Journal of the Nigerian Association of Mathematical Physics
Volume 55 **(February** ***2020 Issue*), pp7 –18**
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FURTHER INVESTIGATION OF INERTIAL HYBRIDALGORITHM FOR FINDING COMMON SOLUTION OF FIXED POINT AND EQUILIBRIUM PROBLEMS IN BANACH SPACE

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Abstract

We formulate a new iterative scheme with an inertial technique that solves a common solution problem of finite family of continuous Bregman quasi-nonexpansive self-mappings and system of equilibrium in a Banach space. This is achieved by demonstrating a strong convergence theorem for it. Our proof finds a common element in the collection of fixed set of finite family of continuous Bregman quasi-nonexpansive self-mappings and the common solution set of the system of finite equilibrium problems. As an improvement to other existing results in this direction, we further justify our theoretical assertions with a numerical experiment.

2000 Mathematics Subject Classification: 47H09; 4705
***Keywords*: C**ontinuous Bregman quasi-nonexpansive mappings; common solution; inertial technique, equilibrium problem, convergence results.

1. Introduction

We denote a reflexive real Banach space by X, the set of real numbers by R, the set of natural numbers by N. Let $\|.\|$ represent a norm function. We represent the dual of X by X^* as the set of all linear functional. Let $d_h : dom\,h \times \text{int}(dom\,h) \to R^+$ represent a bifunction with respect to a convex function denoted by $h : X \to (-\infty,+\infty]$, where $dom\,h = \{u \in X : h(u) < +\infty\}$ is the domain of a convex function and $\text{int}(dom\ h)$ represent the interior domain of $h : X \to (-\infty,+\infty]$.

A function $h : X \to (-\infty,+\infty]$, is G$\hat{a}$teaux differentiable at u if $\lim_{s\to 0^+} \frac{(h(u+sz)-h(u))}{s} = h^\circ(u,z)$ exists for any z in X. By this definition, $h^\circ(u,z) = \nabla h(u)$, which is the gradient of $h : X \to (-\infty,+\infty]$. Let K be a closed, convex subset of X. The function $h : X \to (-\infty,+\infty]$ is uniformly Frechet differentiable whenever the limit is attained uniformly with $\|z\| = 1$ on a subset of $K \subset X$ which is bounded.

Let the convex function $h : X \to (-\infty,+\infty]$ represent a G$\hat{a}$teaux differentiable function, then the bifunction $d_h : dom\,h \times \text{int}(dom\ h) \to R^+$ defined by

$$d_h(z,u) = h(z) - h(u) - \langle \nabla h(u), z\rangle + \langle \nabla h(u), u\rangle, \tag{1.1}$$

is the Bregman function induced by a convex function $h : X \to (-\infty,+\infty]$.

This bifunction $d_h : dom\,h \times \text{int}(dom\ h) \to R^+$ defined by(1.1) has some nice properties like:

P1: The function $d_h(.,u)$ is convex with respect to first variable,

P2: $d_h(u,u) = 0,$

P3: $d_h(z,u) > 0,$

P4: $d_h(z,u) = d_h(z,v) + d_h(v,u) + \langle \nabla h(v), z\rangle - v\rangle - \langle \nabla h(u), z - v\rangle,$

P5: $d_h(u,v) + d_h(v,u) = \langle \nabla h(u), u - v\rangle - \langle \nabla h(v), u - v\rangle,$

P6: $d_h(u,v) \le \|u\|\|\nabla h(u) - \nabla h(v)\| + \|v\|\|\nabla h(u) - \nabla h(v)\|.$

Remark 1: P4 implies P5 and P6 if $u = z.$

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Remark 2: $d_h(,)$ in (1.1) was first called Bregman function in the work of [1]. Though, it was first studied by Bregman as a substitute for the classical distance function, (see [1], [2]) for more details.

A function $h^*: X^* \to R$ defined by

$$h^*(x^*) = \sup\{\langle x, x^*\rangle - h(x),\ x \in X\}, \quad (1.2)$$

is called the conjugate function of $h.$ We see from the conjugate inequality that $h(x) \geq \langle x, x^*\rangle - h^*(x^*),\ \forall x \in X, \forall x^* \in X^*$.

Define a bifunction $V_h : X \times X^* \to R$ by

$$V_h(x, x^*) = h(x) - \langle x, x^*\rangle + h^*(x^*),\ \forall x \in X, x \in X^*. \quad (1.3)$$

In other words,

$$V_h(x, x^*) = d_h(x, \nabla h^*(x^*)) = d_h(x, \nabla h^*(\nabla h(x))),\ \forall x \in X, \nabla h(x) \in X^*. \quad (1.4)$$

It is easy to see that $V_h(x,.)$ is nonnegative and convex with respect to the its second variable (see [3]).

The subdifferential of h at u is defined thus

$$\partial h(u) = \{u^* \in X^* : h(u) + \langle u^*, z\rangle - \langle u^*, u\rangle \leq h(z);\ z \in X\}. \quad (1.5)$$

The function $h: X \to (-\infty, +\infty]$ is Legendre (see [4, 5] and the references contained therein), if the following hold

(1) $\mathrm{int}(dom\ h)$ is non-void, h is differentiable on $\mathrm{int}(dom\,h)$ with $dom\,h = \mathrm{int}(dom\,h),$

(2) $\mathrm{int}(dom\ h^*)$ is non-void, h^* is differentiable on $\mathrm{int}(dom\ h^*)$ with $dom\,h^* = \mathrm{int}(dom\,h^*).$

Remark 3: With $h: X \to (-\infty, +\infty]$ a Legendre function, and X reflexive, then ∇h is a bijection which satisfies $\nabla h = (\nabla h^*)^{-1}$, $range \nabla h = domain \nabla h^* = \mathrm{int}(domain\,h^*)$ and $range \nabla h^* = domain \nabla h = \mathrm{int}(domain\,h),$ where $h: X \to (-\infty, +\infty]$ and $h^*: X^* \to (-\infty, +\infty]$ are strictly convex in the $\mathrm{int}(dom\,h)$. If $\partial h(u) = \{u^* \in X^* : h(u) + \langle u^*, z\rangle - \langle u^*, u\rangle \leq h(z);\ z \in X\}$ of $h: X \to (-\infty, +\infty]$ have a single value, then $\partial h = \nabla h.$ Given $h(u) = t^{-1}\|u\|^2,\ t \in (1, \infty)$, then we have a Legendre function and (1.1) becomes the Lyapunov functional when the space is smooth. If $\partial h = \nabla h = I,$ then (1.1) reduces to metric distance, (see [4], [5] and the references contained therein) for more details).

