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THE D-DIMENSIONAL SCHRÖDINGER EQUATION FOR MULTIPARAMETER-TYPE POTENTIAL (MPP) AND ITS THERMODYNAMIC PROPERTIES USING NIKIFOROV-UVAROV (NU) METHOD

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Abstract

In our study, the Pekeris approximation scheme and Nikiforov-Uvarov (NU) method are applied to obtain the energy eigenvalues of the d-dimensional Schrödinger equation with a multiparameter-type potential (MPP). The normalized energy eigenfunctions were also obtained in terms of the hypergeometric polynomials. Numerical results of the energy eigenvalues were obtained in different dimensions, and they agree with results in the literature. With appropriate values for the potential parameters, the effects of the potential parameters on the energy eigenvalues were evaluated graphically. The eigenvalues of the energy relation for the MPP were employed to evaluate the vibrational partition function and other thermodynamic properties for certain temperatures and upper bound vibrational quantum numbers. Finally, variations of the thermodynamic properties with certain temperatures and upper bound vibrational quantum numbers were analyzed and discussed extensively.

Keywords: Multiparameter potential, Schrödinger equation, Nikiforov-Uvarov method, Vibrational Partition function.
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1. Introduction

The study of non-relativistic wave equation has attracted the interest of many researchers, because of the information that its solution provides regarding any quantum system under investigation. The exact solutions are seen in few cases like Harmonic and Coulomb potentials[1-3]. The applications of these potentials are found in different areas of atomic, nuclear and high energy physics[4-8]. However, for arbitrary ℓ-state ($\ell \neq 0$), an appropriate approximation scheme may be employed to obtain the solution [9-14]. Many analytical methods have been employed while solving the non-relativistic wave equation. They include supersymmetric quantum mechanics (SUSYQM) approach [15-17], exact quantization method and proper quantization method [18-22], factorization method and modified factorization method (MFM) [23-26], asymptotic iteration method (AIM) [27-28], Nikiforov-Uvarov (NU) method [29-33], and others [34-38].

Recently, researchers have considered most potentials in their generalized form. This is done to explain the various interactions that exist within these potentials [39-41]. A typical example of such potential that is of interest to us is the multiparameter-type potential (MPP) of the form [42]:

$$V(r) = \frac{A + Be^{-b(r-L)}}{\left(q + pe^{-b(r-L)}\right)} + \frac{Ce^{-b(r-L)} + Ge^{-2b(r-L)}}{\left(q + pe^{-b(r-L)}\right)^2} \qquad (1),$$

where the potential parameters all take real values. The parameters A, B, C, G, p, and q can take either positive or negative value, while parameters b and L must take positive values. Many authors have studied different forms MPP, due to its usefulness in describing the interactions in the structure of the diatomic and polyatomic molecules. The authorsin [43] studied the scattering state of the MPP with an improved approximation for the centrifugal termin d-dimensions. In another development, the d-dimensional Klein–Gordon equation for multiparameter exponential-type potential was studied, using SUSYQM [44]. With a new developed approximation, the authors in[45] obtained the energy eigenvalues and eigenfunctions of the bound and scattering states for a hyperbolic-type potential, using the NU method. The solutions for the bound and scattering states of the Klein-Gordon equation with the MPP, using a standard method and a Pekeris approximation scheme for the centrifugal term have been explored [46].

For some decades now, the thermodynamic functions of exponential-type potentials and other related potential have been investigated[47-52]. The authors in [53] examined the thermodynamic functions of exponential-type molecule potentials in D dimensions. Also, the solutions of the Schrödinger equation with modified Mobius square potential energy, together with its thermodynamic functions have been investigated [54]. In another development, the thermodynamic functions of quadratic exponential-type potential (QEP) model were studied in D-dimensions via the MFM[55].

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Hence, our aim is to obtain the partition function in a closed form and other thermodynamic functions of the MPP, which has not been documented before in the literature so far. To help us actualize this, we adopt the conventional NU method to obtain the approximate eigenvalues and the eigenfunctions of the d-dimensional Schrödinger equation with the MPP. Numerical results of the energy eigenvalues are also obtained and compared with the previously related works inthe literature. We have arranged this article as follows: In section 2, the theoretical description of the NU method is presented briefly. We obtain the analytical expression of the eigenvalues of the energy for the d-dimensional MPP and its corresponding normalized energy eigenfunctions in section 3. In section 4, we evaluate both the vibrational partition function and other thermodynamic functions of the MPP. An elaborate discussion of the result is done in section 5. Finally, the summary and conclusion of our study are presented in section 6.

