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# SOLVABILITY OF LINEAR AND NONLINEAR KLEIN-GORDON EQUATION BY VARIATIONAL ITERATION METHOD, NEW ITERATIVE METHOD AND ADOMIAN DECOMPOSITION METHOD

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***Abstract***

***In this study, Variational Iteration Method (VIM) developed by Ji-Huan He, Adomian Decomposition Method (ADM) by Adomian, New Iterative Method (NIM) developed by Daftardar Gejji and Jafari and the modified Adomian Decomposition method by Wazwaz have been employed to solve the linear and nonlinear Klein-Gordon equations. The solution is calculated is calculated in the form of series in which its component are computed by applying a recursive relation. To illustrate the ability and reliability of the method some examples are provided. In this study, we compare numerical results with the exact solution. The results show that the Variational Iteration Method, Adomian Decomposition Method and New Iterative Method are powerful tolls in solving the Klein-Gordon equation and they can be used to solve other linear and nonlinear equations***

***Keywords*****:** Klein-Gordon equation, Variational iteration Method, Lagrange multiplier, New Iterative Method, Relativistic Wave Equation.

## 1. Introduction

Klein-Gordon equation plays an important role in mathematical physics, it appears in quantum field theory, dispersive wave-phenomena, plasma physics, nonlinear optics and applied and physical sciences.

We consider the Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + Nu(x,t) = f(x,t) \tag{1}$$

Subject to the initial conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x) \tag{2}$$

where $u$ is a function of x and t, $Nu(x,t)$ is a nonlinear function and $f(x,t)$ is a known or given analytical function [1]. Because of the importance of Klein-Gordon equation in quantum mechanics, several techniques have been developed in order to compute the solution of these equations, such as Sumudu Decomposition Method [2], Perturbation Method [3]. Variational Iteration Method [4], Homotopy Pertubation Transform Method [5], Modified New Iterative Method [6], Homotopy Pertubation Method [7], Adomian Decomposition Method and its convergence [8], Differential Transform Method [9], Homotopy Analysis Method [10], Laplace Decompostion Method [11], New Perturbation Iteration Transform Method [12].

In this study, three (3) of these methods were applied to solve the Klein-Gordon equation, these methods are Variational Iteration Method (VIM), New Iterative Method (NIM) and Adomian Decomposition Method (ADM) with its modification. These methods have been widely used in solving different types of differential equations in Physics, Engineering and Modelling. They have been proved to be powerful, reliable, which can effectively easily and accurately solve higher order linear and nonlinear differential problems with rapid convergence with number of iterations [13].

## 2. METHODS

### I. VARIATIONAL ITERATION METHOD (VIM)

Consider the following differential equation:

$$Lu + Nu = g(t) \tag{3}$$

Where, $L$ and $N$ are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term.
The variational iteration method presents a correction functional as follows:

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$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\varepsilon)\big(Lu_n(\varepsilon) + N\tilde{u}_n(\varepsilon) - g(\varepsilon)\big)d\varepsilon \quad (4)$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of the variational iteration method require first the determination of the Lagrange multiplier $\lambda(\varepsilon)$ that will be identified optimally. Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\varepsilon)$. In other words, carrying out the integration as follows can yield:

$$\int \lambda(\varepsilon)\, u'_n(\varepsilon)d\varepsilon = \lambda(\varepsilon)u_n(\varepsilon) - \int \lambda'(\varepsilon)\, u_n(\varepsilon)d\varepsilon,$$

$$\int \lambda(\varepsilon)\, u''_n(\varepsilon)d\varepsilon = \lambda(\varepsilon)'u_n(\varepsilon) - \lambda'(\varepsilon)u_n(\varepsilon) + \int \lambda''(\varepsilon)\, u_n(\varepsilon)d\varepsilon \quad (5)$$

Having determined the Lagrange multiplier $\lambda(\varepsilon)$, the successive approximations $u_{n+1}$, $n \geq 0$, of the solution $u$ will be readily obtained upon using any selective function $u_0$.