The modulus of total convexity of h at $u \in \mathrm{int}\,dom\,h$ is the function $W_h(u,.): \mathrm{int}(dom\,h) \times R^+ \to R^+$ defined by

$$W_h(u, s) = \inf\{d_h(z, u) : z \in dom\,h, \|z - u\| = s\}. \quad (1.6)$$

If $W_h(x, s)$ is positive, then $h: X \to (-\infty, +\infty]$ becomes totally convex at u for positive value of $s.$ For more information (see [6, 7] and references in them).

Let K represent a non-void, closed as well as convex subset of $\mathrm{int}\,dom\,h.$ Let $T: K \to K$ represent a map. $T: K \to K$ is nonexpansive if $\|Tu - Tz\| \leq \|u - z\|,\ \forall u, z \in K;$ $T: K \to K$ is (quasi)-nonexpansive if $\|Tu - z^0\| \leq \|u - z^0\|,$ and $Fix(T) = \{z^0 \in K : Tu = u\}$ is the collection of fixed point of $T: K \to K.$ An element $u^* \in K$ is asymptotic fixed point of $T: K \to K$ when $\{u_n\}$ is contained in K and converges weakly to u so that $\|u_n - Tu_n\| = 0.$ It is represented by the collection $\hat{F}ix(T) = \{u \in K : \|u_n - Tu_n\|\}.$

A map $T: K \to \mathrm{int}(dom\,h),$ with respect to a convex function $h: X \to (-\infty, +\infty]$ is

(i) Bregman relatively nonexpansive (BRNE) [8] if
$d_h(z^0, Tu) \leq d_h(z^0, u),\ \forall u \in K, \forall z^0 \in Fix(T)$ and $\hat{F}ix(T) = Fix(T).$

(ii) Bregman quasi-nonexpansive (BQNE) [8] if
$d_h(z^0, Tu) \leq d_h(z^0, u),\ \forall u \in K, \forall z^0 \in Fix(T)$.

Remark 4:

1. Any Bregman relatively nonexpansive mapping is Bregman quasi-nonexpansive mapping (see [9]),
2. We note here that weak convergence of sequence $\{u_n\}$ need not imply strong convergence of the sequence $\{u_n\}$ (see [9]),
3. If a sequence $\{u_n\}$ in K converges strongly to a point u in K, then $\{u_n\}$ also converges weakly to $u.$
4. Every nonexpansive mapping defined on a closed convex subset of a Hilbert space such that the fixed point of the mapping is non-void is relatively nonexpansive defined on a closed and convex subset to itself and hence Bregman relatively nonexpansive mapping with respect to $h(u) = \|u\|^2$ (see [3]).

A mapping $\psi: K \times K \to R$ is called a bifunction so that the equilibrium problem with respect to $\psi: K \times K \to R$ is to find $z^0 \in K$ such that

$$\psi(z^0, z) \geq 0\ \ \forall z \in K. \quad (1.7)$$

The collection of solution of (1.7) is represented by $EP(K, \psi) = \{z^0 \in K : \psi(z^0, z)\ \geq 0\ \ \forall z \in K\}$ (see [10], [11]).

To solve a problem of the form (1.7), certain conditions are imposed on the bifunction $\psi: K \times K \to R$ as follows [10], [11]:

(A1): $\psi(x, x) = 0,\ \forall x \in k,$

(A2): $\psi : K \times K \to R$ is monotone

(A3): $\limsup_{t \downarrow 0} \psi((1-t)x + tz, y) \le \psi(x, y), \forall x, y, z \in K,$

(A4): The function $y \mapsto \psi(x, y)$ is convex and lower-semicontinuous.

The Resolvent of a bifunctions $\psi : K \times K \to R$ [12] is the operator $\operatorname{Re}s_{\psi}^{h} : X \to 2^{K}$ defined by

$$\operatorname{Re}s_{\psi}^{h}(x) = \left\{z^0 \in K : \psi\left(z^0, z\right) + \langle \nabla h(z^0) - \nabla h(x), z - z^0 \rangle \ge 0, \forall y \in K\right\}$$

Over the years, smooth convex minimization problem involving equilibrium and fixed point problems have attracted the interest of many authors seeking common solution of these minimum problem in infinite-dimensional space. Iterative approximation methods have always been used to solve this problem. Furthermore, most of the results obtained in this direction only focused on the weak or strong convergence of the formulated schemes to the (common) fixed point sets (see e.g. [1, 3, 10, 13-18] and the many references contained in them). However, very few authors have recently paid attention to the speed or the rate of convergence of sequence of iterates of Bregman nonexpansive-type operators to their (common) fixed point sets when they exists. Thus, to increase the rate of convergence of iterations, a two-step iterative method originally introduced in [19], are now being studied (see [11, 20, 21]). It is defined as

$$u_{n+1} = u_n + \beta_n (u_n - u_{n-1}). \tag{1.8}$$

for all non-negative integers n, where $\beta_n \in (0,1)$.

Very recently, the following method of solving common point problem involving the fixed point of finite family of nonexpansive mappings and system of finite equilibrium problems in Hilbert space was introduced in [11]. Below is their algorithm:

$$\begin{cases} w_n = x_n + \alpha_n (x_n - x_{n-1}), \\ y_n^i = T_{r_n}^{f_i}(w_n), \ i = 1,2,..., N, \\ t_n = \dfrac{y_n^1 + y_n^2 + ... + y_n^N}{N}, \\ x_{n+1} = (1-\lambda_n) w_n + \lambda_n T_m^n T_{m-1}^n ... T_1^n t_n \quad n \ge 1, \end{cases} \tag{1.9}$$

satisfying certain conditions, they proved that the sequence $\{x_n\}$ generated by their algorithm (1.9) converges weakly to a common solution of the problem.

In 2016, a new CQ algorithm for nonexpansive mapping in a real Hilbert space was introduced in [21]. Set $x_0, x_1 \in H$ arbitrarily. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} z_n = x_n + \alpha_n (x_n - x_{n-1}), \\ y_n = (1-\beta_n) z_n + \beta_n T z_n, \\ C_n = \left\{ u \in H : \| y_n - u \| \le \| z_n - u \| \right\}, \\ Q_n = \{ u \in H : \langle x_n - u, x_n - x_0 \rangle \ge 0 \}, \\ x_{n+1} = P_{C_n^i \cap Q_n}\left(x_0\right), \quad n \ge 0. \end{cases} \tag{1.10}$$

then satisfying certain conditions, the iterative sequence $\{x_n\}$ generated by the algorithm (1.10) converges strongly to $P_{F(T)}(x_0)$, where $P_{F(T)}(x_0)$ is the metric projection onto nonempty fixed point of T.

Remark 5: We note here that the work was done in Hilbert space and for a single nonexpansive self-mapping on H. It contains an inertial term which speeds up convergence of sequences in a smooth convex minimization problem. However, the algorithm has two closed half sets C_n and Q_n which complicates the computation of the metric Projection at each interval of the iteration.

