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ANALYTICAL SOLUTION OF THE THIRD-ORDER KORTEWEG-DE VRIES (KDV) EQUATION

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Abstract

In this paper, the approximate solution of the third-order Kortewed-de Vries (KdV)equations is obtained by the Variational Iteration Method (VIM) developed by Ji-Huan He and the New Iterative Method (NIM) developed by Daftardar Gejji and Jafari. These methods provide the solution in the form of a convergent series. To illustrate the ability and the effectiveness of the methods, some examples were provided. The results showed that the methods are very simple, effective, powerful and can easily be applied to other linear and nonlinear PDEs.

***Keywords*:** Variational iteration Method, New Iterative Method, Korteweg-de Vriesequations, Lagrange multiplier.

1. INTRODUCTION

The Korteweg-de Vries equation (KdV) was derived by D.J. Korteweg and G. de Vries in 1895. The KdV equation is used to study the unusual water waves that occur in shallow, narrow channels such as canals [1]. It also occurs in many physical applications such as non-linear plasma waves which exhibit certain dissipative effects, propagation of waves and propagation of bores in shallow water waves. [2]

The third order Korteweg-de Vries (KdV) equation is of the form:

$$u_t(x,t) - auu_x(x,t) + bu_{xxx}(x,t) = 0 \quad (1)$$

Subject to the initial conditions

$$u(x,0) = f(x) \quad (2)$$

where *a* and *b* are constants,$u_x$and u_t are partial derivatives of function *u* with respect to space *x* and *t* respectively and the nonlinear term uu_x tends to localize the wave, whereas the wave was spread out by dispersion. The formulation of solitons that have a single humped waves was define by delicate balance between uu_x and u_{xxx}. the displacement which describes how waves evolve under the competing but comparable effects of weak nonlinearity and weak dispersion is denoted by *u(x,t)*[3]

Over the years, Scientists have made several attempts to obtain the analytical and numerical solutions of KdV equation[4,5,6], Some of the methods used are Homotopy Perturbation Method [7,8,9,10], Elzaki Transform Method [11], Adomian Polynomial and Elzaki Transform Method [12], New Cubic B-Spline Method [13], Hyperbola Function Method [Luwai], Finite Difference Method and Adomian Decomposition Method [14], Multiquadric Quasi-Interpolation Method [15],Meshfree Method [16], Collocation and Radial Basis Function Method , Method of Lines [Schiesser], Small Time Solution [17.], Finite Difference Method, Homotopy Analysis Method, Variational Iteration Method, Homotopy Perturbation and Homotopy Analysis Method [18] and many others.

In this paper, the solution of third order Korteweg-de Vries (KdV) equationsare obtained by Variational Iteration Method (VIM) and New Iterative Method (NIM) are obtained. These methods give the solutions as an approximate analytical solution in series form and it yields exact solution with few iterations.

2. VARIATIONAL ITERATION METHOD (VIM)

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t) \quad (3)$$

Where, *L* and *N* are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term.

The variational iteration method presents a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\varepsilon)\big(Lu_n(\varepsilon) + N\tilde{u}_n(\varepsilon) - g(\varepsilon)\big)d\varepsilon \quad (4)$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$is a restricted variation which means $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of the variational iteration method require first the determination of the Lagrange multiplier $\lambda(\varepsilon)$ that will be identified optimally. Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\varepsilon)$. In other words, carrying out the integration as follows can yield:

$$\int \lambda(\varepsilon)\, u'_n(\varepsilon)d\varepsilon = \lambda(\varepsilon)u_n(\varepsilon) - \int \lambda'(\varepsilon)\, u_n(\varepsilon)d\varepsilon, \quad (5)$$

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Having determined the Lagrange multiplier $\lambda(\varepsilon)$, the successive approximations u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0.
However, for fast convergence, the function $u_0(x,t)$ should be selected by using the initial conditions as follows:
$u_0(x,t) = u(x,0)$ for first order
Consequently, the solution
$u = \lim_{n\to\infty} u_n$ (6)
In other words, equation (5) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations [3,18].