However, for fast convergence, the function $u_0(x,t)$ should be selected by using the initial conditions as follows:

$u_0(x,t) = u(x,0)$ for first order

$u_0(x,t) = u(x,0) + tu_t(x,0)$ for second order (6)

Consequently, the solution

$$u = \lim_{n\to\infty} u_n \quad (7)$$

In other words, equation (5) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations (Wazwaz, 2009).

## II. NEW ITERATIVE METHOD

To illustrate the idea of the NIM, we consider the following general functional equation:

$$u = f + N(u) \quad (8)$$

where $N$ is a nonlinear operator and $f$ is a given function. We can find the solution of equation having the series form

$$u = \sum_{i=0}^{\infty} u_i \quad (9)$$

The nonlinear operator N can be decomposed as:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \quad (10)$$

Substituting equations (9) and (10) into equation (8) gives

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \quad (11)$$

We define the recurrence relation of equation in the following way:

$$u_0 = f$$
$$u_1 = N(u_0)$$
$$u_2 = N(u_0 + u_1) - N(u_0) \quad (12)$$
$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$$
$$u_{n+1} = N(u_0 + u_1 + \dots + u_n) - N(u_0 + u_1 + \dots + u_{n-1}); \quad n = 1, 2, 3$$

Then

$$u_1 + \cdots + u_{m+1} = N(u_0 + u_1 + \cdots + u_m); \quad m = 1,2,3$$

and

$$\sum_{i=0}^{\infty} u_i = f + N\left(\sum_{j=0}^{\infty} u_j\right) \quad (13)$$

The m-term approximate solution of (8) is given by $u = u_0 + u_1 + u_2 \dots + u_{m-1}$

## III. ADOMIAN DECOMPOSITION METHOD

To give a clear overview of Adomian decomposition method,

Considering the following equation:

$$Lu + Ru = g \quad (14)$$

where $L$ is, mostly, the lower order derivative which is assumed to be invertible, $R$ is a linear differential operator, and $g$ is a source term.

Applying the inverse operator ($L^{-1}$) to both sides of (14) and using the initial condition to obtain

$$u = f - L^{-1}Ru \quad (15)$$

where the function $f$ represents the terms arising from integrating the source term $g$ and noting the prescribed conditions.

The Adomian Decomposition Method assumes that the unknown function $u$ can be expressed by an infinite series of the form

$$u(x,y) = \sum_{n=0}^{\infty} (u_n(x,y)) \quad (16)$$

or equivalently,

$$u = u_0 + u_1 + u_2 + \dots \quad (17)$$

where the components $u_0, u_1, u_2, \cdots$ are usually recurrently determined.

Substituting equation (16) into equation (15) yields

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R(\sum_{n=0}^{\infty} u_n)\right) \qquad (18)$$

For simplicity, Equation (18) can be re-written as

$$u_0 + u_1 + u_2 + \cdots = \sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R(\sum_{n=0}^{\infty} u_0 + u_1 + u_2 + \cdots)\right) \qquad (19)$$

To construct the recursive relation needed for the determination of the components $u_0, u_1, u_2, \cdots$, it is important to note that Adomian method suggests that the zeroth component $u_0$ is
usually defined by the function $f$ described above, that is, by all terms that are not included under the inverse operator $L^{-1}$, which arise from the initial data and from integrating the inhomogeneous term. Accordingly, the formal recursive relation is defined by

$$u_0 = f$$
$$u_{k+1} = -L^{-1}\big(R(u_k)\big) \quad , \quad k \geq 0 \qquad (20)$$

or equivalently,

$$u_0 = f$$
$$u_1 = -L^{-1}\big(R(u_0)\big)$$
$$u_2 = -L^{-1}\big(R(u_1)\big)$$
$$u_3 = -L^{-1}\big(R(u_2)\big) \qquad (21)$$

It is clearly seen that the relations (21) reduced the differential equation under consideration into an elegant determination of computable components. Having determined the relations (21), a series form solution is obtained by substituting relations (21) into equations (16).
The approximate solution is given by $u = u_0 + u_1 + u_2 \ldots + u_{m-1}$