2.0 Theoretical Framework

2.1 Brief description of the Nikiforov-Uvarov (NU) method

The NU method as proposed in [29] is often used in converting a nonrelativistic-like equation into a differential equation of the second order using a coordinate transformation $t = t(r)$, given as

$$\psi''(t)+\frac{\tilde{\tau}(t)}{\sigma(t)}\psi'(t)+\frac{\tilde{\sigma}(t)}{\sigma^2(t)}\psi(t)=0 \tag{2}$$

where $\tilde{\sigma}(t)$, $\sigma(t)$ are polynomials, at most second degree, and $\tilde{\tau}(t)$ is a first-degree polynomial. To obtain the exact solution of Eq. (2), we employ the transformation given below:

$$\psi(t)=\Phi(t)y_{n\ell}(t) \tag{3}$$

Eq. (3) reduces Eq. (2) into a hypergeometric-type equation given as

$$\sigma(t)y''_{n\ell}(t)+\tau(t)y'_{n\ell}(t)+\xi y_{n\ell}(t)=0 \tag{4}$$

We also define the function $\Phi(t)$ as the logarithm derivative given by [29]

$$\frac{\Phi'(t)}{\Phi(t)}=\frac{\pi(t)}{\sigma(t)} \tag{5}$$

where $\pi(t)$ is a first-degree polynomial. The second part of $\psi(t)$ being $y_{n\ell}(t)$ in Eq. (3) is the hypergeometric function and its solution is obtained in the form of a polynomial, using the Rodrigues relation of the form

$$y_{n\ell}(t)=\frac{B_{n\ell}}{\rho(t)}\frac{d^n}{dt^n}\left[\sigma^n(t)\rho(t)\right] \tag{6}$$

Here, $B_{n\ell}$ and $\rho(t)$ are known to be the normalization constant and the weight function, respectively, and they satisfy the condition

$$\frac{d}{dt}\left[\sigma(t)\rho(t)\right]=\tau(t)\rho(t) \tag{7}$$

with

$$\tau(t)=\tilde{\tau}(t)+2\pi(t) \tag{8}$$

We also state here that the derivative of $\tau'(t)$ should be negative for bound state solutions. The eigenfunctions and eigenvalues can be obtained with the help of the function $\pi(t)$ and parameter ξ, respectively, as given below:

$$\pi(t)=\frac{\sigma'(t)-\tilde{\tau}(t)}{2}\pm\sqrt{\left(\frac{\sigma'(t)-\tilde{\tau}(t)}{2}\right)^2-\tilde{\sigma}(t)+k\sigma(t)} \tag{9}$$

and

$$\xi=k+\pi'(t). \tag{10}$$

To obtain the value of k, we set the discriminant of the square root in Eq. (9) equal to zero; hence obtaining the new eigenvalue equation as

$$\xi+n\tau'(t)+\frac{n(n-1)}{2}\sigma''(t)=0,\ (n=0,1,2,...). \tag{11}$$

2.2 Solutions of D-Dimensional Schrödinger Equation with Multiparameter-type Potential (MPP)

The radial part of the Schrödinger equation in d-dimensions reads [56]:

$$\frac{d^2\varphi(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E_{n\ell}-V(r)-\frac{\hbar^2}{2\mu r^2}\left(\frac{(d-1)(d-3)}{4}+\ell(\ell+d-2)\right)\right]\varphi(r)=0 \tag{12}$$

where μ is the reduced mass and $E_{n\ell}$ is the non-relativistic energy eigenvalues to be determined.

Substituting Eq. (1) into Eq. (12) gives,

$$\frac{d^2\varphi}{dr^2}+\left\{\begin{array}{l}\dfrac{2\mu E_{n\ell}}{\hbar^2}-\dfrac{2\mu A}{\hbar^2\left(q+pe^{-b(r-L)}\right)}-\dfrac{2\mu Be^{-b(r-L)}}{\hbar^2\left(q+pe^{-b(r-L)}\right)}\\ -\dfrac{2\mu Ce^{-b(r-L)}}{\hbar^2\left(q+pe^{-b(r-L)}\right)^2}-\dfrac{2\mu Ge^{-2b(r-L)}}{\hbar^2\left(q+pe^{-b(r-L)}\right)^2}+\dfrac{\omega}{r^2}\end{array}\right\}\varphi=0 \tag{13}$$

where

$$\omega=\frac{(d-1)(d-3)}{4}+\ell(\ell+d-2)\cdot \tag{14}$$

By letting $r=z+L,\gamma=\frac{p}{q}$, we can find the analytical solution of Eq. (13) by invoking the Pekeris approximation [9] of the form:

$$\frac{1}{r^2}=\frac{1}{(z+L)^2}\approx c_0+\frac{c_1}{\left(p+qe^{bz}\right)}+\frac{c_2}{\left(p+qe^{bz}\right)^2} \tag{15}$$

with the coefficients c_0, c_1, c_2 being obtained as [42]

$$\begin{aligned}c_0&=\frac{(3+bL)p^2-2(-3+bL)pq+\left(3-3bL+b^2L^2\right)q^2}{b^2q^2L^4},\\ c_1&=\frac{2(p+q)^2\left[(3+bL)p+(3-2bL)q\right]}{b^2q^2L^4},\\ c_2&=\frac{(p+q)^3\left[(3+bL)p+(3-bL)q\right]}{b^2q^2L^4}.\end{aligned} \tag{16}$$

By using the expression $p+qe^{bz}=qe^{bz}\left(1+\gamma e^{-bz}\right)$ and the coordinate transformation $t=-\gamma e^{-bz}$, Eq. (13) reduces to

$$\frac{d^2\varphi(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{d\varphi(s)}{ds}+\frac{1}{s^2(1-s)^2}\left\{-\upsilon_1s^2+\upsilon_2s-\upsilon_3\right\}\varphi(s)=0 \tag{17}$$