Following [21], in 2018a new inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps was introduced in [20]. The authors used the Lyapunov functional induced by the norm to prove a strong convergence result of their sequence generated by their algorithm in uniformly convex and uniformly smooth Banach spaces. Set $u_0, u_1 \in X$ and define a sequence $\{u_n\}$ by the following algorithm:

$$\begin{cases} K_0 = X \\ z_n = u_n + \alpha_n (u_n - u_{n-1}), \\ y_n = J^{-1}\left((1-\beta) J(z_n) + \beta J T z_n\right), \\ C_{n+1} = \left\{ u^* \in K : \phi\left(u^*, y_n\right) \le \phi\left(u^*, z_n\right)\right\}, \\ u_{n+1} = \Pi_{C_{n+1}}\left(u_0\right). \end{cases} \tag{1.11}$$

They showed that their method converge strongly to a mutual element of $Fix(T) = \bigcap_{i=1}^{\infty} Fix(G_i)$.

Our justification for this study is the results of [11, 20, 21]. Weformulate a new iterative scheme with an inertial technique that solves a common fixed point problems of finite family of continuous Bregman quasi-nonexpansive self-mappings and equilibrium problem in a reflexive and (real) Banach space. This is achieved by demonstrating a convergence theorem for it. Our proof finds a common element in the collection of fixed set of finite family of continuous Bregman quasi-nonexpansive self-mappings and the common solution set of the system of finite equilibrium problems. As an improvement to other existing results in this direction, we further justify our theoretical assertions with a numerical experiment.

2. Preliminaries

The following lemmas shall be used in the sequel.

Lemma 2.1 (see [6]). The function h is totally convex on bounded sets if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that either $\{x_n\}$ or $\{y_n\}$ is bounded, then

$\lim_{n\to\infty} d_h(y_n, x_n) = 0 \Rightarrow \| y_n - x_n \| = 0$.

Lemma 2.2 (see [17]). Let K be a non-void, closed, convex subsets of $\text{int}(dom\ h)$ and $T: K \to K$ be a Bregman quasi nonexpansive mapping with respect to h. Then $Fix(T)$ is closed and convex.

Lemma 2.3 (see [11]). Let X be a reflexive Banach space and let h be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:

(1) h is bounded on bounded subsets and uniformly smooth on bounded subsets of X

(2) h^* is Fréchet differentiable and ∇h^* is uniformly norm-to-norm continuous on bounded subsets of X^*.

(3) $dom\ h^* = X^*$, h^* is strongly coercive and uniformly convex on bounded subsets of X^*.

Lemma 2.4 (see [15]). Let $h: X \to (-\infty, +\infty]$ be a $G\hat{a}teaux$ differentiable on $\text{int}(dom\ h)$ such that ∇h^* is bounded on bounded subsets of $dom\, h^*$. Let $x_0 \in X$ and $\{x_n\}$ is a sequence in X. If $\{d_h(x_0, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 2.5(see [12]).Let $h: X \to (-\infty, +\infty]$ be a Legendre function and K a non-void, closed and convex subset of X. If the bifunctions $\psi: K \times K \to R$ satisfies condition (A1)-(A4), then the following hold:

(1) $\text{Re}\, s_\psi^h$ is single valued

(2) $Fix(\text{Re}\, s_\psi^h) = EP(K, \psi)$

(3) $d_h(p, \text{Re}\, s_\psi^h x) + d_h(\text{Re}\, s_\psi^h x, x) \leq d_h(p, x)\ \forall p \in Fix(\text{Re}\, s_\psi^h)$

(4) $EP(K, \psi)$ is closed and convex.

The Bregman Projection $u \in \text{int}(dom\ h)$ onto $K \subset dom\, h$, is the unique $u_0 \in K$ such that the mapping $P_K^h : \text{int}\, dom\, h \to K$ satisfy

$d_h(u_0, u) = \min\{d_h(z, u) : z \in K\}$

and $P_K^h(u) = u_0$. The Bregman Projection mapping satisfy the following results:

Lemma 2.6 (see [9]). Let K be non-void, closed, convex subsets of X. Let $h: X \to (-\infty, +\infty]$ be $G\dot{a}teaux$ differentiable and totally convex function and let $x \in X$, then

(1) $z = P_K^h(x)$ if and if $\langle \nabla h(x) - \nabla h(z), y - z \rangle \leq 0,\ \forall y \in K,$

(2) $d_h(y, P_K^h(x)) + d_h(P_K^h(x), x) \leq d_h(y, x)\ \ \forall y \in K.$

3. Main Results

Lemma 3.1: Let $h: X \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then, for all $u \in X$ we have

$$d_h(u, \nabla h^*(\frac{1}{N}\sum_{i=1}^N \nabla h(x_i))) \leq \frac{1}{N}\sum_{i=1}^N d_h(u, x_i). \quad (3.1)$$

Proof:

Using (1.3) and (1.4), we have

$$d_h(u, \nabla h^*(\frac{1}{N}\sum_{i=1}^N \nabla h(x_i))) = V_h(u, \frac{1}{N}\sum_{i=1}^N \nabla h(x_i)) \quad (3.2)$$

$$= h(u) - \langle u, \frac{1}{N}\sum_{i=1}^N \nabla h(x_i)\rangle + h^*(\frac{1}{N}\sum_{i=1}^N \nabla h(x_i))$$

$$\leq h(u) - \frac{1}{N}\sum_{i=1}^N \langle u, \nabla h(x_i)\rangle + \frac{1}{N}\sum_{i=1}^N h^*(\nabla h(x_i)) \quad (3.3)$$

$$= \frac{1}{N}\sum_{i=1}^N \left[h(u) - \langle u, \nabla h(x_i) + h^*(\nabla h(x_i))\right]$$

$$= \frac{1}{N}\sum_{i-1}^N V_h(u, \nabla h(x_i))$$

$$= \frac{1}{N}\sum_{i-1}^N d_h(u, x_i). \quad (3.4)$$

This ends the proof. ∎

Theorem 3.2: Let K be a non-void, closed, convex subset of $\text{int}(dom\, h)$. Let $h: X \to R$ be a Legendre function which is bounded, uniformly Fr$\acute{e}$chet differentiable and totally convex on bounded subsets of a reflexive real Banach space X. Let $\{\psi_i\}_{i=1}^N : K \times K \to R$ be N -bifunctions which meets properties A1–A4. Let $\{T_j\}_{j=1}^m : K \to K$ be m -finite family of continuous Bregman quasi-nonexpansive mappings induced by a convex function h. Assume that $F = \bigcap_{i=1}^N EP(\psi_i) \cap \left(\bigcap_{j=1}^M Fix(T_j)\right)$ is non-void. Set ;; Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0 \in K \\ z_n = \nabla h^*\left(\nabla h(x_n) + \alpha_n \nabla h(x_n - x_{n-1})\right), \\ y_n^j = \nabla h^*\left((1-b_n)\nabla h(z_n) + b_n \nabla h(T_j z_n)\right),\ j = 1,2,\ldots,M, \\ w_n^i = \operatorname{Res}_{\psi_i}^h (y_n^j),\ i, j = 1,2,\ldots,N,M, \\ t_n = \nabla h^*\left(\sum_{i=1}^N \frac{1}{N} \nabla h(w_n^i)\right), \\ K_{n+1} = \{u \in K_n : d_h(u,t_n) \le d_h(u,z_n)\}, \\ x_{n+1} = P_{K_{n+1}}^h(x_0),\ \ n \ge 1, \end{cases} \tag{3.5}$$

suppose $\{\alpha_n\}, \{b_n\} \subset (0,1)$, then the sequence $\{x_n\},\{z_n\}$ converges strongly to a common element of F.