## IV. MODIFIED ADOMIAN DECOMPOSITION METHOD

Modified Adomian decomposition method developed by Wazwaz (2009). The modified decomposition method will further accelerate the convergence of the series solution. It is to be noted that in this study, the modified decomposition method will be applied to linear inhomogeneous and nonlinear Klein-Gordon equations.
The decomposition method admits the use of the recursive relation,

$$u_0 = f,$$
$$u_{k+1} = -L^{-1}(Ru_k), \quad k \geq 0 \qquad (22)$$

the components $u_n, \ \ n \geq 0$ is obtained.
The modified decomposition method introduces a slight variation to the recursive relation (22) that will lead to the determination of the components of $u$ in a faster and easier way.
For specific cases, the function $f$ can be set as the sum of two partial functions, namely $f_1$ and $f_2$.
In other words, we have

$$f = f_1 + f_2. \qquad (23)$$

Using equation (23), we introduce a qualitative change in the formation of the recursive relation (22). To reduce the size of calculations, we identify the zeroth component $u_0$ by one part of $f$ , namely $f_1$ or $f_2$. The other part of $f$ can be added to the component $u_1$ among other terms. In other words, the modified recursive relation can be identified by

$$u_0 = f_1$$
$$u_1 = f_2 - L^{-1}\big(R(u_0)\big) \qquad (24)$$
$$u_{k+1} = -L^{-1}\big(R(u_k)\big)$$

The success of this modification depends only on the choice of $f_1$ and $f_2$, and this can be made through trials. Second, if $f$ consists of one term only, the standard decomposition method should be employed in this case.
The approximate solution is given by $u = u_0 + u_1 + u_2 \ldots + u_{m-1}$

## 3. NUMERICAL APPLICATIONS

In this section, we use the three methods in solving linear and nonlinear Klein-Gordon equations.

### 3.1 Linear Klein-Gordon Equation

**Example 1**

Consider the following Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0$$

With the initial conditions

$$u(x,0)=0, \ u_t(x,0) = x \qquad (25)$$

and the exact solution is

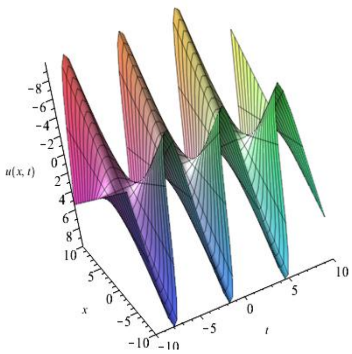
$$u(x,t) = x \,\mathrm{Sin}\, t$$

Following the procedures in section 2 and by substituting the obtained coefficient in the equation, the solution for the three methods become

$u(x,t) = x\,\mathrm{Sin}\,t$ (26)

Which is the exact solution.

**Table 1** shows the numerical results of the three methods for example (1)



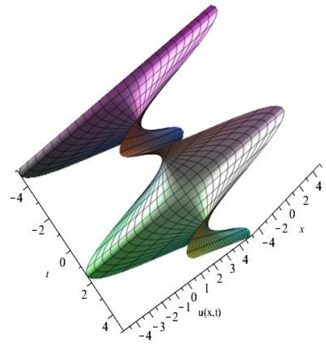




**Figure 1a: 3D plot of example 1 at x=-10..10, t=-10..10** **Figure 1b: 3D plot of example 1 at x=-5..5, t=-5..5**

**Example 2**

Consider the Klein-Gordon equation

$u_{tt} - u_{xx} - u = 0$

With initial conditions

$u(x,0) = \ Sin(x)\ + 1,\ u_t(x,0) = 0$ (27)

and the exact solution is

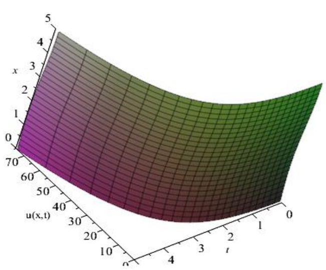
$u(x,t) = sin(x) + cosh(t)$

Following the procedures for the three methods after 3 iterations, the solution becomes

$u(x,t) = sin(x) + cosh(t)$ (28)

Which is the exact solution.