Where υ_1, υ_2, υ_3 are given respectively as

$$\begin{aligned}\upsilon_1&=\frac{\omega}{b^2\gamma^2q^2}\left(\gamma^2q^2c_0+\gamma qc_1+c_2\right)+\frac{2}{b^2\gamma^2q^2\hbar^2}(G+\gamma qB)-\frac{2\mu E_{n\ell}}{b^2},\\ \upsilon_2&=\frac{\omega}{b^2\gamma q}\left(2\gamma qc_0+c_1\right)+\frac{2\mu}{\gamma b^2q^2\hbar^2}\left(q\gamma A+qB+C\right)-\frac{4\mu E_{n\ell}}{b^2},\\ \upsilon_3&=\frac{\omega}{b^2}c_0+\frac{2\mu A}{qb^2\hbar^2}-\frac{2\mu E_{n\ell}}{b^2}.\end{aligned} \tag{18}$$

By comparing Eq. (17) and Eq. (2), we have

$$\tau(t)=1-t;\qquad \sigma(t)=t(1-t);\qquad \bar{\sigma}(t)=-\upsilon_1t^2+\upsilon_2t-\upsilon_3. \tag{19}$$

Substituting eq. (19) into Eq. (9), $\pi(s)$ becomes

$$\pi(t)=-\frac{t}{2}\pm\sqrt{\left(\frac{1}{4}+\upsilon_1-k\right)t^2+(k-\upsilon_2)t+\upsilon_3}. \tag{20}$$

To find the constant k, the discriminant of the expression under the square root of Eq. (20) must be zero. As such, we have that

$$k_\pm=\upsilon_2-2\upsilon_3\pm2\sqrt{\upsilon_3\left(\frac{1}{4}+\upsilon_3+\upsilon_1-\upsilon_2\right)} \tag{21}.$$

Substituting Eq. (21) into Eq. (20), we have

$$\pi(t)_\pm=-\frac{t}{2}\pm\left\{\begin{array}{l}\left(\sqrt{\upsilon_3}+\sqrt{\dfrac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}\right)t-\sqrt{\upsilon_3};\ for\ k_+=\upsilon_2-2\upsilon_3+2\sqrt{\upsilon_3\left(\dfrac{1}{4}+\upsilon_3+\upsilon_1-\upsilon_2\right)}\\ \left(\sqrt{\upsilon_3}+\sqrt{\dfrac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}\right)t-\sqrt{\upsilon_3};\ for\ k_-=\upsilon_2-2\upsilon_3-2\sqrt{\upsilon_3\left(\dfrac{1}{4}+\upsilon_3+\upsilon_1-\upsilon_2\right)}\end{array}\right. \tag{22}$$

Here, we choose the expression $\pi(t)_{-}$ which the function $\tau(t)$ has a negative derivative. This is given by

$$\pi_{-}(t)=-\frac{t}{2}-\left(\left(\sqrt{\upsilon_3}+\sqrt{\frac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}\right)t-\sqrt{\upsilon_3}\right) \tag{23}$$

with $\tau(t)$ being obtained as

$$\tau(t)=1-2t-2\left(\sqrt{\upsilon_3}+\sqrt{\frac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}\right)t-\sqrt{\upsilon_3}. \tag{24}$$

From Eq. (10), we define the constant ξ to be

$$\xi=\upsilon_2-2\upsilon_3-\frac{1}{2}-\sqrt{\frac{1}{4}+\upsilon_1+\upsilon_3-\upsilon_2}-\left(1+2\sqrt{\frac{1}{4}+\upsilon_1+\upsilon_3-\upsilon_2}\right)\sqrt{\upsilon_3}. \tag{25}$$

Substituting Eq. (25) into Eq. (11) and carrying out simple algebra, where

$$\tau'(t)=-2\left(1+\sqrt{\upsilon_3}+\sqrt{\frac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}\right),\ \tau'(t)=-2 \tag{26}$$

we have

$$\upsilon_3=\left\{\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\upsilon_1+\upsilon_3-\upsilon_2}\right)^2+\upsilon_3-\upsilon_1}{2\left(\frac{1}{2}+n+\sqrt{\frac{1}{4}+\upsilon_1+\upsilon_3-\upsilon_2}\right)}\right\}^2. \tag{27}$$

Substituting Eq. (18) into Eq. (27) and rearranging the terms yields the d-dimensional energy eigenvalues of the (MPP) in the form

$$E_{n,\ell}=\frac{\beta c_0}{2\mu}+\frac{A}{q\hbar^2}-\frac{b^2}{2\mu}\left\{\frac{(n+\sigma)}{2}-\frac{(\upsilon_1-\upsilon_3)}{2(n+\sigma)}\right\}^2 \tag{28}$$

where

$$\sigma=\frac{1}{2}\left(1+\sqrt{1+4(\upsilon_1+\upsilon_3-\upsilon_2)}\right) \tag{29}$$

We also evaluate the corresponding eigenfunctions by substituting $\pi(t)_{-}$ and $\sigma(t)$ from Eqs. (23) and(19) respectively into Eq. (5) and solving the first order differential equation. This gives

$$\Phi(t)=t^{\sqrt{\upsilon_3}}(1-t)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}} \tag{30}$$

