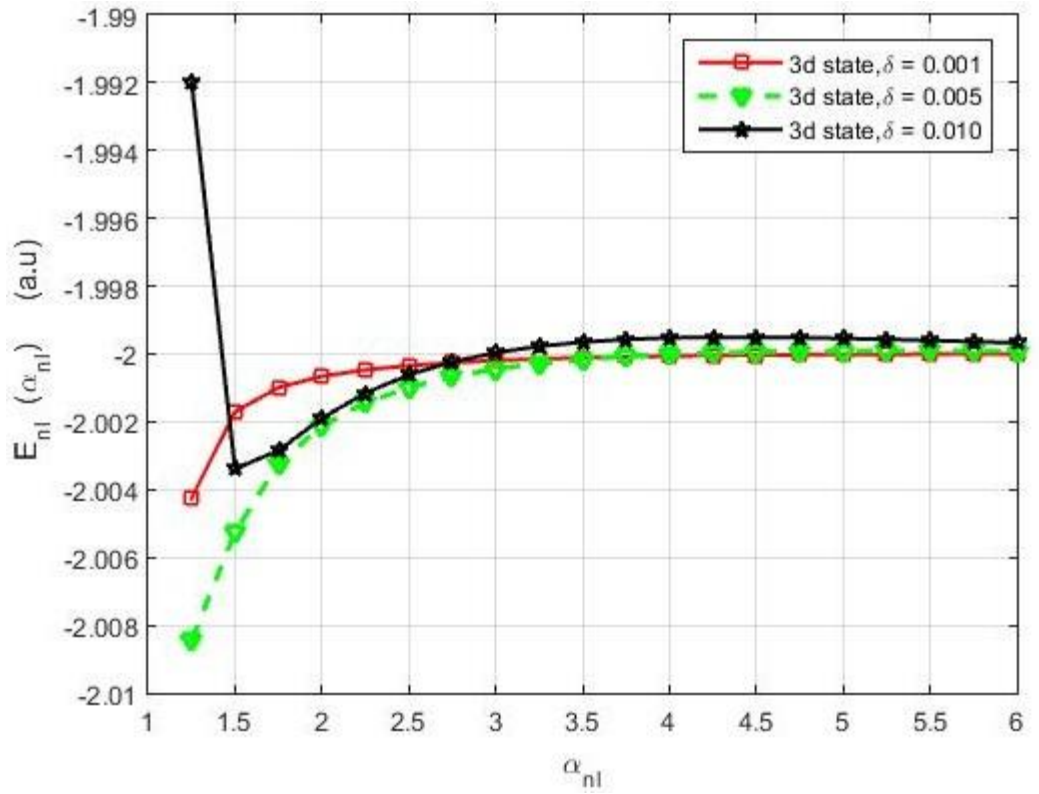


Figure 3. Plot of Energy Eigenvalues vs. Energy Determining parameter at δ = 0.010

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