**Table 2** shows the numerical results of the three methods for example (2)



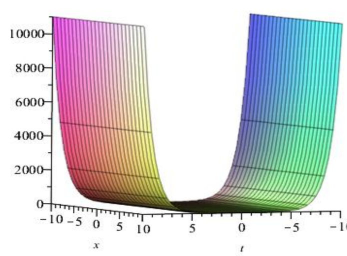




**Figure 2a: 3D plot of example 2 at x=0..5, t=0..5** **Figure 2b: 3D plot of example 2 at x=-10..10, t=-10..10**

**Example 3**

Consider the inhomogeneous linear Klein-Gordon equation

$u_{tt} - u_{xx} + u = 2\sin x$ (29)

With the initial conditions

$u(x,0) = \sin x, \quad u_t(x,0) = 1$

and the exact solution is

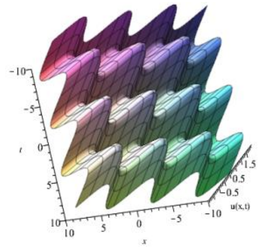
$u(x,t) \ = \ sin\,x \ + sin\,t$

Following the procedures for the three methods after 3 iterations, the solution becomes

$u(x,t) \ = \ sin\,x \ + sin\,t$ (30)

Which is the exact solution

**Table 3** shows the numerical results of the three methods for example (3)



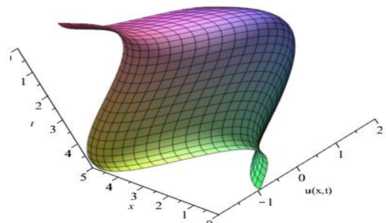


**Figure 3a: 3D plot of example 3 at x=-10..10, t=-10..10** **Figure 3b: 3D plot of example 3 at x= 0..5, t= 0..5**

### 3.2 Nonlinear Klein-Gordon Equations

**Example 4**

Consider the following Nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = x^2t^2 \quad (31)$$

With the boundary conditions

$u(x,0) = 0, \quad u_t(x,0) = x$

The exact solution is

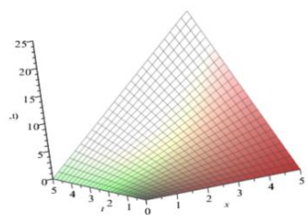
$u(x,t) = xt$

Following the procedures for the three methods after 3 iterations, the solution becomes

$$u(x,t) = xt \quad (32)$$

Which is the exact solution

**Table 4** shows the numerical results of the three methods for example (4)



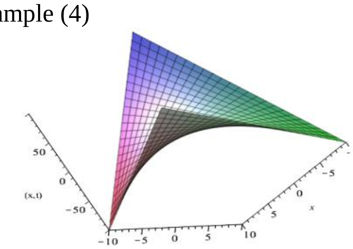


**Figure 4a: 3D plot of example 4 at x=0..5, t= 0..5** **Figure 4b: 3D plot of example 4 at x= -10..10, t= -10..10**

**Example 5**

Given the following nonlinear inhomogeneous Klein-Gordon equation:

$$u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2 \quad (33)$$

With the initial conditions

$u(x,0) = 1, \qquad u_t(x,0) = x$

The Exact solution is

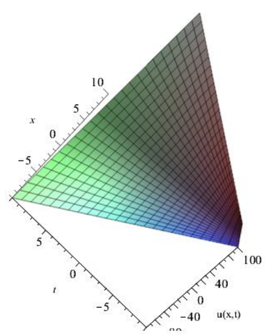
$u(x,t) = 1 + xt$

Following the procedures for the three methods after 3 iterations, the solution becomes