The weight function $\rho(t)$ from Eq. (7) can be obtained as

$$\rho(t)=t^{2\sqrt{\upsilon_3}}(1-t)^{2\sqrt{\frac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}} \tag{31}$$

From the Rodrigues relation of Eq. (6), we obtain

$$y_{n\ell}(t)=B_{n\ell}t^{-2\sqrt{\upsilon_3}}(1-t)^{-2\sqrt{\frac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}}\frac{d^n}{dt^n}\left[t^{n+2\sqrt{\upsilon_3}}(1-t)^{n+2\sqrt{\frac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}}\right] \tag{32},$$

$$y_{n\ell}(t)\equiv B_{n\ell}P_n^{\left(2\sqrt{\upsilon_3},\,2\sqrt{\frac{1}{4}+\upsilon_1-\upsilon_2+\upsilon_3}\right)}(1-2t) \tag{33}$$

where $P_n^{(\theta,\vartheta)}$ is the Jacobi Polynomial which is defined as [57]

$$P_n^{(\theta,\vartheta)}(x)=\frac{\Gamma(n+\theta+1)}{n!\Gamma(\theta+1)}\,{}_2F_1\left(-n,\theta+\vartheta+n+1,\theta+1;\frac{1-x}{2}\right) \tag{34}$$

Substituting $\Phi(t)$ and $y_{n\ell}(t)$ from Eqs.(30) and(33) respectively into Eq. (3), we obtain

$$\psi(t) = B_{n\ell} t^{\sqrt{\upsilon_3}} (1-t)^H P_n^{\left(2\sqrt{\upsilon_3}, 2H-1\right)}(1-2t) \tag{35}$$

where

$$H = \frac{1}{2}\left(1+\sqrt{1+4\left(\upsilon_1-\upsilon_2+\upsilon_3\right)}\right) \tag{36}$$

and $B_{n\ell}$ is the normalization constant.

In terms of hypergeometric Polynomials, Eq. (35) can be written as

$$\psi(t) = B_{n\ell} t^{\sqrt{\upsilon_3}} (1-t)^H \frac{\Gamma\left(n+2\sqrt{\upsilon_3}+1\right)}{n!\Gamma\left(2\sqrt{\upsilon_3}+1\right)} {}_2F_1\left(-n,\, 2\sqrt{\upsilon_3}+2H+n,\, 2\sqrt{\upsilon_3}+1;\, t\right) \tag{37}$$

To find the normalization constant, we write the radial wave function as

$$\left[\psi(t)\right]^2 = B_{n\ell}^2 t^{2\sqrt{\upsilon_3}} (1-t)^{2H} \left[P_n^{\left(2\sqrt{\upsilon_3}, 2H-1\right)}(1-2t)\right]^2,\ t = e^{-\alpha r} \tag{38}$$

Eq. (38) also represents the probability density $\rho(t)$. Effecting the normalization condition given by

$$\int_0^\infty |\psi(r)|^2\, dr = 1, \tag{39}$$

We have that

$$-\frac{B_{n\ell}^2}{\alpha}\int_1^0 t^{2\sqrt{\upsilon_3}} (1-t)^{2H} \left[P_n^{\left(2\sqrt{\upsilon_3}, 2H-1\right)}(1-2t)\right]^2 \frac{dt}{t} = 1,\ t = e^{-\alpha r} \tag{40}$$

Carrying out a coordinate transformation $Z = 1-2t$, Eq. (40) becomes

$$\frac{B_{n\ell}^2}{2\alpha}\int_{-1}^1 \left(\frac{1-Z}{2}\right)^{2\sqrt{\upsilon_3}} \left(\frac{1+Z}{2}\right)^{2H} \left[P_n^{\left(2\sqrt{\upsilon_3}, 2H-1\right)}(Z)\right]^2 dZ = 1 \tag{41}$$

Using the standard integral [58],

$$\int_{-1}^1 \left(\frac{1-u}{2}\right)^g \left(\frac{1+u}{2}\right)^h \left[P_n^{(g,h-1)}(u)\right]^2 du = \frac{2^{g+h+1}\Gamma(g+n+1)\Gamma(h+n+1)}{n!\Gamma(g+h+n+1)\Gamma(g+h+2n+1)} \tag{42}$$

we have the normalization constant in Eq. (41) as

$$B_{n\ell} = \sqrt{\frac{2\alpha(n!)\Gamma\left(1+n+2H+2\sqrt{\upsilon_3}\right)\Gamma\left(1+2n+2H+2\sqrt{\upsilon_3}\right)}{2^{\left(1+2H+2\sqrt{\upsilon_3}\right)}\Gamma\left(1+n+2\sqrt{\upsilon_3}\right)\Gamma\left(1+n+2H\right)}} \tag{43}$$

2.3 Partition Function and Thermodynamic Properties of MPP

To obtain the vibrational partition function of the MPP, we first reduce Eq. (28) to the form

$$E_{n\ell} = R - \frac{b^2}{2\mu}\left[\frac{(n+\sigma)}{2} - \frac{T}{2(n+\sigma)}\right]^2 \tag{44}$$

where

$$R = \frac{\beta c_0}{2\mu} + \frac{A}{q\hbar^2},\ \ T = \left(\upsilon_1 - \upsilon_3\right). \tag{45}$$

For any bound state system, the vibrational partition function at absolute temperature T is defined as [59]

$$Z(\beta,\lambda) = \sum_{n=0}^{\lambda} e^{-\beta E_{nl}},\ \ \beta = \frac{1}{k_B T} \tag{46}$$

where $n = 0,1,2,\ldots < \lambda,$ and λ being the maximum quantum number is given by

$\lambda = -\sigma + \sqrt{R} \pm \sqrt{R-T}$; k_B being the Boltzmann constant.