3. NEW ITERATIVE METHOD (NIM)

To illustrate the idea of the NIM, we consider the following general functional equation:
$u = f + N(u)$ (7)
where N is a nonlinear operator and f is a given function. We can find the solution of equation (7) having the series form
$u = \sum_{i=0}^{\infty} u_i$ (8)
The nonlinear operator N can be decomposed as:
\$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)\}$ (9)
Substituting equations (8) and (9) into equation (7) gives
$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)\}$ (10)
We define the recurrence relation of equation in the following way:
$u_0 = f$
$u_1 = N(u_0)$
$u_2 = N(u_0 + u_1) - N(u_0)$ (11)
$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$
$u_{n+1} = N(u_0 + u_1 + \ldots + u_n) - N(u_0 + u_1 + \ldots + u_{n-1})$; n =1, 2, 3
Then
$u_1 + \cdots + u_{m+1} = N(u_0 + u_1 + \cdots + u_m)$; m = 1,2, 3
And
$\sum_{i=0}^{\infty} u_i = f + N(\sum_{j=0}^{\infty} u_j)$ (12)
The m-term approximate solution of (7) is given by $u = u_0 + u_1 + u_2 \ldots + u_{m-1}$
(Bhaleka S. & Daftardar-Gejji V.)

4. NUMERICAL APPLICATIONS

Example 4.1
Consider the following KDV equation
$u_t - 6uu_x + u_{xxx} = 0$ (13)
With the initial condition
$u(x,0) = 6x$
The exact solution is given by$u(x,t) = \frac{6x}{1-36t}$

Method 1: VIM
The correction functionalfor example 4.1 is
$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6\tilde{u}_n(x,\varepsilon)\frac{\partial \tilde{u}_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 \tilde{u}_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$ (14)
Taking variations on both sides gives
$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6\tilde{u}_n(x,\varepsilon)\frac{\partial \tilde{u}_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 \tilde{u}_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$
$\widetilde{u_n}$ is a restrictive vibration, therefore $\delta\tilde{u}_n = 0$
$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon}\right] d\varepsilon$
Simplify by integration by parts
$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\lambda(\varepsilon)u_n(x,\varepsilon) - \int_0^t \lambda'(\varepsilon)\delta u_n(x,\varepsilon)\, d\varepsilon$
Making the correction functional stationary
$\delta u_{n+1}(x,t) = 0$
We have
$0 = \delta u_n(x,t) + \delta\lambda(\varepsilon)u_n(x,\varepsilon) - \int_0^t \lambda'(\varepsilon)\delta u_n(x,\varepsilon)\, d\varepsilon$
The stationary points are $1 + \lambda = 0$ and $\lambda' = 0$
Solving this, we have$\lambda = -1$

Hence, the iterative formula becomes

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6u_n(x,\varepsilon)\frac{\partial u_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon \qquad (15)$$

From the initial condition, we have

$u_0 = 6x$

Equation (15) at n = 0 becomes

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\varepsilon)}{\partial \varepsilon} - 6u_0(x,\varepsilon)\frac{\partial u_0(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_0(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$$

$u_1(x,t) = 6x + 6^3xt = 6x(1 + 36t)$

When n = 1

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,\varepsilon)}{\partial \varepsilon} - 6u_1(x,\varepsilon)\frac{\partial u_1(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_1(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$$

$u_2(x,t) = 6x + 6^3t + 6^5xt^2 = 6x(1 + 36t + (36t)^2)$

Consequently, the successive approximations are obtained

$u_1(x,t) = 6x(1 + 36t)$

$u_2(x,t) = 6x(1 + 36t + (36t)^2)$

$u_3(x,t) = 6x(1 + 36t + (36t)^2 + (36t)^3)$

Hence, the closed form and exact solution is given as

$$u(x,t) = 6x(1 + 36t + (36t)^2 + (36t)^3 + (36t)^4 + \cdots \ldots) \qquad (16)$$

$$u(x,t) = \frac{6x}{1-36t} \qquad (17)$$

Method 2: NIM

Considering Example 4.1

Using NIM, the equation (13) is equivalent to the following integral equation

$$u = f + \int_0^t (6uu_x - u_{xxx})dt \qquad (18)$$

From the initial condition, we have

$u_0 = 6x$

$N(u_k) = \int_0^t (6u_k u_{kx} - u_{kxxx})dt$

When k = 0,

$N(u_0) = u_1 = \int_0^t (6u_0 u_{0x} - u_{0xxx})dt$

$= \int_0^t (6(6x)(6) - 0)dt = \int_0^t 6^3x dt = 6^3xt$

$u_1 = 6^3xt$

$u_2 = N(u_0 + u_1) - N(u_0)$

$u_2 = 6^3xt + 6^5xt^2 - 6^3xt = 6^5xt^2$

$u_3 = 6^3xt + 6^5xt^2 + 6^7xt^3 - 6^3xt - 6^5xt^2 = 6^7xt^3$

The successive approximations are:

$u_0 = 6x$

$u_1 = 6^3xt$

$u_2 = 6^5xt^2$

$u_3 = 6^7xt^3$

Thus the series solution of example 4.1 is given by

$u(x,t) = \sum_{n=1}^{\infty} u_n = 6x + 6^3xt + 6^5xt^2 + 6^7xt^3$

$u(x,t) = 6x + 6^3xt + 6^5xt^2 + 6^7xt^3$

$u(x,t) = 6x(1 + 36t + (36t)^2 + (36t)^3 + (36t)^4 + \cdots \ldots)$

$$u(x,t) = \frac{6x}{1-36t} \qquad (19)$$

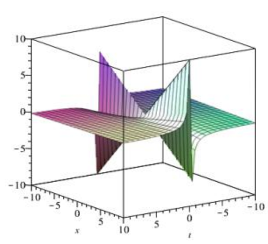

Figure 1: The exact solution of example 4.1 (u(x,t)=6x/(1-36t)

Example 4.2
Consider the following KDV equation
$u_t + 6uu_x + u_{xxx} = 0$ (20)
With the initial condition
$u(x,0) = x$
The exact solution is given by $u(x,t) = \frac{x}{1+6t}$

Method 1: VIM
The correction functional for example 4.2 is
$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} + 6\tilde{u}_n(x,\varepsilon)\frac{\partial \tilde{u}_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 \tilde{u}_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$ (21)
Taking variations on both sides gives
$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} + 6\tilde{u}_n(x,\varepsilon)\frac{\partial \tilde{u}_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 \tilde{u}_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$
$\widetilde{u_n}$ is a restrictive vibration, therefore $\delta\tilde{u}_n = 0$
$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon}\right] d\varepsilon$
From previous calculation,$\lambda = -1$
Hence, the iterative formula becomes
$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} + 6u_n(x,\varepsilon)\frac{\partial u_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$
From the initial condition, we have
$u_0 = x$
When n = 0
$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\varepsilon)}{\partial \varepsilon} + 6u_0(x,\varepsilon)\frac{\partial u_0(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_0(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$
$$u_1(x,t) = x - \int_0^t 6x\, d\varepsilon = x - 6xt$$
When n = 1
$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,\varepsilon)}{\partial \varepsilon} + 6u_1(x,\varepsilon)\frac{\partial u_1(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_1(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$
$u_2(x,t) = x - 6xt + 6^2xt^2$
$u_3(x,t) = x - 6xt + 6^2xt^2 - 6^3xt^3$
Consequently, the successive approximations are obtained
$u_1(x,t) = x(1-6t)$
$u_2(x,t) = x(1-6t+(6t)^2)$
$u_3(x,t) = x(1-6t+(6t)^2-(6t)^3)$

Hence, the closed form and exact solution is given as
$u(x,t) = x(1-6t+(6t)^2-(6t)^3+(6t)^4+\cdots\ldots)$
$\mathrm{u(x,t)} = \frac{x}{1+6t}$ (22)

Method 2: NIM
Considering Example 4.2
Using NIM, the equation is equivalent to the following integral equation
$u = f + \int_0^t(-6uu_x - u_{xxx})dt$ (23)
From the initial condition, we have
$u_0 = x$
$N(u_k) = \int_0^t(-6u_k u_{kx} - u_{kxxx})dt$
When k = 0,
$N(u_0) = u_1 = \int_0^t(-6u_0 u_{0x} - u_{0xxx})dt$
$$= \int_0^t (-6x)dt = -6xt$$
$u_1 = -6xt$
$u_2 = N(u_0+u_1) - N(u_0)$
$u_2 = -6xt + 6^2xt^2 + 6xt = 6^2xt^2$
$u_3 = -6^3xt^3$