$$u(x,t) = 1 + xt \quad (34)$$

Which is the exact solution

**Table 5** shows the numerical results of the three methods for example (5)



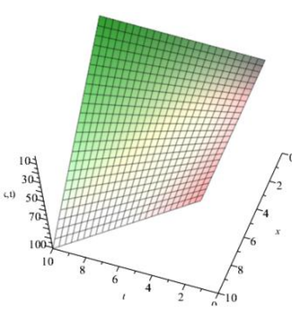


**Figure 5a: 3D plot of example 5 at x= -10..10, t= -10..10** **Figure 5b: 3D plot of example 5 at x= 0..10, t= 0..10**

**Example 6**

Given the following nonlinear inhomogeneous Klein-Gordon equation:

$$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4t^4 \quad (35)$$

With the initial conditions

$u(x,0) = 0, \qquad u_t(x,0) = 0$

And the exact solution is

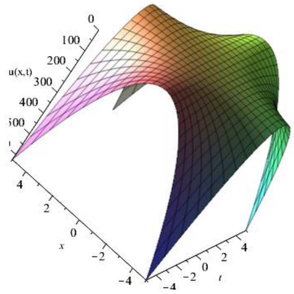
$u(x,t) = x^2t^2$

Following the procedures for the three methods after 3 iterations, the solution becomes

$$u(x,t) = x^2t^2 \tag{36}$$

Which is the exact solution

**Table 6** shows the numerical results of the three methods for example (6)



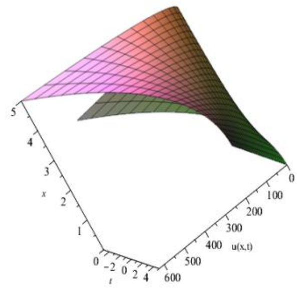




**Figure 6a: 3D plot of example 5 at x= -5..5, t= -5..5** **Figure 6b: 3D plot of example 5 at x= 0..5, t= -5..5**

## 4. RESULTS

**Table 1:** comparison of the3 methods for example 1 at x = 0..2, t = 0.3 after 2nd iteration

| X | VIM | NIM | ADM |
|---|---|---|---|
| 0.0 | 0 | 0 | 0 |
| 0.2 | 0.05910405 | 0.05910405 | 0.05910405 |
| 0.4 | 0.11820810 | 0.11820810 | 0.11820810 |
| 0.6 | 0.17731215 | 0.17731215 | 0.17731215 |
| 0.8 | 0.23641620 | 0.23641620 | 0.23641620 |
| 1.0 | 0.29552025 | 0.29552025 | 0.29552025 |
| 1.2 | 0.35462430 | 0.35462430 | 0.35462430 |
| 1.4 | 0.41372835 | 0.41372835 | 0.41372835 |
| 1.6 | 0.47283240 | 0.47283240 | 0.47283240 |
| 1.8 | 0.53193645 | 0.53193645 | 0.53193645 |
| 2.0 | 0.59104050 | 0.59104050 | 0.59104050 |

**Table 2:**comparison of the3 methods for example 2 for x = 0..2, t = 0.3 after 2nd iteration

| x | VIM | NIM | ADM |
|---|---|---|---|
| 0 | 1.04533750000 | 1.04533750000 | 1.04533750000 |
| 0.2 | 1.24400683080 | 1.24400683080 | 1.24400683080 |
| 0.4 | 1.43475584231 | 1.43475584231 | 1.43475584231 |
| 0.6 | 1.60997997340 | 1.60997997340 | 1.60997997340 |
| 0.8 | 1.76269359090 | 1.76269359090 | 1.76269359090 |
| 1.0 | 1.88680848481 | 1.88680848481 | 1.88680848481 |
| 1.2 | 1.97737658597 | 1.97737658597 | 1.97737658597 |
| 1.4 | 2.03078722999 | 2.03078722999 | 2.03078722999 |
| 1.6 | 2.04491110304 | 2.04491110304 | 2.04491110304 |
| 1.8 | 2.01918513088 | 2.01918513088 | 2.01918513088 |
| 2.0 | 1.95463492683 | 1.95463492683 | 1.95463492683 |