Substituting Eq. (44) into Eq. (46) gives

$$Z(\beta,\lambda) = \sum_{n=0}^{\lambda} e^{\left[Q(n+\sigma)^2\beta + \frac{M\beta}{(n+\sigma)^2} + N\beta\right]} \tag{47}$$

where

$$Q=\frac{b^2}{8\mu},\ M=\frac{b^2T^2}{8\mu},\ N=-\left(\frac{b^2T}{4\mu}+R\right). \tag{48}$$

Replacing the sum in Eq. (47) by an integral, we obtain:

$$\int_{\sigma}^{\lambda+\sigma} e^{\left(Q\rho^2\beta+\frac{M\beta}{\rho^2}+N\beta\right)}d\rho,\ \rho=(n+\sigma) \tag{49}$$

By the evaluation of Eq. (49), we obtain the vibrational partition function to be

$$Z(\beta,\lambda)=-e^{\left(Q\rho^2\beta+N\beta\right)}\left[-\sqrt{M\beta}\sqrt{\pi}\left(erfi\left(\frac{\sqrt{M\beta}}{\sigma}\right)-erfi\left(\frac{\sqrt{M\beta}}{(\lambda+\sigma)}\right)\right)+\sigma e^{\frac{M\beta}{\sigma^2}}-(\sigma+\lambda)e^{\frac{M\beta}{(\lambda+\sigma)^2}}\right] \tag{50}$$

and the imaginary error function is defined as [60]:

$$erfi(x)=\frac{erf(ix)}{i}=\frac{2}{\sqrt{\pi}}\int_0^x e^{t^2}dt. \tag{51}$$

Other thermodynamic functions can be deduced from the vibrational partition function of Eq. (50), via the following relations [52]:

$$\begin{aligned}
F(\beta,\lambda)&=-\frac{1}{\beta}\ln Z(\beta,\lambda),\\
S(\beta,\lambda)&=-k_B\frac{\partial}{\partial\beta}F(\beta,\lambda),\\
U(\beta,\lambda)&=-\frac{\partial}{\partial\beta}\ln Z(\beta,\lambda),\\
C(\beta,\lambda)&=k_B\frac{\partial}{\partial\beta}F(\beta,\lambda),
\end{aligned} \tag{52}$$

where $F(\beta,\lambda)$, $S(\beta,\lambda)$, $U(\beta,\lambda)$, and $C(\beta,\lambda)$ represents the Helmholtz free energy, entropy, internal energy, and specific heat capacity, respectively.

3.0 Results and discussion

In this paper, the energy eigenvalues for the d-dimensional MPP are computed. Table 1 shows the 3-dimensional energy eigenvalues, with appropriate potential parameters being chosen. Our results agree with that obtained in [42], for vibrational and rotational quantum numbers. Table 2 presents the energy eigenvalues for the d-dimensional MPP in higher dimensions. It can be seen that for any quantum number n, the energy eigenvalue decreases as ℓ increases. Also, there is a decrease in the energy eigenvalue as the dimension increases at a specific quantum number.

We also investigate the effects of the potential parameters on the energy eigenvalues, as presented in Figures 1-8, for $2p$, $3d$, and $4f$ quantum states. As the potential parameters A, B, C, q, and d increases, the energy eigenvalues increases also (see Figures 1-5). We observe a monotonic decrease in energy eigenvalues as the potential parameter b increases (see Figure 6). The variation of the energy eigenvalue with the potential parameter p tends to be unsteady. As seen in Figure 7, the energy eigenvalues increases as p increases, and later becomes constant with continuous increase in the potential parameter. An inversely proportional relationship is observed between the energy eigenvalues and potential parameter G, as shown in Figure 8 for various quantum states.

We plot the vibrational partition function for the d-dimensional MPP for various upper bound quantum numbers λ=10, 50, and 100, and temperature $\beta=0.01$, 0.02, and 0.1, as shown in Figures 9 and 10 respectively. We observe that the vibrational partition function increases as β and λ increase for the potential understudy. The vibrational internal energy shows a monotonous decrease as β and λ increases, as observed in Figures 11 and 12. Figure 13 and Figure 14show the dependence of the vibrational Helmholtz free energy on β and λ, respectively. Also, as β and λ increases, the vibrational entropy of the d-dimensional MMP increases also, as seen in Figures 15 and 16. In Figure 17, the vibrational specific heat capacity and the temperature β increase simultaneously.

