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# MATHEMATICAL MODEL OF INVENTORY SYSTEM FOR DETERIORATING ITEMS WITH RAMP – TYPEDEMAND CONSIDERING TRADE CREDIT FINANCING AND TIME VALUE OF MONEY

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***Abstract***

***The demands for some items newly introduced into the market tends to grow with time due to some certain promotional activities such as permissible delay in payment, adverts, cash discount and so on and later stabilizes at a certain stage $\mu$during the replenishment cycle. This type of demand has not been reported in the literature of two – warehouse inventory system considering trade credit financing using time value of money. Therefore, in this paper, we gave a mathematical model of this type of inventory system for deteriorating items whose demand is a ramp type under upstream permissible delay in payment using time value of money. With the help of several realistic cases and sub – cases, we developed the model for all the possible scenarios to reflect the realities of the present market.***

***Keywords***: Ramp – Type Demand, Deteriorating, Time Value of Money, Trade Credit Period.

## 1. Introduction

In the classical inventory control theory, it is in most cases assumed that goods are paid for at the placement of the order, which in reality is not always the case as some suppliers adopt some promotional tools one of which is trade credit in order to stimulate the demand of retailers by the supplier. Trade credit is a business transaction in which a certain period is offered as a grace where payments of items ordered are not expected. During the period, the retailer deposits the sales revenue accrued to him/her in an interest bearing account so that an interest is earned over the sales made. At the expiration of the period, the retailer is charged an interest rate over the unsold items.
In the study of permissible delay in payment, [1] was the first, in his maiden paper, he assumed selling and purchasing prices to be the same. In his reply to the author of [1], [2] pointed out it as a mistake to assume the purchasing and selling prices to be the same. The researchers in [3] considered permissible delay in payment on deteriorating items as an extension of [1]. Also, [4] extended the work by allowing shortages to occur. Different purchasing cost and selling price was used in [5] to correct the model in [1].
The demands of items sometimes behave like a constant function, linear trend, stock dependent or sensitive, price dependent, quality dependent, and so on. In [6], the demand is considered to be a ramp type. A ramp type demand is a demand that changes at a certain stage during the replenishment cycle. For literature that considers ramp type demand, see [7– 12] among others.
As a result of permissible delay in payment, the retailer may like to order bulk quantity so as to earn much profit over the sales he/she will make during the allowed period or if the retailer fears about the scarcity of the item in nearer future or that the item is seasonal. As a result, the retailer might have excess after exceeding the maximum stocking capacity of his/her own warehouse (OW) which will necessitate renting another warehouse(s) (RW) with unlimited capacity. For the authors who worked on two – warehouses with condition of permissible delay in payment; see [13 – 16] and many others.
In this paper, we used the approach in [16] and the ramp type demand used in [12] to look at the two – warehouse inventory system so as to come up with an appropriate inventory decision policies.
The structure of this work is, assumptions and notation in chapter 2, model formulation in 3, derivation of the physical quantities in 4 whereas summary and conclusion are in 5.

## 2. Assumptions and Notations

The following are assumptions made in building the model

a. The model considers single deteriorating item stocked in two – warehouse, rented warehouse RW and own warehouse OW.
b. The demand is a ramp type demand i.e. the demand possesses two stages (growth and stable stages).
c. Deterioration rates are constants in both warehouse.
d. The dispatching policy is last – in first – out (LIFO).
e. The model considers trade credit financing.
f. Shortage is not allowed and lead time is assumed to be zero.

The following are the notations used in the model

$I_r(t), I_o(t)$are the inventory levels of the *RW* and *OW* respectively at time *t*.

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$D$ is the demand rate of the item given as $D(t) = \begin{cases} a + bt & t < \mu \\ D', \quad a\ constant & t \geq \mu \end{cases}$
$\mu$is the breakpoint at which the demand changes from growth stage to stable stage.
$W$is the maximum quantity of the item that can be stored in $OW$.
$Q$ is the rest of the quantity of item stored in $RW$ after exceeding the stocking capacity in $OW$.
$t_w$is the time at which inventory at $RW$ drops to zero.
$T$ is the replenishment cycle for the model.
$\alpha, \beta$are the constant deterioration rates at OW and RW respectively with $\alpha > \beta$.
$h_r, h_o$are holding costs per unit per unit time excluding interest charges of $RW$ and $OW$ respectively with $h_r \geq h_o$.
$A$ is the ordering cost per unit per unit time.
$I_P$ , $I_e$ The interest payable and interest earned return rates respectively.
$c$, $p$ are the purchasing cost and selling price per unit of the item respectively.
$M$ is the trade credit period offered to the retailer by the supplier (Upstream, i.e. the retailer is the only beneficiary while the retailer's customers are not).
TRC is the total relevant costs of the model to be minimized.
All other notations not defined here will be defined appropriately.

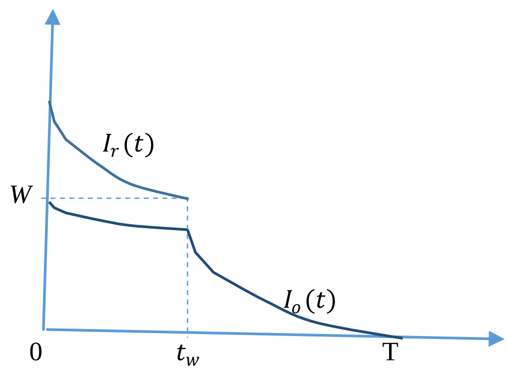

Fig 1: Graphical representation of the model.