Proof: We demonstrate the analytical proof theorem 3.2 in the steps below.

Step 1: The algorithm (3.5) is well-defined in terms of $\{x_n\}$ for each $n \ge 1$.

We first demonstrate that $F = \bigcap_{i=1}^N EP(\psi_i) \cap \left(\bigcap_{j=1}^M Fix(T_j)\right)$ is closed, convex. Lemma 2.2 gives that $Fix(T)$ is closed, convex and consequently $\bigcap_{j=1}^M Fix(T_j)$ is closed, convex. Lemma 2.5 gives that $EP(g)$ is closed, convex and sois $\bigcap_{i=1}^N EP(\psi_i)$. So $F = \bigcap_{i=1}^N EP(\psi_i) \cap \left(\bigcap_{j=1}^M Fix(T_j)\right)$ is closed, convex since the intersection of closed and convex sets is itself closed andconvex.

Next is to demonstrate that that K_n is closed, convex for each $n \ge 1$. This can be seen from definition of K_n, that K_n is closed. Moreover, since $d_h(u,t_n) \le d_h(u,z_n)$ is equivalent to

$$\langle \nabla h(z_n) - \nabla h(t_n), u \rangle + \langle \nabla h(z_n) - \nabla h(t_n), t_n - z_n \rangle \le h(t_n) - h(z_n),$$

which is convex, it follows that K_n is a half space and hence convex for each $n \ge 1$.

In addition to closedness and convexity of $F = \bigcap_{i=1}^N EP(\psi_i) \cap \left(\bigcap_{j=1}^M Fix(T_j)\right)$, we demonstrate concretely that $F \subset K_n$ for each $n \ge 1$. It is clear from the initial assumption that $F \subset K_0 = K$. Now suppose that $F \subset K_n$ for some positive $n \ge 1$, then for $p \in F$, and using Lemma 3.1 we obtain

$$\begin{aligned} d_h(p,t_n) &= d_h\left(p, \nabla h^*\left(\sum_{i=1}^N \frac{1}{N} \nabla h(w_n^i)\right)\right) \\ &\le \frac{1}{N} \sum_{i=1}^N d_h\left(p, w_n^i\right)\ \forall i = 1,2,\ldots,N, \end{aligned} \tag{3.6}$$

In addition and invoking Lemma 2.5 we get

$$\begin{aligned} d_h(p,w_n^i) &= d_h(p, \operatorname{Res}_{\psi_i}^h y_n) \\ &\le D_f(p,y_n), \quad \forall i = 1,2,\ldots,N \end{aligned} \tag{3.7}$$

Furthermore,

$$\begin{aligned} d_h(p,y_n) &= d_h\left(p, \nabla h^*\left((1-b_n)\nabla h(z_n) + b_n \nabla h(T_j z_n)\right)\right) \\ &= V_h\left(p, (1-b_n)\nabla h(z_n) + b_n \nabla h(T_j z_n)\right) \\ &= h(p) - \langle p, (1-b_n)\nabla h(z_n) + b_n \nabla h(T_j z_n)\rangle + h^*((1-b_n)\nabla h(z_n) + b_n \nabla h(T_j z_n)) \\ &\le (1-b_n)\left[h(p) - \langle p, \nabla h(z_n)\rangle + h^*(\nabla h(z_n))\right] + \\ &\quad b_n\left[h(p) - \langle p, \nabla h(T_j z_n)\rangle + h^*(\nabla h(T_j z_n))\right] \\ &= (1-b_n)V_h(p, \nabla h(z_n)) + b_n V_h(p, \nabla h(T_j z_n)) \\ &= (1-b_n)d_h\left(p, z_n\right) + b_n d_h\left(p, T_j z_n\right) \\ &\le (1-b_n)d_h\left(p, z_n\right) + b_n d_h\left(p, z_n\right) \\ &= d_h\left(p, z_n\right). \end{aligned}$$

Thus,

$d_h(p,y_n) \le d_h(p,z_n)$. (3.8)

Using (3.8) in (3.7) we obtain

$d_h(p,w_n^i) \le d_h(p,z_n)$, for each $i = 1,2,...,N$. (3.9)

Consequently (3.9) in (3.6) gives

$d_h(p,t_n) \le d_h(p,z_n)$. (3.10)

So $p \in K_{n+1}$ and $K_{n+1} \subset K_n$. This implies by set induction that $F \subset K_n$. Thus, the algorithm (3.5) is well-defined in terms of $\{x_n\}$ for each $n \ge 1$.

Step 2: We demonstrate that

(i) $\lim_{n\to\infty} d_h(x_{n+1},x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \| x_{n+1} - x_n \| = 0,$

(ii) $\lim_{n\to\infty} \| x_n - z_n \| = 0 \quad \Rightarrow \quad \lim_{n\to\infty} d_h(x_n,z_n) = 0$

(iii) $\lim_{n\to\infty} d_h(x_{n+1},t_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \| x_{n+1} - t_n \| = 0,$

(iv) $\lim_{n\to\infty} d_h(z_n,t_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \| z_n - t_n \| = 0.$

We notice that $x_n = P_{K_n}^h(x_0)$ and $x_{n+1} = P_{K_{n+1}}^h(x_0) \in K_{n+1} \subset K_n$. Thus we get

$d_h(x_n,x_0) \le d_h(x_{n+1},x_0) - d_h(x_{n+1},x_n)$

$d_h(x_n,x_0) \le d_h(x_{n+1},x_0)$. (3.10)

(3.10) demonstrates that $\{d_h(x_n,x_0)\}$ is a monotone non decreasing sequence. Again we get from Lemma 2.6 that

$d_h(x_n,x_0) = d_h(P_{K_n}^h(x_0),x_0) \le d_h(p,x_0) - d_h(p,P_{K_n}^h(x_0)) \le d_h(p,x_0) \quad \forall n \ge 1,\ p \in F,$

implying that

$d_h(x_n,x_0) \le d_h(p,x_0)$. (3.11)

(3.11) demonstrates that $\{d_h(x_n,x_0)\}$ is bounded and from Lemma 2.4 we get that $\{x_n\},\{y_n\},\{z_n\},\{w_n^i\},\{t_n\}$ for each $i = 1,2,...,N$ are bounded. Combining (3.10) and (3.11) we get that $\lim_{n\to\infty} d_h(x_n,x_0)$ exist. Now wlog, let

$\lim_{n\to\infty} d_h(x_n,x_0) = l$ (3.12)

In addition to (3.12) and Lemma 2.6 we get that for any positive integer, μ,

$d_h(x_{n+\mu},x_n) = d_h(x_{n+\mu},P_{K_n}^h(x_0))$

$\le d_h(x_{n+\mu},x_0) - d_h(x_n,x_0) \to 0$ as $n \to \infty$.