Table 1. The energy eigenvalues for the multiparameter-type potential for different quantum states.

n	ℓ	$E_{n\ell}$	$E_{n+1,\ell}$ [42]	n	ℓ	$E_{n\ell}$	$E_{n+1,\ell}$ [42]
0	0	-1.481428659	-1.48143	5	0	-18.55021385	-18.5502
1	0	-2.347432522	-2.34743		1	-18.51953115	
	1	-2.330023408			2	-18.45768587	
2	0	-4.806389509	-4.80639		3	-18.36366403	
	1	-4.786890497			4	-18.23579471	
	2	-4.747710656			5	-18.07152133	
3	0	-8.363783599	-8.36378	6	0	-25.15521853	-25.1552
	1	-8.340910484			1	-25.12044421	
	2	-8.294877769			2	-25.05032235	
	3	-8.225078453			3	-24.94364243	
4	0	-12.95098275	-12.9510		4	-24.79841127	
	1	-12.92429849			5	-24.61158232	
	2	-12.87054512			6	-24.37856375	
	3	-12.78890891		7	0	-32.76333562	
	4	-12.67804679			1	-32.72441813	

Table 2. The energy eigenvalues for the multiparameter-type potential for different quantum states at different dimensions.

n	ℓ	$E_{n\ell}\ (D=4)$	$E_{n\ell}\ (D=5)$	$E_{n\ell}\ (D=6)$
0	0	-1.473671949	-1.460749546	-1.442669653
1	0	-2.340906561	-2.330023408	-2.314772907
	1	-2.288589515	-2.260164739	-2.227285168
2	0	-4.799084327	-4.786890497	-4.769779611
	1	-4.740336666	-4.708277938	-4.671068219
	2	-4.641124001	-4.588323244	-4.529872980
3	0	-8.355217141	-8.340910484	-8.320818836
	1	-8.286202991	-8.248448093	-8.204542701
	2	-8.169143656	-8.106577693	-8.037095230
	3	-8.000420690	-7.910501812	-7.812181019

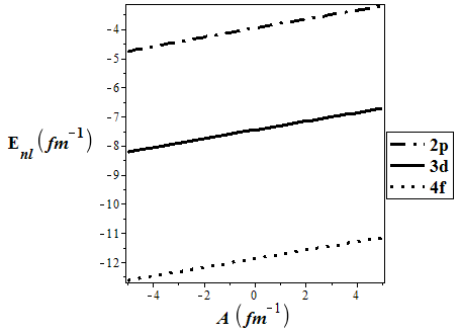

Figure 1. Energy eigenvalues versus A for different quantum states, with $B=2\ fm^{-1};\ C=1\ fm^{-1};\ G=1\ fm^{-1};\ p=1.5;\ q=3.5;\ b=2\ fm^{-1};\ d=3;\ L=7\ fm;\ \hbar=\mu=1.$

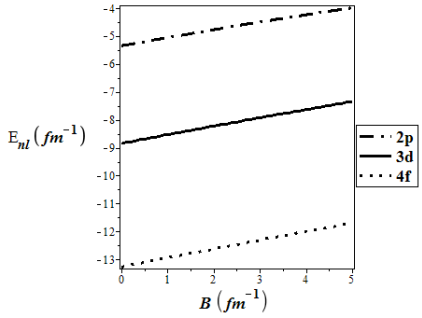

Figure 2. Energy eigenvalues vs B for different quantum states, with $A=-5\ fm^{-1};\ C=1\ fm^{-1};\ G=1\ fm^{-1};\ p=1.5;\ q=3.5;\ b=2\ fm^{-1};\ d=3;\ L=7\ fm;\ \hbar=\mu=1.$

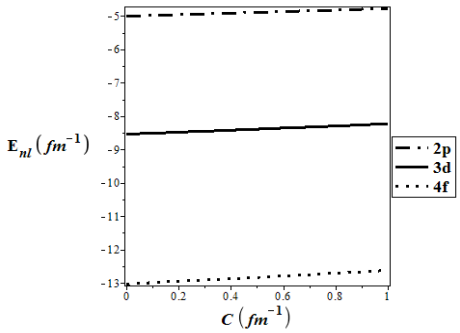

Figure 3. Energy eigenvalues vs C for different quantum states, with $A=-5\ fm^{-1};\ B=2\ fm^{-1};\ G=1\ fm^{-1};\ p=1.5;\ q=3.5;\ b=2\ fm^{-1};\ d=3;\ L=7\ fm;\ \hbar=\mu=1.$

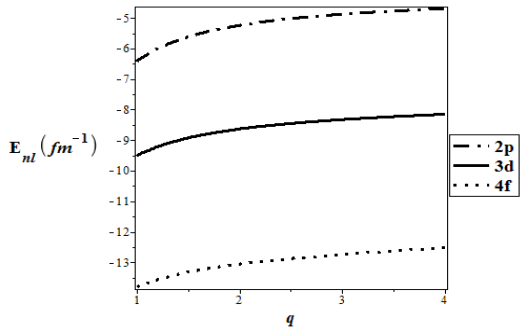

Figure 4. Energy eigenvalues vs q for different quantum states, with $A=-5\ fm^{-1};\ B=2\ fm^{-1};\ C=1\ fm^{-1};\ G=1\ fm^{-1};\ p=1.5;\ b=2\ fm^{-1};\ d=3;\ L=7\ fm;\ \hbar=\mu=1.$