### 3. Mathematical Model Formulation

Assuming the demand as $D(t) = \begin{cases} a + bt & t < \mu \\ D' & t \geq \mu \end{cases}$
where$D(t) = a + bt$ is an increasing function with time where$a > 0, b > 0$ and $D'$ is the constant demand after the demand reaches the stability stage $\mu$.
Then, we explore the possible relations between $t_w$ and $\mu$ which will give us two models
(i) Model I $t_w < \mu$ and (ii) Model II $t_w \geq \mu$
For these cases, there are different holding and deterioration costs.
I. Model I$t_w < \mu$ (when the breakpoint time is longer than the time at which the inventory level at RW drops to zero).
The differential equations governing model I are given by

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -(a + bt)\ 0 \leq t \leq t_w \tag{1}$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = 0 0 \leq t \leq t_w \tag{2}$$

with the condition $I_r(t_w) = 0$ and $I_o(0) = W$ for (1) and (2) respectively.

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -(a + bt) t_w < t < \mu \tag{3}$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -D' \mu \leq t < T \tag{4}$$

with the boundary conditions $I_o(T) = 0$

II. Model II $t_w \geq \mu$ (when the break point time is shorter than the period at which the inventory at $RW$ drops to zero)
The differential equations governing the phenomena in model II are given by

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -(a + bt) 0 < t < \mu \tag{5}$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -D' \mu \leq t < t_w \tag{6}$$

with the boundary condition $I_r(t_w) = 0$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = 0\ 0 < t < \mu \tag{7}$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = 0 \mu \leq t < t_w \tag{8}$$

and the initial condition $I_o(0) = W$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -D' t_w < t \leq T \tag{9}$$

With the boundary condition $I_o(T) = 0$
The solutions of equations (1) - (4) are respectively given as

$$I_r(t) = \frac{1}{\beta^2}\left[(\beta(a+bt_w)-b)e^{\beta(t_w-t)} - (\beta(a+bt)-b)\right] 0 \le t \le t_w \tag{10}$$

$$I_o(t) = We^{-\alpha t}\, 0 \le t \le t_w \tag{11}$$

$$I_o(t) = \frac{1}{\alpha^2}\left[(\alpha(a+b\mu)-b)e^{\alpha(\mu-t)} - (\alpha(a+bt)-b)\right] + \frac{D'}{\alpha}\left(e^{\alpha(T-\mu)}-1\right) t_w \le t \le \mu \tag{12}$$

$$I_o(t) = \frac{D'}{\alpha}\left(e^{\alpha(T-t)}-1\right)\mu \le t \le T \tag{13}$$

From continuity of OW at $t = t_w$, by using (11)and (12) we get

$$We^{-\alpha t_w} = \frac{D'}{\alpha}\left(e^{\alpha(T-\mu)}-1\right) + \frac{1}{\alpha^2}\left[(\alpha(a+b\mu)-b)e^{\alpha(\mu-t_w)} - (\alpha(a+bt)-b)\right]$$

Since the value of $\alpha \in (0,1)$, $t_w < 100\%$ and $T - t_w < 100\%$ then $\alpha t_w \ll 1$ and $\alpha(T-t_w) \ll 1$ as in [13], the above immediate equation can be reduced to the following by using series expansion,

$$W\left(1-\alpha t_w+\alpha^2\frac{t_w^2}{2!}\right) = \frac{D'}{\alpha}\left(\left(1+\alpha(T-t_w)+\alpha^2\frac{(T-t_w)^2}{2!}\right)-1\right) + \frac{1}{\alpha^2}\left[\left((a\alpha+b\alpha\mu-b)\left(1+\alpha(\mu-t_w)+\alpha^2\frac{(\mu-t_w)^2}{2!}\right)\right)-a\alpha-b\alpha t_w+b\right]$$

Simplifying and collecting like terms, we get

$$\left(\frac{1}{2}\alpha D'\right)T^2 + (D'-\alpha D'\mu)T - (a+b-(a+b)t_w-D')\mu + \frac{1}{2}(\alpha(a+b)-b)(\mu^2-2\mu t_w+t_w^2) + \frac{1}{2}\alpha D'\mu^2 - W\left(1-\alpha t_w+\frac{1}{2}\alpha^2 t_w^2\right) = 0$$

Thus we get

$$T = \left(-(D'-\alpha D'\mu) \pm \sqrt{\{(D'-\alpha D'\mu)^2-4ac\}}\right)\Big/_{\alpha D'} \tag{14}$$

where$a = \frac{1}{2}\alpha D'$ and $c = \frac{1}{2}(\alpha(a+b)-b)(\mu^2-2\mu t_w+t_w^2)+\frac{1}{2}\alpha D'\mu^2-(a+b-(a+b)t_w-D')\mu - W\left(1-\alpha t_w+\frac{1}{2}\alpha^2 t_w^2\right)$

The solutions of (5) - (9) are respectively given as

$$I_r(t) = \frac{D'}{\beta}\left(e^{\beta(\mu-t)}-1\right) + \frac{1}{\beta^2}\left[(\beta(a+b\mu)-b)e^{\beta(\mu-t)} - (\beta(a+bt)-b)\right],\ t \le \mu \tag{15}$$

$$I_r(t) = \frac{D'}{\beta}\left(e^{\beta(t_w-t)}-1\right)\mu \le t \le t_w \tag{16}$$

$$I_o(t) = W\left(e^{\alpha(\mu-t)}-e^{-\alpha\mu}\right) 0 \le t \le \mu \tag{17}$$

$$I_o(t) = We^{-\alpha t}\, \mu \le t \le t_w \tag{18}$$

$$I_o(t) = \frac{D'}{\alpha}\left(e^{\alpha(T-t)}-1\right) t_w \le t \le T \tag{19}$$