So that

$\lim_{n\to\infty} d_h(x_{n+\mu},x_n) = 0.$

In particular,

$\lim_{n\to\infty} d_h(x_{n+1},x_n) = 0.$ (3.13)

By Lemma 2.1, (3.13) implies that

$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0.$ (3.14)

This establishes (i).

From (3.14) we conclude that the sequence $\{x_n\}$ is a Cauchy sequence in K. Using the fact that X is complete and K is closed, we get that $x_n \to z_0 \in K$ as $n \to \infty$.

Now, from the uniform continuity of ∇h we get

$\lim_{n\to\infty} \| \nabla h(x_{n+1}) - \nabla h(x_n) \| = 0.$ (3.15)

From the definition of z_n, and together with (3.10) we have that

$\| \nabla h(x_n) - \nabla h(z_n) \| = \| \nabla h(x_n) - \nabla h(x_n) - \alpha_n \nabla h(x_n - x_{n-1}) \|$

$= \| \alpha_n \nabla h(x_{n-1} - x_n) \|$

$\le \| \nabla h(x_{n-1} - x_n) \| \to 0$ as $n \to \infty$.

This implies that

$$\lim_{n\to\infty}\|\nabla h(x_n)-\nabla h(z_n)\|=0. \tag{3.16}$$

By Lemma 2.3, we obtain that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.17}$$

This establishes (ii) and shows that $z_n\to z_0$ as $n\to\infty$.

Moreso, since $\{z_n\}$ is bounded and using (P6) and (3.17), we have that

$$\lim_{n\to\infty}d_h(x_n,z_n)=0. \tag{3.18}$$

In addition, since $x_{n+1}\in K_{n+1}\subset K_n$, we have from the definition of the half space that

$$d_h(x_{n+1},t_n)\le d_h(x_{n+1},z_n). \tag{3.19}$$

$$0\le d_h(x_{n+1},z_n)=d_h(x_{n+1},x_n)+d_h(x_n,z_n)+\langle\nabla h(x_n)-\nabla h(z_n),x_{n+1}-x_n\rangle$$

$$\le d_h(x_{n+1},x_n)+d_h(x_n,z_n)+\|\nabla h(x_n)-\nabla h(z_n)\|\|x_{n+1}-x_n\|\to 0 \text{ as } n\to\infty.$$

This demonstrate that

$$\lim_{n\to\infty}d_h(x_{n+1},z_n)=0. \tag{3.20}$$

This implies that

$$\lim_{n\to\infty}d_h(x_{n+1},t_n)=0. \tag{3.21}$$

Thus, by Lemma 2.1, (3.20) and (3.21) implies that

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0$$

and

$$\lim_{n\to\infty}\|x_{n+1}-t_n\|=0. \tag{3.22a}$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-w_n^i\|=0 \text{ for all } i=1,2,\ldots,N. \tag{3.22b}$$

This establishes (iii).

In addition, we have from our definition that

$$d_h(z_n,t_n)=d_h\left(z_n,\nabla h^*\left(\sum_{i=1}^{N}\frac{1}{N}\nabla h(w_n^i)\right)\right)\le\frac{1}{N}\sum_{i=1}^{N}d_h(p,w_n^i)-d_h(p,z_n),\ \forall i=1,2,\ldots,N$$

$$\le d_h(p,z_n)-d_h(p,z_n)$$

$$\to 0 \text{ as } n\to\infty.$$

This demonstrate that

$$\lim_{n\to\infty}d_h(z_n,t_n)=0 \tag{3.23}$$

Thus by lemma 2.1, (3.19) implies that

$$\lim_{n\to\infty}\|z_n-t_n\|=0. \tag{3.24}$$

This implies that

$$\lim_{n\to\infty}\|z_n-w_n^i\|=0 \text{ for all } i=1,2,\ldots,N.$$

This establishes (iv).

Step 3: We demonstrate that $z_0\in\left(\bigcap_{i=1}^{N}EP(\psi_i)\right)\bigcap\left(\bigcap_{j=1}^{M}Fix(T_j)\right)$

First, we demonstrate that $z_0\in\bigcap_{i=1}^{N}EP(\psi_i)$. Using Lemma 2.5 and the fact that $p\in F$, we have

$$d_h(y_n,w_n^1)=d_h(y_n,\mathrm{Res}_{\psi_1}^h y_n)\le d_h(p,\mathrm{Res}_{\psi_1}^h y_n)-d_h(p,y_n)$$

$$\le d_h(p,y_n)-d_h(p,y_n)\to 0 \text{ as } n\to\infty. \tag{3.25}$$

$$d_h(y_n,w_n^2)=d_h(y_n,\mathrm{Res}_{\psi_2}^h(y_n))\le d_h(p,\mathrm{Res}_{\psi 2}^h(y_n))-d_h(p,y_n)$$

$$\le d_h(p,y_n)-d_h(p,y_n)\to 0 \text{ as } n\to\infty. \tag{3.26}$$

continuing this process, we get

$$d_h(y_n,w_n^N)=d_h(y_n,\mathrm{Res}_{\psi_n}^h(y_n))\le d_h(p,\mathrm{Res}_{\psi_n}^h(y_n))-d_h(p,y_n)$$

$$\le d_h(p,y_n)-d_h(p,y_n)\to 0 \text{ as } n\to\infty. \tag{3.27}$$

Hence in general, we arrive at

$$\lim_{n\to\infty} d_h(y_n, w_n^i) = 0,\ \forall i = 1,2,...,N. \tag{3.28}$$

By Lemma 2.1, (3.28) implies that

$$\lim_{n\to\infty} \| w_n^i - y_n \| = 0,\ \forall\, i = 1,2,...,N. \tag{3.29}$$