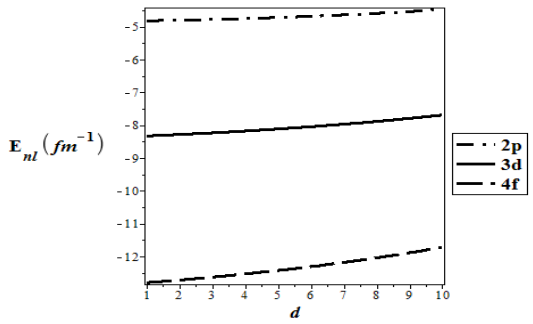

Figure 5. Energy eigenvalues vs d for different quantum states, with $A=-5\ fm^{-1};\ B=2\ fm^{-1};\ C=1\ fm^{-1};\ G=1\ fm^{-1};\ p=1.5;\ q=3.5;\ b=2\ fm^{-1};\ L=7\ fm;\ \hbar=\mu=1.$

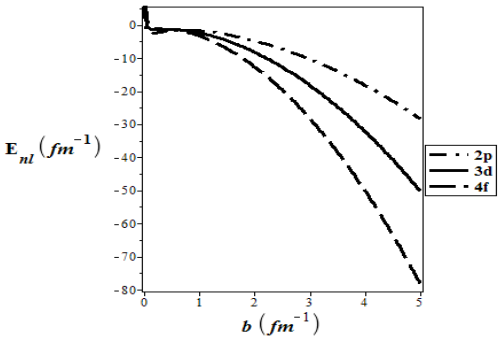

Figure 6. Energy eigenvalues vs b for different quantum states, with $A=-5\ fm^{-1};\ B=2\ fm^{-1};\ C=1\ fm^{-1};\ G=1\ fm^{-1};\ p=1.5;\ q=3.5;d=3;L=7\ fm;\ \hbar=\mu=1.$

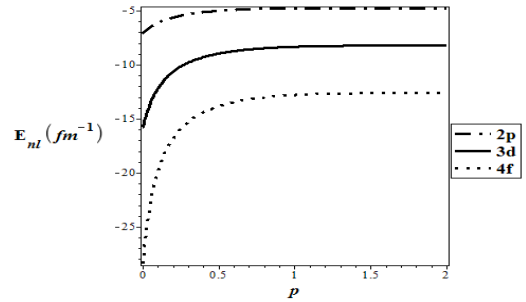

Figure 7. Energy eigenvalues vs p for different quantum states, with $A=-5\ fm^{-1};\ B=2\ fm^{-1};\ C=1\ fm^{-1};\ G=1\ fm^{-1};\ q=3.5;\ b=2\ fm^{-1};d=3;L=7\ fm;\ \hbar=\mu=1.$

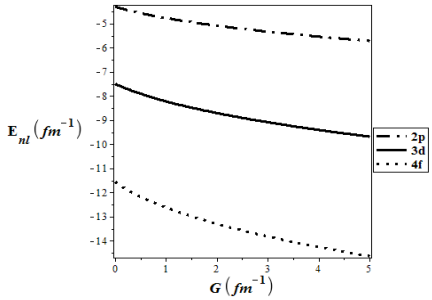

Figure 8. Energy eigenvalues vs G for different quantum states, with $A=-5\ fm^{-1};\ B=2\ fm^{-1};\ C=1\ fm^{-1};\ p=1.5;\ q=3.5;\ b=2\ fm^{-1};d=3;L=7\ fm;\ \hbar=\mu=1.$

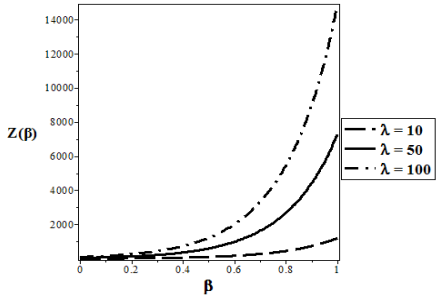

Figure 9. Vibrational partition function versus β for various values of λ.

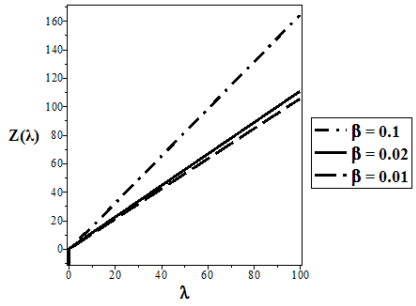

Figure 10. Vibrational partition function versus λ for various values of β.

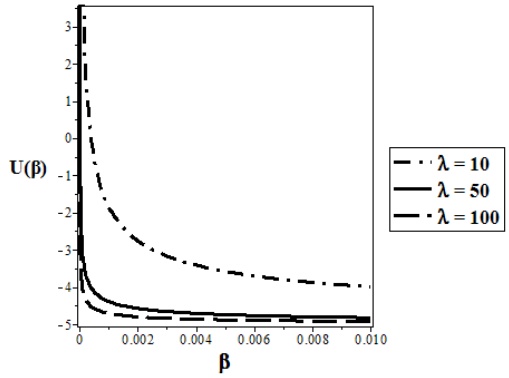

Figure 11. Vibrational mean energy versus β for various values of λ.