From continuity of OW at $t = t_w$, using (17) and (19) we get

$$W\left(e^{\alpha(\mu-t_w)}-e^{-\alpha\mu}\right) = \frac{D'}{\alpha}\left(e^{\alpha(T-t_w)}-1\right)$$

After simplification and collecting like terms, we get

$$T = t_w + \frac{1}{\alpha}\ln\left(1+\frac{\alpha W}{D'}\left(e^{\alpha(\mu-t_w)}-e^{-\alpha\mu}\right)\right) \tag{20}$$

## 4. Derivation of the Physical Quantities

For model I, the holding and deteriorating costs are respectively given as

$$HC1 = \alpha^3 h_r\left(b-\beta(a+bt_w)\right)\left(1-e^{\beta t_w}\right) + \left(b-\beta(a+bt_w)\right)(\alpha^3\beta h_r - \alpha\beta^3 h_o)t_w + \alpha^2\beta^3 W h_o(1-e^{-\alpha t_w}) + \beta^3 h_o\left(b-\alpha(a+b\mu)\right)\left(1-e^{\alpha(\mu-t_w)}\right) + \alpha\beta^3 h_o\left(b-\alpha\left(a+\frac{1}{2}b\mu\right)\right)\mu + \alpha^2\beta^3 D' h_o\left(e^{\alpha(T-\mu)}-1\right)(\mu-t_w) + \alpha\beta^3 D' h_o\left(e^{\alpha(T-t)}-\alpha(T-t_w)-1\right) \tag{21}$$

and

$DC1 = c\alpha^2 h_r\left(b-\beta(a+bt_w)\right)\left(1-e^{\beta t_w}\right) + c\left(b-\beta(a+bt_w)\right)(\alpha^2\beta h_r - \alpha\beta^2 h_o)t_w + c\alpha^2\beta^2 W h_o(1-e^{-\alpha t_w}) + c\beta^2 h_o\left(b-\alpha(a+b\mu)\right)(1-$
*eαμ−tw+cαβ2hob−αa+12bμμ+cα2β2D′hoeαT−μ−1μ−tw+cαβ2D′hoeαT−t−αT−tw−1* (22)

For model II, the holding and deterioration costs are respectively given as

$HC2 = \alpha^3\beta D' h_r\left(e^{\beta\mu}-\beta\mu-1\right) + \alpha^3 h_r\left(\left(b-\beta(a+b\mu)\right)\left(1-e^{\beta\mu}\right) + \beta\left(b-\beta\left(a+\frac{1}{2}b\mu\right)\right)\mu\right) + \alpha^2\beta^3 W h_o(e^{\alpha\mu} - (\alpha\mu +$
*1e−αμ−e−αtw−1+αβ3hoD′eαT−tw−αT−tw−1* (23)

and

$DC2 = c\alpha^3\beta^2 D' h_r\left(e^{\beta\mu} - \beta\mu - 1\right) + c\alpha^3\beta h_r\left(\left(b - \beta(a + b\mu)\right)\left(1 - e^{\beta\mu}\right) + \beta\left(b - \beta\left(a + \frac{1}{2}b\mu\right)\right)\mu\right) + c\alpha^3\beta^3 W h_o(e^{\alpha\mu} -$

*αμ+1e−αμ−e−αtw−1+cα2β3hoD′eαT−tw−αT−tw−1* (24)

The ordering cost OC is $A$ (25)

To get the total profit, we calculate the amount with the retailer together with the interest earned after paying the interest incurred due to unsold items or exceeding the trade credit period without settling his/her account with the supplier and also deducting the holding, deterioration and ordering costs. This is done as follows:

I. For $M \leq \mu$

In this situation, the offered credit period $M$ to the retailer is shorter than the break point $\mu$, therefore we explore the relation between $M, t_w, \mu$ and $T$. There are two cases to consider in this situation $t_w \leq M \leq \mu < T$ and $M \leq \mu \leq t_w < T$.

Case 1: $t_w \leq M \leq \mu < T$

The total quantity of the items ordered by the retailer is $W + Q$

Therefore the total cost of the order or amount payable by the retailer is $c(W + Q)$ (26)

Let $G_1$ be the total amount of revenue generated by the retailer due to sales and interest on the sales revenue during the allowed credit period$[0, M]$. It is therefore given by

$$G_1 = p\int_0^M (a + bt)dt + pI_e\int_0^M t(a + bt)dt = pM\left(a\left(1 + \frac{1}{2}I_e M\right) + bM\left(\frac{1}{2} + \frac{1}{3}I_e M\right)\right) \quad (27)$$

In this situation, two sub – cases occur,

Sub – case 1.1: $G_1 \geq c(W + Q)$ Amount generated by the retailer due to sales and interest earned during the allowed trade credit period is greater than or equals to the total amount payable to the retailer, and

Sub – case 1. 2: $G_1 < c(W + Q)$ Amount generated by the retailer due to sales and interest earned during the allowed trade credit period is less than the amount payable to the retailer.