Consequently, we get

$$\lim_{n\to\infty} \| z_n - y_n \| = 0. \tag{3.30}$$

Now, from the uniform continuity of $\nabla h,$ (3.29) and (3.30) becomes

$$\lim_{n\to\infty} \| \nabla h(w_n^i) - \nabla h(y_n) \| = 0, \tag{3.31}$$

and

$$\lim_{n\to\infty} \| \nabla h(z_n) - \nabla h(y_n) \| = 0. \tag{3.32}$$

By definition, we have for $i = 1,2,...,N$, that

$$\psi_i(w_n^i, y) + \langle \nabla h(w_n^i) - \nabla h(y_n), y - w_n^i \rangle \geq 0,\ \forall\, y \in K,$$

$$\langle \nabla h(w_n^i) - \nabla h(y_n), y - w_n^i \rangle \geq \psi_i(y, w_n^i),\ \forall\, y \in K,$$

$$\| \nabla h(w_n^i) - \nabla h(y_n) \| \| y - w_n^i \| \geq \langle \nabla h(w_n^i) - \nabla h(y_n), y - w_n^i \rangle - \psi_i(y, w_n^i)$$

This implies that

$$\| \nabla h(w_n^i) - \nabla h(y_n) \| \| y - w_n^i \| \geq \psi_i(y, w_n^i)\ \forall\, y \in K. \tag{3.33}$$

Since $\psi_i(y, w_n^i)\ \forall\, y \in K, \forall i = 1,2,....,N$ is convex and lower semicontinuous and the fact that $w_n^i \to z_0\ \forall i = 1,2,....,N$, we get that

$$\psi_i(y, z_0) \leq 0\ \forall\, y \in K. \tag{3.34}$$

We set $\lambda \in (0,1)$ and $w_\lambda = \lambda y + (1-\lambda) z_0$ so that $w_\lambda \in K.$ This demonstrates that $\psi_i(w_\lambda, z_0) \leq 0\ \forall\, y \in K$. Using this, together with (A1) and (A4), we get

$$0 = \psi_i(w_\lambda, w_\lambda) = \psi_i(w_\lambda, \lambda y + (1-\lambda) z_0) \leq \lambda \psi_i(w_\lambda, y) + (1-\lambda) \psi_i(w_\lambda, z_0) \leq \lambda \psi_i(w_\lambda, y)$$

This implies that

$$\psi_i(w_\lambda, y) \geq 0. \tag{3.35}$$

By (A3), we get that

$\psi_i(z_0, y) \geq 0,\ y \in K,\ i = 1,2,...,N.$ We conclude that $z_0 \in \bigcap_{i=1}^N EP(\psi_i).$

Next, we demonstrate that $z_0 \in \bigcap_{j=1}^M Fix(T_j)$. Since $y_n = \nabla h^*\big((1-b_n)\nabla h(z_n) + b_n \nabla h(T_j z_n)\big)$, we obtain that

$$\| \nabla h(z_n) - \nabla h(y_n) \| = \| \nabla h(z_n) - \nabla h(z_n) + b_n(\nabla h(T_j z_n) - \nabla h(z_n)) \| = b_n \| \nabla h(T_j z_n) - \nabla h(z_n) \|.$$

Using (3.32) we have that

$$\lim_{n\to\infty} \| \nabla h(T_j z_n) - \nabla h(z_n) \| = 0.$$

Since h is strongly coercive and uniformly convex on bounded subsets of X, h^* is uniformly convex on bounded subsets of $X,$ so we obtain

$$\lim_{n\to\infty} \| z_n - T_j z_n \| = 0. \tag{3.36}$$

Using the fact that $z_n \to z_0$ (a Cauchy sequence), we have from (3.36) that

$$z_0 = \lim_{n\to\infty} T_j z_n. \tag{3.37}$$

If we pick a subsequence say $\{i_k\} \subset N$ such that $T_{i_k} = T_1\ \forall\, k \geq 1,$ then by implication $z_{n_k} \to z_0$ as $k \to \infty,$ and the continuity of T_1 (3.37) gives

$$z_0 = \lim_{k\to\infty} T_{i_k} z_{n_{k+1}} = T_1 \lim_{k\to\infty} z_{n_k} = T_1 z_0.$$

In addition, if we pick another subsequence say $\{i_{k+1}\} \subset N$ such that $T_{i_{k+1}} = T_2\ \forall\, k \geq 1,$ then

$$z_0 = \lim_{k\to\infty} T_{i_{k+1}} z_{n_{k+1}} = T_2 \lim_{k\to\infty} z_{n_{k+1}} = T_2 z_0.$$

Furthermore, the process yields $z_0 = T_j z_0,\ j \geq 3$. This demonstrate that $z_0 \in \bigcap_{j=1}^M Fix(T_j)$.

Thus $z_0 \in \left(\bigcap_{i=1}^N EP(\psi_i)\right) \cap \left(\bigcap_{j=1}^M Fix(T_j)\right)$.

Step 4: We demonstrate that $x_n \to z_0 = P_F^h(x_0)$. Since $x_n = P_{K_n}^h(x_0)$ and from step 1, $F \subset K_n$ so that from Lemma 2.6, we have

$$d_h(x_0, x_{n+1}) + d_h(x_{n+1}, P_F^h(x_0)) \le d_h(x_0, P_F^h(x_0)) \qquad (3.38)$$

Since $x_n \to z_0$ and by taking limit on both sides of (3.38), we get

$$d_h(x_0, z_0) + d_h(z_0, P_F^h(x_0)) \le d_h(x_0, P_F^h(x_0)).$$

This implies

$$d_h(x_0, z_0) \le d_h(x_0, P_F^h(x_0)). \qquad (3.39)$$

On the other hand, we get using Lemma 2.6 that

$$d_h(x_0, P_F^f(x_0)) + d_h(P_F^h(x_0), z_0) \le d_h(x_0, z_0).$$

This implies

$$d_h(x_0, P_F^h(x_0)) \le d_h(x_0, z_0) \qquad (3.40)$$

By combining (3.39) and (3.40), we have

$$d_h(x_0, P_F^h(x_0)) = d_h(x_0, z_0) \qquad (3.41)$$

By the uniqueness property of $P_F^h(x_0)$, we conclude that $x_n \to z_0 = P_F^h(x_0)$. This ends the proof of Theorem 3.2.

■

Corollary 3.3. Let K be a non-void, closed, convex subset of $\mathrm{int}(dom\, h)$. Let $h: X \to R$ be a Legendre function which is bounded, uniformly Fr*é*chet differentiable and totally convex on bounded subsets of a reflexive real Banach space X. Let $\{T\}_{i=1}^N : K \to K$ be an N-finite family of continuous Bregman quasi-nonexpansive mappings induced by a convex function h. Assume that $\bigcap_{i=1}^N Fix(T_i)$ is non-void. Set $x_0, x_1 \in K$. Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0 \in K \\ z_n = \nabla h^*\left(\nabla h(x_n) + \alpha_n \nabla h(x_n - x_{n-1})\right), \\ y_n^i = \nabla h^*\left((1-b_n)\nabla h(z_n) + b_n \nabla h(Tz_n)\right), i = 1,2,...N \\ t_n = \nabla h^*\left(\sum_{i=1}^N \frac{1}{N} \nabla h(y_n^i)\right), \\ K_{n+1} = \{u \in K_n : d_h(u, t_n) \le d_h(u, z_n)\}, \\ x_{n+1} = P_{K_{n+1}}^h(x_0), \ n \ge 1, \end{cases} \qquad (3.5)$$

suppose $\{\alpha_n\}, \{b_n\} \subset (0,1)$, then the sequence $\{x_n\}, \{z_n\}$ converges strongly to a common element of $\bigcap_{i=1}^N Fix(T_i)$.