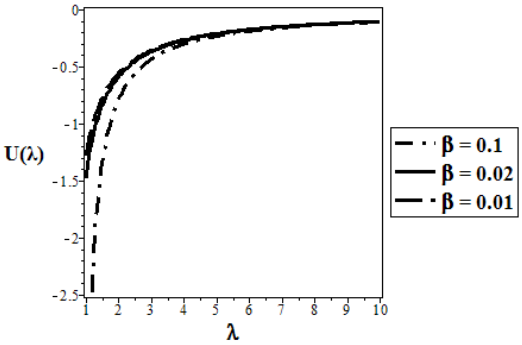

Figure 12. Vibrational mean energy versus λ for various values of β.

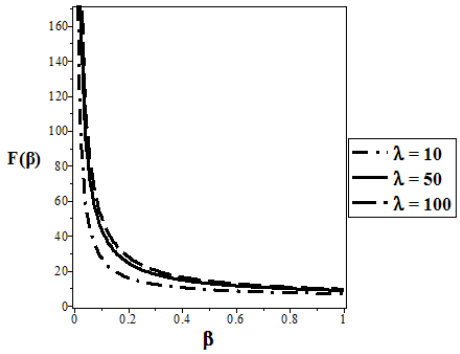

Figure 13. Vibrational free energy versus β for various values of λ .

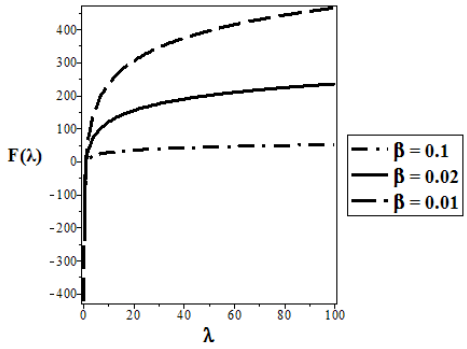

Figure 14. Vibrational free energy versus λ for various values of β .

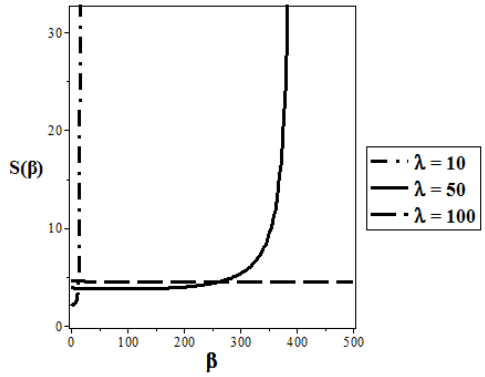

Figure 15. Vibrational entropy versus β for various values of λ .

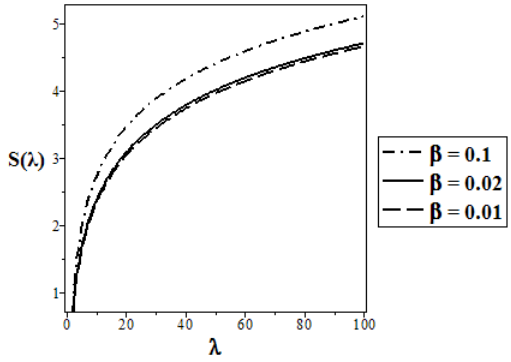

Figure 16. Vibrational entropy versus λ for various values of β .

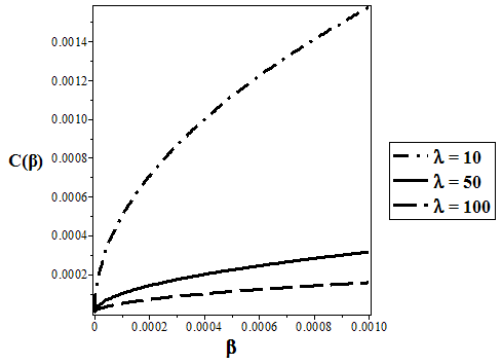

Figure 17. Vibrational specific heat capacity versus β for various values of λ .

4.0 Conclusion

In the present study, the Schrödinger equation was solved in d-dimensions, using the NU method. The eigenvalue of the energy equation of theMPPand the normalized wave function were deduced. By applying the Pekeris approximation scheme for the centrifugal potential, we obtained the analytical solutions for the d-dimensional Schrödinger equation with the MPP. The numerical solutions of the d-dimensional Schrödinger equation with the MPP were also obtained. Our analysis show that our results agree with the results obtained in literatures for the 3-dimensional case. The effects of the different potential parameters on the energy eigenvalues were demonstrated graphically. Furthermore, we calculated the vibrational partition function, using the energy eigenvalues obtained for the MPP. Also, other thermodynamic properties like the vibrational free energy, vibrational mean energy, vibrational entropy, and vibrational specific heat capacity were obtained via the vibrational partition function. The variation of these thermodynamic functions with temperature and upper bound quantum numbers were considered graphically. This novel aspect of our study has been discussed extensively. The study of the thermodynamic properties of the MPP has not been reported before in literature. As such, this aspect of our study could not be compared with any previous work.

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