Sub – case 1.1: $G_1 \geq c(W + Q)$

Now, at $t = M$, the retailer settle his/her account with the supplier. The excess amount, using equations (26) and (27) is given by

$$G_1 - c(W + Q) = pM\left(a\left(1 + \frac{1}{2}I_e M\right) + bM\left(\frac{1}{2} + \frac{1}{3}I_e M\right)\right) - c(W + Q) \quad (28)$$

After $t = M$, during the period$[M, T]$, the retailer will generate revenue due to sales and earn interests on the excess and the sales revenue generated.

The revenue generated due to sales and the interest earned on the revenue generated during the period $[M, T]$ is

$p\left(\int_M^\mu (a + bt)dt + \int_\mu^T D' dt\right) + pI_e\left(\int_M^\mu (a + bt)tdt + \int_\mu^T D' tdt\right) = p\left(a\left(1 + \frac{1}{2}I_e(\mu + M)\right)(\mu - M) + b\left(\frac{1}{2}(\mu^2 - M^2) + \frac{1}{3}I_e(\mu^3 -\right.\right.$

*M3+D′1+12IeT+μT−μ* (29)

The excess amount after paying the supplier at $t = M$ and the interest earned on the excess during the period $[M, T]$, using (29), is

$G_1 - c(W + Q) + I_e\int_M^T\left(G_1 - c(W + Q)\right)dt = \left(G_1 - c(W + Q)\right)\left(1 + I_e(T - M)\right) = \left(pM\left(a\left(1 + \frac{1}{2}I_e M\right) + bM\left(\frac{1}{2} + \frac{1}{3}I_e M\right)\right) -\right.$

*cW+Q1+IeT−M* (30)

Therefore, the total profit for the retailer $TP_1$ is given by

The excess amount after paying the supplier at $t = M$ + the interest earned on the excess during $[M, T]$ + the revenue generated due to sales during $[M, T]$ + the interest earned on the revenue generated due to sales during the period $[M, T]$ – ordering cost – holding cost – deterioration cost

Using equations (21), (22), (25), (29) and (30), we get

$TP_1 = \left(pM\left(a\left(1 + \frac{1}{2}I_e M\right) + bM\left(\frac{1}{2} + \frac{1}{3}I_e M\right)\right) - c(W + Q)\right)\left(1 + I_e(T - M)\right) + p\left(a\left(1 + \frac{1}{2}I_e(\mu + M)\right)(\mu - M) + b\left(\frac{1}{2}(\mu^2 -\right.\right.$

*M2+13Ieμ3−M3+D′1+12IeT+μT−μ−OC−HC1−DC1* (31)

Sub – case 1.2: $G_1 < c(W + Q)$

In this sub – case, we look at whether the supplier will accept partial payment at time $t = M$ or full payment at a time later than $M$.

Sub – case 1.2.1: when the supplier accept partial payment at $t = M$ from the retailer and the remaining balance to be paid at $t = M_{11}$ where $\mu > M_{11} > M$

After depositing the part payment in the retailer's account with the supplier at time $t = M$, the rest amount to be paid at time $t = M_{11}$ is

$$c(W + Q) - G_1 = c(W + Q) - pM\left(a\left(1 + \frac{1}{2}I_e M\right) + bM\left(\frac{1}{2} + \frac{1}{3}I_e M\right)\right) \quad (32)$$

For this the retailer must pay interest on the remaining balance for the period $[M, M_{11}]$

The total balance to be paid by the retailer at $t = M_{11}$ using (32) will be

$$\left(c(W + Q) - G_1\right) + \left(c(W + Q) - G_1\right)I_p\int_M^{M_{11}} dt = c(W + Q) - pM\left(a\left(1 + \frac{1}{2}I_e M\right) + bM\left(\frac{1}{2} + \frac{1}{3}I_e M\right)\right)\left(1 + I_p(M_{11} - M)\right) \quad (33)$$

The sales and interest generated on the sales revenue by the retailer during $[M\ ,M_{11}]$ is $p\int_{M}^{M_{11}}(a+bt)dt + pI_e\int_{M}^{M_{11}}(a+bt)tdt = p\left(a(M_{11}-M)+\frac{1}{2}b(M_{11}^2-M^2)+I_e\left(\frac{1}{2}a(M_{11}^2-M^2)+\frac{1}{3}b(M_{11}^3-M^3)\right)\right)$ (34)

At time $t = M_{11}$, the amount payable to the supplier should be equals to amount available with the retailer. Using the equations (33) and (34), we get

$$c(W+Q) - pM\left(a\left(1+\frac{1}{2}I_eM\right)+bM\left(\frac{1}{2}+\frac{1}{3}I_eM\right)\right)\left(1+I_p(M_{11}-M)\right)$$
$$= p\left(a(M_{11}-M)+\frac{1}{2}b(M_{11}^2-M^2)+I_e\left(\frac{1}{2}a(M_{11}^2-M^2)+\frac{1}{3}b(M_{11}^3-M^3)\right)\right)$$

After some small algebraic simplification we have

$$\frac{1}{3}I_ebpM_{11}^3+\frac{1}{2}p(aI_e+b)M_{11}^2+\left(ap-I_p(c(W+Q)-G_1)M_{11}-(c(W+Q)-G_1)(1-I_pM\right)+G_1=0 \quad (35)$$

**Lemma 1**: Solution to (35) will determine the value of $M_{11}$

Now, the revenue generated on the sales and the interest on the sales revenue for the period $[M_{11},T]$ is given by