Corollary 3.4. Let K be a non-void, closed, convex subset of $\mathrm{int}(dom\, h)$. Let $h: X \to R$ be a Legendre function which is bounded, uniformly Fr*é*chet differentiable and totally convex on bounded subsets of a reflexive real Banach space X. Let $\{T\}_{i=1}^N : K \to K$ be an N-finite family of Bregman relatively nonexpansive mappings induced by a convex function h. Assume that $\bigcap_{i=1}^N Fix(T_i)$ is non-void. Set $x_0, x_1 \in K$. Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0 \in K \\ z_n = \nabla h^*\left(\nabla h(x_n) + \alpha_n \nabla h(x_n - x_{n-1})\right), \\ y_n^i = \nabla h^*\left((1-b_n)\nabla h(z_n) + b_n \nabla h(Tz_n)\right), i = 1,2,...N \\ t_n = \nabla h^*\left(\sum_{i=1}^N \frac{1}{N} \nabla h(y_n^i)\right), \\ K_{n+1} = \{u \in K_n : d_h(u, t_n) \le d_h(u, z_n)\}, \\ x_{n+1} = P_{K_{n+1}}^h(x_0), \ n \ge 1, \end{cases} \qquad (3.5)$$

suppose $\{\alpha_n\}, \{b_n\} \subset (0,1)$, then the sequence $\{x_n\}, \{z_n\}$ converges strongly to a common element of $\bigcap_{i=1}^N Fix(T_i)$.

Corollary 3.5. Let K be a non-void, closed, convex subset of a Hilbert space. Let $\{\psi_i\}_{i=1}^N : K \times K \to R$ be N-bifunctions which meets properties (A1)–(A4). Let $\{T_j\}_{j=1}^m : K \to K$ be m-finite family of nonexpansive mappings. Assume that $F = \bigcap_{i=1}^N EP(\psi_i) \cap \left(\bigcap_{j=1}^M Fix(T_j)\right)$. Set $x_0, x_1 \in K$. Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} x_0 \in K \\ z_n = x_n + \alpha_n (x_n - x_{n-1}), \\ y_n^j = (1-b_n) z_n + b_n T_j z_n, \ j = 1,2,...N \\ w_n^i = \mathrm{Re}\, s_{\psi_i}(y_n^j), \ i, j = 1,2,...N, M, \\ t_n = \nabla h^*\left(\sum_{i=1}^N \frac{1}{N} \nabla h(w_n^i)\right), \\ K_{n+1} = \{u \in K_n : \| t_n u \| \le \| z_n - u \|\}, \\ x_{n+1} = P_{K_{n+1}}(x_0), \ n \ge 1, \end{cases} \qquad (3.5)$$

suppose $\{\alpha_n\}, \{b_n\} \subset (0,1)$, then the sequence $\{x_n\},\{z_n\}$ converges strongly to a common element of F.

4 Numerical Example

We present a numerical example to justify our theoretical assertions made in section 3 of this paper. Our codes were written in Python and run on PC with intel(R) Core(TM)2 Duo CPU @ 3.10 GHz processor.

Example 1: Let $X = R,\ K = [0,1].$ Also consider $M = N = 30$. Consider the convex function $h: K \to R$ defined by $h(x) = (2/3)x^2$, such that $\nabla h(x) = (4/3)x$.

(i) We define the mappings $T_j : K \to K$ by $T_j(x) = -(1/2)x^j + x^{j-1},\ j = 1,2,...,M,\ \forall x \in K$. It is easy to check that $Fix(T_1) = \{2/3\}$ for $j = 1$ and $Fix(T_j) = \{0\}$ for $j \geq 2.$ To see this, for $j = 1$ and $T_1(x) := -(1/2)x + 1,$ gives $x = -(1/2)x + 1 \Rightarrow 3x = 2 \Rightarrow x = 2/3$. Hence $Fix(T_1) = \{2/3\}$. In addition, for $j = 2$ and $T_2(x) := -(1/2)x^2 + x,$ gives $x = -(1/2)x^2 + x \Rightarrow -x^2 = 0 \Rightarrow x = 0.$ Thus, $Fix(T_2) = \{0\}$. Continuing the process and for $j \geq 3$, we conclude that $Fix(T_j) = \{0\} \forall j \geq 2.$ But $\bigcap_{j=1}^{M} Fix(T_j) = \emptyset.$

(ii) If we define the mappings $T_j : K \to K$ by $T_j(x) = -(1/2)x^j,\ j = 1,2,...,M,\ \forall x \in K$, we get that $Fix(T_j) = \{0\} \forall j \geq 1.$ Thus $\bigcap_{j=1}^{M} Fix(T_j) = \{0\}$.

Next, we check if $T_j(x) = -(1/2)x^j + x^{j-1},\ j \geq 1,$ and $T_j(x) = -(1/2)x^j,\ j \geq 1\ \forall x \in K$ are Bregman quasi-nonexpansive mappings and continuous.

Now for $j = 1$ and $T_1(x) := -(1/2)x + 1,\quad p = \{2/3\},$ we get from the definition of Bregman bifunctions that

$$d_h(p, T_1x) = h(p) - h(T_1x) - \langle \nabla h(T_1x), p\rangle + \langle \nabla h(T_1x), T_1x\rangle,$$

$$= \frac{8}{27} - \frac{2}{3}\left(-\frac{1}{2}x + 1\right)^2 - \left(-\frac{4}{9}x + \frac{8}{9}\right) + \left(-\frac{2}{3}x + \frac{4}{3}\right) * \left(-\frac{1}{2}x + 1\right)$$

$$= \frac{1}{6}x^2 - \frac{2}{9}x + \frac{2}{27}.$$

$$d_h(p, x) = h(p) - h(x) - \langle \nabla h(x), p\rangle + \langle \nabla h(x), x\rangle,$$

$$= \frac{8}{27} - \frac{2}{3}x^2 - \frac{8}{9}x + \frac{4}{3}x^2$$

$$= \frac{2}{3}x^2 - \frac{8}{9}x + \frac{8}{27}.$$

Thus,

$$d_h(p, T_1x) < d_h(p, x)\ \ \forall x \in [0,1].$$

Similarly, for $j = 2$ and $T_2(x) := -(1/2)x^2 + x,\quad p = \{0\},$

$$d_h(p, T_2x) = h(p) - h(T_2x) - \langle \nabla h(T_2x), p\rangle + \langle \nabla h(T_2x), T_2x\rangle,$$