**Table 3:**comparison of the3 methods for example 3 for x = 0..2, t = 0.3 after 2nd iteration

| x | VIM | NIM | ADM |
|---|---|---|---|
| 0 | 0.2955202500 | 0.2955202500 | 0.2955202500 |
| 0.2 | 0.4941895808 | 0.4941895808 | 0.4941895808 |
| 0.4 | 0.6849385923 | 0.6849385923 | 0.6849385923 |
| 0.6 | 0.8601627234 | 0.8601627234 | 0.8601627234 |
| 0.8 | 1.0128763409 | 1.0128763409 | 1.0128763409 |
| 1.0 | 1.1369912348 | 1.1369912348 | 1.1369912348 |
| 1.2 | 1.2275593360 | 1.2275593360 | 1.2275593360 |
| 1.4 | 1.2809699800 | 1.2809699800 | 1.2809699800 |
| 1.6 | 1.2950938530 | 1.2950938530 | 1.2950938530 |
| 1.8 | 1.2693678809 | 1.2693678809 | 1.2693678809 |
| 2.0 | 1.2048176768 | 1.2048176768 | 1.2048176768 |

**Table 4:**comparison of the3 methods for example 4 for x = 0..2, t = 0.3 after 2nd iteration

| x | VIM | NIM | MADM |
|---|---|---|---|
| 0 | 0.000 | 0.000 | 0.000 |
| 0.2 | 0.060 | 0.060 | 0.060 |
| 0.4 | 0.120 | 0.120 | 0.120 |
| 0.6 | 0.180 | 0.180 | 0.180 |
| 0.8 | 0.240 | 0.240 | 0.240 |
| 1.0 | 0.300 | 0.300 | 0.300 |
| 1.2 | 0.360 | 0.360 | 0.360 |
| 1.4 | 0.420 | 0.420 | 0.420 |
| 1.6 | 0.480 | 0.480 | 0.480 |
| 1.8 | 0.540 | 0.540 | 0.540 |
| 2.0 | 0.600 | 0.600 | 0.600 |

**Table 5:**comparison of the3 methods for example 5 for x = 0..2, t = 0.3 after 2nd iteration

| **x** | VIM | NIM | MADM |
|---|---|---|---|
| 0 | 1.000 | 1.000 | 1.000 |
| 0.2 | 1.060 | 1.060 | 1.060 |
| 0.4 | 1.120 | 1.120 | 1.120 |
| 0.6 | 1.180 | 1.180 | 1.180 |
| 0.8 | 1.240 | 1.240 | 1.240 |
| 1.0 | 1.300 | 1.300 | 1.300 |
| 1.2 | 1.360 | 1.360 | 1.360 |
| 1.4 | 1.420 | 1.420 | 1.420 |
| 1.6 | 1.480 | 1.480 | 1.480 |
| 1.8 | 1.540 | 1.540 | 1.540 |
| 2.0 | 1.600 | 1.600 | 1.600 |

**Table 6:**comparison of the3 methods for example 6 for x = 0..2, t = 0.3 after 2nd iteration

| X | VIM | NIM | MADM |
|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.0036 | 0.0036 | 0.0036 |
| 0.4 | 0.0144 | 0.0144 | 0.0144 |
| 0.6 | 0.0324 | 0.0324 | 0.0324 |
| 0.8 | 0.0576 | 0.0576 | 0.0576 |
| 1.0 | 0.0900 | 0.0900 | 0.0900 |
| 1.2 | 0.1296 | 0.1296 | 0.1296 |
| 1.4 | 0.1764 | 0.1764 | 0.1764 |
| 1.6 | 0.2304 | 0.2304 | 0.2304 |
| 1.8 | 0.2916 | 0.2916 | 0.2916 |
| 2.0 | 0.3600 | 0.3600 | 0.3600 |

## 5. CONCLUSION

In this paper, the methods of VIM, NIM and ADM have been successfully performed for Klein-Gordon equations. The obtained results show that the three methods yielded the same results and they were excellent agreement with the exact solutions. It is capable to converge to exact solutions with fewest number of iterations.

Tables 1-6 and figures 1-6 justify that the methods are reliable and efficient and can be applied to linear and nonlinear equations of different parameters.

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