$p\left(\int_{M_{11}}^{\mu}(a+bt)dt+\int_{\mu}^{T}D'dt\right)+pI_e\left(\int_{M_{11}}^{\mu}(a+bt)tdt+\int_{\mu}^{T}D'tdt\right)=p\left(a\left(1+\frac{1}{2}I_e(\mu+M_{11})\right)(\mu-M_{11})+b\left(\frac{1}{2}(\mu^2-M_{11}^2)+\right.\right.$

*13Ieμ3−M113+D′1+12IeT+μT−μ* (36)

Therefore, the total profit = revenue due to sales during $[M_{11},T]$ + interest earned over the sales revenue during $[M_{11},T]$ – *OC – HC1– DC1*

Using equations (21), (22), (25) and (36), we get

$$TP_2 = p\left(a\left(1+\frac{1}{2}I_e(\mu+M_{11})\right)(\mu-M_{11})+b\left(\frac{1}{2}(\mu^2-M_{11}^2)+\frac{1}{3}I_e(\mu^3-M_{11}^3)\right)+D'\left(1+\frac{1}{2}I_e(T+\mu)\right)(T-\mu)\right)-OC-HC1-DC1 \quad (37)$$

Sub – case 1.2.2: When the supplier rejected partial payment, the full payment is to be made after the time $t = M$.

Let the possible time for the payment be $M_{12}$ where $\mu > M_{12} > M$

The amount that will be paid to the supplier is his/her money + interest on the money during the period $[M,M_{12}]$ given by

$$c(W+Q)+c(W+Q)I_p\int_{M}^{M_{12}}dt = c(W+Q)\left(1+I_p(M_{12}-M)\right) \quad (38)$$

The amount earned by the retailer due to sales and interest on the sales revenue during the period $[0\ ,M_{12}]$ is given by

$$p\int_{0}^{M_{12}}(a+bt)dt+pI_e\int_{0}^{M_{12}}(a+bt)tdt$$
$$= pM_{12}\left(a\left(1+\frac{1}{2}I_eM_{12}\right)+bM_{12}\left(\frac{1}{2}+\frac{1}{3}I_eM_{12}\right)\right) \quad (39)$$

Amount payable to the supplier should be equals to the amount available with the retailer at the time $t = M_{12}$. Using the equations (38) and (39) we get

$$c(W+Q)\left(1+I_p(M_{12}-M)\right)=pM_{12}\left(a\left(1+\frac{1}{2}I_eM_{12}\right)+bM_{12}\left(\frac{1}{2}+\frac{1}{3}I_eM_{12}\right)\right)$$

After some algebraic simplification, we get

$$\frac{1}{3}I_ebpM_{12}^3+\frac{1}{2}p(aI_e+b)M_{12}^2+\left(ap-I_p(c(W+Q)\right)M_{12}-\left(c(W+Q)\right)\left(1-I_pM\right)=0 \quad (40)$$

**Lemma 2:** The solution to equation (40) determine the value of $M_{12}$

The sales revenue and the interest on the sales revenue generated during the period $[M_{12}\ ,T]$ is given by

$p\left(\int_{M_{12}}^{\mu}(a+bt)dt+\int_{\mu}^{T}D'dt\right)+pI_e\left(\int_{M_{12}}^{\mu}(a+bt)tdt+\int_{\mu}^{T}D'tdt\right)=p\left(a\left(1+\frac{1}{2}I_e(\mu+M_{12})\right)(\mu-M_{12})+b\left(\frac{1}{2}(\mu^2-M_{12}^2)+\right.\right.$

*13Ieμ3−M123+D′1+12IeT+μT−μ* (41)

The total profit of the retailer is total earning after paying the supplier – *OC – HC1 – DC1*

Therefore, using (21), (22), (25) and (41), we get the total profit to be

$$TP_3 = p\left(a\left(1+\frac{1}{2}I_e(\mu+M_{12})\right)(\mu-M_{12})+b\left(\frac{1}{2}(\mu^2-M_{12}^2)+\frac{1}{3}I_e(\mu^3-M_{12}^3)\right)+D'\left(1+\frac{1}{2}I_e(T+\mu)\right)(T-\mu)\right)-OC-HC1-DC1 \quad (42)$$

Case II: $M \le \mu \le t_w < T$

Let the total revenue for the retailer at time $t = M$ be $G_2$

$G_2 =$ Revenue due to sales during $[0,M]$ + interest earned on the revenue due to sales during the period $[0,M]$

$$G_2 = p\int_{0}^{M}(a+bt)dt+pI_e\int_{0}^{M}t(a+bt)dt = pM\left(a\left(1+\frac{1}{2}I_eM\right)+bM\left(\frac{1}{2}+\frac{1}{3}I_eM\right)\right) \quad (43)$$

In this case, two sub – case arise
Sub – case 2.1: $G_2 \geq c(W+Q)$ when the amount generated by the retailer exceeds or equals to the amount payable to the supplier during the allowed credit period $t = M$, and
Sub – case 2.2: $G_2 < c(W+Q)$ when the amount generated by the retailer is less than the amount payable to the supplier at during the allowed credit period $t = M$.
Sub – case 2.1: $G_2 \geq c(W+Q)$
After paying the supplier at $t = M$, the excess amount available with the retailer is $G_2 - c(W+Q)$
The retailer will make sales during the period $[M\ ,T]$ and also earned interest on the sales revenue and the excess amount.
The revenue for the retailer from excess during $[M\ ,T]$ is $\big(G_2 - c(W+Q)\big)\big(1 + pI_e(T-M)\big)$
The revenue for the retailer due to sales and interest on the sales during $[M\ ,T]$ is

$$p\left(a\left(1+\frac{1}{2}I_e(\mu+M)\right)(\mu-M)+b\left(\frac{1}{2}(\mu^2-M^2)+\frac{1}{3}I_e(\mu^3-M^3)\right)+D'\left(1+\frac{1}{2}I_e(T+\mu)\right)(T-\mu)\right)$$