$$= \frac{1}{6}x^4 - \frac{2}{3}x^3 + \frac{2}{3}x^2$$

$$d_h(p, x) = h(p) - h(x) - \langle \nabla h(x), p\rangle + \langle \nabla h(x), x\rangle,$$

$$= \frac{2}{3}x^2.$$

Thus,

$$d_h(p, T_2x) \leq d_h(p, x)\ \ \forall x \in [0,1].$$

Continuing the process, we get that $d_h(p, T_jx) \leq d_h(p, x)\ \forall j \geq 3,\ \forall x \in [0,1].$ Therefore, $T_j(x) = -(1/2)x^j + x^{j-1},\ j = 1,2,...,M,\ \forall x \in K$ are Bregman quasi-nonexpansive mappings and continuous. Similarly, repeating the steps for $T_j(x) = -(1/2)x^j,\ j = 1,2,...,M,\ \forall x \in K$ with $p = \{0\}$ we conclude that $T_j(x) = -(1/2)x^j,\ j = 1,2,...,M,\ \forall x \in K$ are Bregman quasi-nonexpansive mappings as well as continuous.

Furthermore, we define the bifunctions $\psi_i : K \times K \to R$ for $i = 1,2...,N$ by $\psi_i(u, z) := i(2z^2 + uz - 3u^2)$.

It is clear that ψ_i satisfies the conditions (A1) – (A4). So by Lemma 2.5, $\mathrm{Res}_{\psi_i}^{h}(y)$ is nonempty and single-valued for each $y \in K.$ Hence there exist $u \in K$ such that

$$\psi_i(u, z) + \langle \nabla h(u) - \nabla h(y), z - u\rangle \geq 0,\ \ \forall z \in K,$$

$$i(2z^2 + uz - 3u^2) + \langle \frac{4}{3}u - \frac{4}{3}y, z - u\rangle \geq 0, z \in K,$$

which is equivalent to

$$2iz^2+\left(iu+\frac{4}{3}u-\frac{4}{3}y\right)z-3iu^2-\frac{4}{3}u^2+\frac{4}{3}yu\ \geq 0,\ z\in K.$$

Set $R(z):=2iz^2+\left(iu+\frac{4}{3}u-\frac{4}{3}y\right)z-3iu^2-\frac{4}{3}u^2+\frac{4}{3}yu$. This function is a quadratic function with respect to z. Now using the discriminant of $R,$ we get

$$D1:=\frac{1}{9}(15iu+4u-4y)^2.$$

Since $R(z)\geq 0\ \forall z\in K$ and since it has at most one solution in $R,$ we get that $D1:=\frac{1}{9}(15iu+4u-4y)^2\leq 0$ so that equality holds and solving for $u,$ we get

$u:=\frac{4}{15i+4}y.$ This implies that $\mathrm{Res}_{\psi_i}^h(y):=\frac{4}{15i+4}y.$

We assume for our purpose that

$$\alpha_n=\frac{n}{4n^2+10},\ b_n=\frac{n}{2n+1},\ (1-b_n)=\frac{n+1}{2n+1}.$$

Using the above, we simplify our scheme of theorem 3.3 for particular cases of $i=j=2.$

Case 1: for $i=j=2.$

$$z_n:=x_n+\frac{3}{4}\frac{n\left(\frac{4}{3}x_n-\frac{4}{3}x_{n-1}\right)}{4n^2+10};$$

$$y_n:=-\frac{1}{2}*\frac{nz_n^2-4nzn-2z_n}{2n+1};$$

$$w_n:=\frac{2}{17}*y_n;$$

$$t_n:=\frac{1}{2}\sum_{i=1}^{2}\frac{4}{15i+4}*y_n;$$

$$K_{n+1}:=\{u\in K_n:d_h(u,t_n)\leq d_h(u,z_n)\};$$

$$\therefore K_{n+1}:\left\{u\in K_n:u\leq\frac{1}{2}z_n+\frac{106}{323}y_n;\right\}$$

$$x_{n+1}:=P_{K_{n+1}}^h(x_0)=\frac{1}{2}z_n+\frac{106}{323}y_n.$$

Table 1: Values of $x[n]$ and $x2[n]$ with initials $x[0]=0.25$ $x[1]=0.33$

Itera[n]	$x[n]$	$x2[n]$	||x[n]-x2[n]||
0	0.250000	0.000000	0.250000
1	0.333333	0.333333	0.000000
2	0.274691	0.269981	0.004711
3	0.218965	0.218807	0.000158
4	0.175070	0.177843	0.002773
5	0.140849	0.144978	0.004130
6	0.113912	0.118499	0.004588
7	0.092505	0.097075	0.004570
8	0.075363	0.079673	0.004310
9	0.061554	0.065493	0.003939
10	0.050378	0.053906	0.003528
11	0.041300	0.044417	0.003117
12	0.033903	0.036630	0.002726
13	0.027863	0.030230	0.002367
14	0.022920	0.024964	0.002044
15	0.018868	0.020625	0.001757
16	0.015543	0.017047	0.001505
17	0.012810	0.014095	0.001285
18	0.010563	0.011657	0.001094
19	0.008714	0.009643	0.000930
20	0.007190	0.007979	0.000789
21	0.005935	0.006603	0.000668
22	0.004900	0.005465	0.000565
23	0.004047	0.004523	0.000477
24	0.003342	0.003745	0.000402
25	0.002761	0.003100	0.000339
26	0.002281	0.002567	0.000285
27	0.001885	0.002125	0.000240
28	0.001558	0.001760	0.000202
29	0.001288	0.001457	0.000169

Key: $x[n]$ represent iterates with inertial component while $x2[n]$ represent iterates without the inertial component. ||x[n]-x2[n]|| represents the error difference between the iterates. Itera[n] represent the no of iterations.

Remark: From the computed results as shown in the table above, we can see that the iterates with inertial component converges faster to its fixed point.

5. Conclusion

We formulated a new iterative scheme with an inertial component that solves a common solution problem of finite family of continuous Bregman quasi-nonexpansive self-mappings and system of equilibrium in a reflexive and (real) Banach space. Our proof finds a common element in the collection of fixed set of finite family of continuous Bregman quasi-nonexpansive self-mappings and the common solution set of the system of finite equilibrium problems. As an improvement to other existing results in this direction, we further justified our theoretical assertions with a numerical experiment as seen above.

Conflict of Interest

The authors identified void conflict of interest.

Data Availability

No data were used to support this study, except the codes written in Maple 18 and Python Programs for our numerical experiment.

Acknowledgement

Thanks to the Editor-in-Chief and anonymous referees, for their careful reading, and valuable suggestions that improved the quality of this manuscript.

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