The successive approximations are:
$u_0 = x$
$u_1 = -6xt$
$u_2 = 6^2xt^2$
$u_3 = -6^3xt^3$
Thus the series solution of example 4.2 is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n = x - 6xt + 6^2xt^2 - 6^3xt^3$$

$$u(x,t) = x - 6xt + 6^2xt^2 - 6^3xt^3$$

$$u(x,t) = x(1 - 6t + (6t)^2 - (6t)^3 + (6t)^4 + \cdots \ldots)$$

$$\mathrm{u(x,t)} = \frac{x}{1+6t} \tag{24}$$

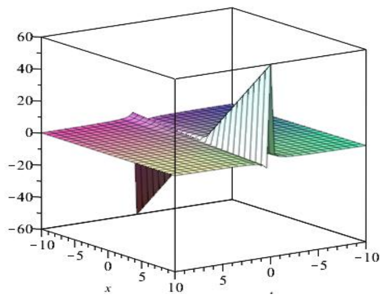

Figure 2: The exact solution of example 4.2 (u(x,t)=x/1+6t)

Example 4.3

Consider the following KDV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{25}$$

With the initial condition

$$u(x,0) = \frac{2}{x^2}$$

The exact solution is given by $u(x,t) = \frac{2}{x^2}$

Method 1: VIM

The correction functional for example 4.3 is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6\tilde{u}_n(x,\varepsilon)\frac{\partial \tilde{u}_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 \tilde{u}_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$$

Taking variations on both sides gives

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6\tilde{u}_n(x,\varepsilon)\frac{\partial \tilde{u}_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 \tilde{u}_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$$

$\widetilde{u_n}$ is a restrictive vibration, therefore $\delta\tilde{u}_n = 0$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon}\right] d\varepsilon$$

From previous calculation,$\lambda = -1$

Hence, the iterative formula becomes

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6u_n(x,\varepsilon)\frac{\partial u_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$$

From the initial condition, we have

$$u_0 = \frac{2}{x^2}$$

When n = 0

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\varepsilon)}{\partial \varepsilon} - 6u_0(x,\varepsilon)\frac{\partial u_0(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_0(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$$

$$u_1(x,t) = \frac{2}{x^2} - \int_0^t (48x^{-5} - 48x^{-5})\, d\varepsilon = \frac{2}{x^2}$$

When n = 1

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,\varepsilon)}{\partial \varepsilon} + 6u_1(x,\varepsilon)\frac{\partial u_1(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_1(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$$

$$u_2(x,t) = \frac{2}{x^2} - \int_0^t (48x^{-5} - 48x^{-5})\, d\varepsilon = \frac{2}{x^2}$$

$u_3(x,t) = \frac{2}{x^2} - \int_0^t (48x^{-5} - 48x^{-5})\, d\varepsilon = \frac{2}{x^2}$

Consequently, the successive approximations are obtained

$u_1(x,t) = \frac{2}{x^2}$

$u_2(x,t) = \frac{2}{x^2}$

$u_3(x,t) = \frac{2}{x^2}$

Hence, the closed form and exact solution is given as

$u(x,t) = \frac{2}{x^2}$ (26)

Method 2: NIM

Considering Example 4.3

Using NIM, the equation is equivalent to the following integral equation

$u = f + \int_0^t (6uu_x - u_{xxx})dt$ (27)

From the initial condition, we have

$u_0 = \frac{2}{x^2}$

$N(u_k) = \int_0^t (6u_k u_{kx} - u_{kxxx})dt$

When k = 0,

$N(u_0) = u_1 = \int_0^t (6u_0 u_{0x} - u_{0xxx})dt$

$= \int_0^t (-48x^{-5} + 48x^{-5})dt = 0$

$u_1 = 0$

$u_2 = N(u_0 + u_1) - N(u_0)$

$u_2 = \int_0^t (-48x^{-5} + 48x^{-5})dt = 0$

$u_3 = 0$

The successive approximations are:

$u_0 = \frac{2}{x^2}$

$u_1 = 0$

$u_2 = 0$

$u_3 = 0$

Thus the series solution of example 4.3 is given by

$u(x,t) = \sum_{n=1}^{\infty} u_n = \frac{2}{x^2} + 0 + 0 + 0 + \cdots.$

$u(x,t) = \frac{2}{x^2}$ (28)

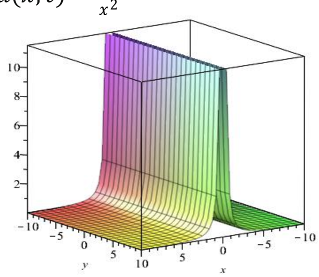

Figure 3: The exact solution of example 4.3 (u(x,t)=2/x^2)