Therefore, the total earning of the retailer for the period $[M\ ,T]$ is given by

$$\big(G_2 - c(W+Q)\big)\big(1 + pI_e(T-M)\big)+p\left(a\left(1+\tfrac{1}{2}I_e(\mu+M)\right)(\mu-M)+b\left(\tfrac{1}{2}(\mu^2-M^2)+\tfrac{1}{3}I_e(\mu^3-M^3)\right)+D'\left(1+\tfrac{1}{2}I_e(T+\right.\right.$$
$$\mu T{-}\mu \qquad (44)$$

The total profit of the retailer $TP_4$ is the sum of the excess money after paying the supplier at $t = M$, the interest on the excess for the period$[M\ \ T]$, the sales revenue after paying the supplier with the interest during the period $[M\ \ T]$ minus the sum of OC, HC2 and DC2.
Therefore, using equations (23), (24), (25) and (44), we get

$$TP_4 = \big(G_2 - c(W+Q)\big)\big(1 + pI_e(T-M)\big)+p\left(a\left(1+\tfrac{1}{2}I_e(\mu+M)\right)(\mu-M)+b\left(\tfrac{1}{2}(\mu^2-M^2)+\tfrac{1}{3}I_e(\mu^3-M^3)\right)+D'\left(1+\right.\right.$$
$$12IeT{+}\mu T{-}\mu{-}OC{-}HC2{-}DC2 \qquad (45)$$

Sub – case 2.2: $G_2 < c(W+Q)$
There arise two scenarios
Sub – case 2.2.1: when the supplier accept partial payment at $t = M$ and the remaining balance to be paid at time $t = M_{21}$where $\mu >$ $M21{>}M$
Now, after the retailer deposited the part payment in the suppliers account at time $t = M$, the remaining balance to be paid at $t = M_{21}$ will by using (43) be

$$c(W+Q) - G_2 = c(W+Q) - pM\left(a\left(1+\tfrac{1}{2}I_eM\right)+bM\left(\tfrac{1}{2}+\tfrac{1}{3}I_eM\right)\right) \qquad (46)$$

Using equation (46), the interest payable to the supplier on the remaining balance for the period $[M\ ,M_{21}]$ is

$$\big(c(W+Q)-G_2\big)I_p\int_M^{M_{21}}dt = c(W+Q) - pM\left(a\left(1+\tfrac{1}{2}I_eM\right)+bM\left(\tfrac{1}{2}+\tfrac{1}{3}I_eM\right)\right)I_p(M_{21}-M)(47)$$

At $t = M_{21}$, the total amount payable to the supplier by the retailer using (46 & 47) is

$$c(W+Q) - pM\left(a\left(1+\tfrac{1}{2}I_eM\right)+bM\left(\tfrac{1}{2}+\tfrac{1}{3}I_eM\right)\right)+c(W+Q) - pM\left(a\left(1+\tfrac{1}{2}I_eM\right)+bM\left(\tfrac{1}{2}+\tfrac{1}{3}I_eM\right)\right)I_p(M_{21}-M)\ \ (48)$$

But during the period $[M\ ,M_{21}]$, the retailer make sales and earn interest on the sales revenue generated.
Therefore, the retailer's total amount available/earned for the period $[M\ ,M_{21}]$ is

$$p\left(\int_M^{M_{21}}(a+bt)dt+I_e\int_M^{M_{21}}t(a+bt)dt\right)=p\left(a(M_{21}-M)+\tfrac{1}{2}b(M_{21}^2-M^2)+I_e\left(\tfrac{1}{2}a(M_{21}^2-M^2)+\tfrac{1}{3}b(M_{21}^3-M^3)\right)\right)(49)$$

Since, the retailer has to settle his/her account at time $t = M_{21}$ then, amount available to the retailer should be equals to the remaining amount payable to the supplier. Using (48) and (49) we get

$$\big(c(W+Q)-G_2\big)\left(1+I_p(M_{21}-M)\right)=p\left(a(M_{21}-M)+\tfrac{1}{2}b(M_{21}^2-M^2)+I_e\left(\tfrac{1}{2}a(M_{21}^2-M^2)+\tfrac{1}{3}b(M_{21}^3-M^3)\right)\right) \qquad (50)$$

After algebraic simplification, we have

$$\tfrac{1}{3}I_ebpM_{21}^3+\tfrac{1}{2}p(aI_e+b)M_{21}^2+\big(ap-I_p(c(W+Q)-G_2\big)M_{21}-\big(c(W+Q)-G_2\big)\big(1-I_pM\big)+G_2=0 \quad (51)$$