Example 4.4

Consider the following KDV equation

$u_t - 6uu_x + u_{xxx} = 0$ (29)

With the initial condition

$u(x,0) = \frac{2}{(x-3)^2}$

The exact solution is given by $u(x,t) = \frac{2}{(x-3)^2}$

Method 1: VIM

The correction functional for example 4.4 is

$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6\tilde{u}_n(x,\varepsilon)\frac{\partial \tilde{u}_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 \tilde{u}_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$

Taking variations on both sides gives

$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6\tilde{u}_n(x,\varepsilon)\frac{\partial \tilde{u}_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 \tilde{u}_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$

$\widetilde{u_n}$ is a restrictive vibration, therefore $\delta\tilde{u}_n = 0$

$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon}\right] d\varepsilon$

From previous calculation,$\lambda = -1$

Hence, the iterative formula becomes

$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\varepsilon)}{\partial \varepsilon} - 6u_n(x,\varepsilon)\frac{\partial u_n(x,\varepsilon)}{\partial x} + \frac{\partial^3 u_n(x,\varepsilon)}{\partial x^3}\right] d\varepsilon$

From the initial condition, we have

$$u_0 = \frac{2}{(x-3)^2}$$

$$u_1(x,t) = \frac{2}{(x-3)^2} - \int_0^t (48(x-3)^{-5} - 48(x-3)^{-5})\, d\varepsilon = \frac{2}{(x-3)^2}$$

$$u_2(x,t) = \frac{2}{(x-3)^2} - \int_0^t (48(x-3)^{-5} - 48(x-3)^{-5})\, d\varepsilon = \frac{2}{(x-3)^2}$$

$$u_3(x,t) = \frac{2}{(x-3)^2} - \int_0^t (48(x-3)^{-5} - 48(x-3)^{-5})\, d\varepsilon = \frac{2}{(x-3)^2}$$

Consequently, the successive approximations are

$u_0(x,t) = \frac{2}{(x-3)^2}$

$u_1(x,t) = \frac{2}{(x-3)^2}$

$u_2(x,t) = \frac{2}{(x-3)^2}$

$u_3(x,t) = \frac{2}{(x-3)^2}$

Hence, the closed form and exact solution is given as

$\mathrm{u(x,t)} = \frac{2}{(x-3)^2}$ (30)

Method 2: NIM

Considering Example 4.4

Using NIM, the equation is equivalent to the following integral equation

$u = f + \int_0^t (6uu_x - u_{xxx})dt$ (31)

From the initial condition, we have

$$u_0 = \frac{2}{(x-3)^2}$$

$N(u_k) = \int_0^t (6u_k u_{kx} - u_{kxxx})dt$

When k = 0,

$N(u_0) = u_1 = \int_0^t (6u_0 u_{0x} - u_{0xxx})dt$

$$= \int_0^t (-48(x-3)^{-5} + 48(x-3)^{-5})dt = 0$$

$u_1 = 0$

$u_2 = N(u_0 + u_1) - N(u_0)$

$$u_2 = \int_0^t (-48(x-3)^{-5} + 48(x-3)^{-5})dt = 0$$

$u_3 = 0$

The successive approximations are:

$$u_0 = \frac{2}{(x-3)^2}$$

$u_1 = 0$

$u_2 = 0$
$u_3 = 0$
Thus the series solution of example 4.4 is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n = \frac{2}{(x-3)^2} + 0 + 0 + 0 + \cdots.$$

$$u(x,t) = \frac{2}{(x-3)^2} \quad (32)$$

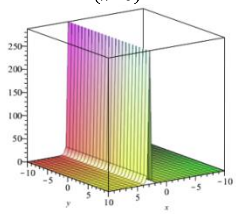

Figure 4: The exact solution of example 4.4 (u(x,t)=2/(x-3)^2

VII. CONCLUSION

In this paper, the analytical solutions of the third-order Korteweg-de Vries (KdV) equations have been obtained using the Variational Iteration Method (VIM) and the New Iterative Method (NIM).

The examples considered showed that Variational Iteration Method (VIM) and the New Iterative Method (NIM) are very powerful, efficient and effective methods for solving third order KdV equations because the solutions obtained in series form converge to exact solutions with few iterations.

Hence, solving other linear and nonlinear differential equations of higher order is very easy by using these powerful methods.

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