**Lemma 3**: The solution to equation (51) will determine the value of $M_{21}$
After the retailer settle his/her account with the supplier at time $M_{21}$ the retailer will continue to generate revenue due to sales of the unsold items and earned interest on the sales revenue generated for the period$[M_{21}\ T]$.
Therefore, the retailer's total earning for this period is

$$p\left(\int_{M_{21}}^{\mu}(a+bt)dt+\int_{\mu}^{T}D'dt\right)+pI_e\left(\int_{M_{21}}^{\mu}(a+bt)tdt+\int_{\mu}^{T}D'tdt\right)=p\left(a\left(1+\tfrac{1}{2}I_e(\mu+M_{21})\right)(\mu-M_{21})+b\left(\tfrac{1}{2}(\mu^2-M_{21}^2)+\right.\right.$$
$$13Ie\mu3{-}M213{+}D'1{+}12IeT{+}\mu T{-}\mu \qquad (52)$$

The total profit of the retailer using (23), (24), (25) and (52) is

$TP_5 = p\left(a\left(1+\frac{1}{2}I_e(\mu+M_{21})\right)(\mu-M_{21}) + b\left(\frac{1}{2}(\mu^2-M_{21}^2)+\frac{1}{3}I_e(\mu^3-M_{21}^3)\right) + D'\left(1+\frac{1}{2}I_e(T+\mu)\right)(T-\mu)\right) - OC - HC2 - DC2$ (53)

Sub – case 2.2.2: when the supplier reject partial payment, but full payment at $t = M_{22}$ where $\mu > M_{22} > M$

Since the retailer only accept full payment at time $t = M_{22}$ then the retailer will pay the supplier his/her money and the interest on the money for the period $[M\ M_{22}]$

This shows the total amount payable to the supplier by the retailer is

$c(W+Q) + c(W+Q)I_p\int_M^{M_{22}} dt = c(W+Q)\left(1+I_p(M_{22}-M)\right)$ (54)

The revenue earned or generated by the retailer due to sales and interest on the sales revenue during the period $[0\ M_{22}]$ is

$p\int_0^{M_{22}}(a+bt)dt + pI_e\int_0^{M_{22}}(a+bt)tdt = pM_{22}\left(a\left(1+\frac{1}{2}I_eM_{22}\right) + bM_{22}\left(\frac{1}{2}+\frac{1}{3}I_eM_{22}\right)\right)$ (55)

Therefore, the amount payable to the supplier = amount available to the retailer during the period$[0\ M_{22}]$. By using (54) and (55) we get

$$c(W+Q)\left(1+I_p(M_{22}-M)\right) = pM_{22}\left(a\left(1+\frac{1}{2}I_eM_{22}\right) + bM_{22}\left(\frac{1}{2}+\frac{1}{3}I_eM_{22}\right)\right)$$

After some algebraic simplification, we found

$\frac{1}{3}I_ebpM_{22}^3 + \frac{1}{2}p(aI_e+b)M_{22}^2 + \left(ap - I_p(c(W+Q)\right)M_{22} - \left(c(W+Q)\right)\left(1-I_pM\right) = 0$ (56)

**Lemma 4:** Solution to equation (56) will determine the value of $M_{22}$

The sales revenue and the interest on the sales revenue generated during the period $[M_{22}\ , T]$ is given by

$p\left(\int_{M_{22}}^{\mu}(a+bt)dt + \int_{\mu}^{T}D'dt\right) + pI_e\left(\int_{M_{22}}^{\mu}(a+bt)tdt + \int_{\mu}^{T}D'tdt\right) = p\left(a\left(1+\frac{1}{2}I_e(\mu+M_{22})\right)(\mu-M_{22}) + b\left(\frac{1}{2}(\mu^2-M_{22}^2) + \right.\right.$

*13Ieμ3−M223+D′1+12IeT+μT−μ* (57)

Using (23), (24), (25) and (57), we find that the total profit is given by

$TP_6 = p\left(a\left(1+\frac{1}{2}I_e(\mu+M_{22})\right)(\mu-M_{22}) + b\left(\frac{1}{2}(\mu^2-M_{22}^2)+\frac{1}{3}I_e(\mu^3-M_{22}^3)\right) + D'\left(1+\frac{1}{2}I_e(T+\mu)\right)(T-\mu)\right) - OC - HC2 -$
$DC2$ (58)

II. For $M > \mu$

In this case, the offered credit period is longer than the break point when the demand will change from the first stage of increasing to the next stage of stability/constant. Therefore we explore the possible relation between the parameters $M, t_w, \mu$ and $T$.

From the relation, we shall consider four cases as shown in figures 3a – 3d above, $t_w \le \mu < M < T,\ \mu < t_w < M < T\ ,\ t_w \le \mu < T < M$ and $\mu < t_w < T < M$.

Sub – case 3: $t_w \le \mu < M < T$ when the demand stabilized after inventory at RW have dropped to zero.

At time $t = M$, the amount payable to the supplier by retailer is $c(W+Q)$

The generated amount of revenue by the retailer due to sales during the period $[0\ , M]$ is

$p\int_0^M D(t)dt = p\left(\int_0^{\mu}(a+bt)dt + \int_{\mu}^{M}D'dt\right) = p\left(a\mu + \frac{1}{2}b\mu^2 + D'(M-\mu)\right)$ (59)

The interest generated/earned during the period $[0\ , M]$ on the sales revenue by the retailer is $pI_e\left(\int_0^{\mu}(a+bt)tdt + \int_{\mu}^{M}D'tdt\right) =$
$pI_e\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \frac{1}{2}D'(M^2-\mu^2)\right)$ (60)

Therefore using (59 & 60) and letting $G_3$ to be the total amount generated by the retailer during the period $[0\ , M]$, we have

$G_3 = p\left(\left(\int_0^{\mu}(a+bt)dt + \int_{\mu}^{M}D'dt\right) + I_e\left(\int_0^{\mu}t(a+bt)dt + \int_{\mu}^{M}tD'dt\right)\right) = p\left(a\mu + \frac{1}{2}b\mu^2 + D'(M-\mu) + I_e\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \right.\right.$

*12D′M2−μ2* (61)

In this case, there are two cases to consider

Sub – case 3.1: $G_3 \ge c(W+Q)$ when the amount of money at the retailer's disposal is larger than the amount needed to settle his/her account with the supplier and

Sub – case 3.2: $G_3 < c(W+Q)$ when the amount of money at the retailer's disposal is smaller than the amount needed to settle his/her account with the supplier

Sub – case 3.1: $G_3 > c(W+Q)$

In this situation, the retailer has excess after paying the supplier.

The excess amount is $G_3 - c(W+Q)$ (62)

After paying the supplier at time $t = M$, the retailer will earned interest over the excess for the period $[M\ , T]$ and is given by

$G_3 - c(W+Q)I_e\int_M^T dt = \left(G_3 - c(W+Q)\right)I_e(T-M)$ (63)

The amount the retailer earned due to excess using (62 & 63) is

$G_3 - c(W+Q) + \left(G_3 - c(W+Q)\right)I_e(T-M) = \left(G_3 - c(W+Q)\right)\left(1+I_e(T-M)\right)$ (64)

After paying the supplier at $t = M$, the retailer will continue to make sales and earned interest over the sales for the period $[M\ , T]$ and is given as

$p\int_M^T D'dt + pI_e\int_M^T tD'dt = pD'(T-M)\left(1+\frac{1}{2}I_e(T+M)\right)$ (65)

The total earning of the retailer using (64) and (65) will be

$$\left(G_3 - c(W+Q)\right)\left(1 + I_e(T-M)\right) + pD'(T-M)\left(1 + \frac{1}{2}I_e(T+M)\right) \tag{66}$$

The total profit $TP_7$ of the retailer using (21), (22), (25) and (66) is given by

$$TP_7 = \left(G_3 - c(W+Q)\right)\left(1 + I_e(T-M)\right) + pD'(T-M)\left(1 + \frac{1}{2}I_e(T+M)\right) - A - HC1 - DC1 \tag{67}$$

Sub – case 3.2: $G_3 < c(W+Q)$

In this sub – case, there arise two scenarios

Sub – case 3.2.1: when the supplier is willing to accept part payment from the retailer at time $t = M$ and the balance (including interest charge) at $t = M_{31}$ where $M_{31} > M > \mu$

After depositing the part payment $G_3$ with the supplier at time $t = M$, the remaining balance to be paid to the supplier at $t = M_{31}$ and the interest on the balance for the period $[M\ M_{31}]$ is

$$c(W+Q) - G_3 + (c(W+Q) - G_3)I_p\int_M^{M_{31}} dt = c(W+Q) - G_3\left(1 + I_p(M_{31} - M)\right) \tag{68}$$

The amount generated due to sales and the interest by the retailer during the period $[M\ , M_{31}]$ is

$$p\int_M^{M_{31}} D'dt + pI_e\int_M^{M_{31}} tD'dt = pD'\left(1 + \frac{1}{2}I_e(M_{31} + M)\right)(M_{31} - M) \tag{69}$$

Therefore, at time $t = M_{31}$, the amount payable to the supplier = amount available to the retailer. Using (68) and (69) we have

$$(c(W+Q) - G_3)\left(1 + I_p(M_{31} - M)\right) = pD'\left(1 + \frac{1}{2}I_e(M_{31} + M)\right)(M_{31} - M)$$

After some algebraic simplification, we get

$$\frac{1}{2}pI_eD'M_{31}^2 + \left(pD' - I_p(c(W+Q) - G_3)\right)M_{31} - pD'M\left(1 + \frac{1}{2}I_eM\right) - (c(W+Q) - G_3)\left(1 + I_pM\right) \tag{70}$$

**Lemma 5**: The solution to equation (70) determines the value of $M_{31}$

The total amount for the retailer after paying the supplier's balance at the time $t = M_{31}$ will be the selling revenue during $[M_{31}, T]$ + the interest earned on the sales revenue for the period $[M_{31}\ \ T]$ is given by

$$p\int_{M_{31}}^T D'dt + pI_e\int_{M_{31}}^T tD'dt = pD'\left(1 + I_e(M_{31} + T)\right)(T - M_{31}) \tag{71}$$

The total annual profit $TP_8$ by the retailer using (21), (22), (25) and (71) is given by

$$TP_8 = pD'\left(1 + I_e(M_{31} + T)\right)(T - M_{31}) - OC - HC1 - DC1 \tag{72}$$

Sub – case 3.2.2: when the supplier reject partial payment but full payment (his/her owed amount + the interest charged) at time $t = M_{32}$ where $M_{32} > M > \mu$

Let the time agreed between the retailer and supplier be $t = M_{32}$, at this time the retailer will pay $c(W+Q)$ and interest of $I_pc(W+Q)\int_M^{M_{32}} dt$

The total amount payable by the retailer to the supplier is

$$c(W+Q) + I_pc(W+Q)\int_M^{M_{32}} dt = c(W+Q)\left(1 + I_p(M_{32} - M)\right) \tag{73}$$

The amount generated due to sales and interest earned on the sales revenue by the retailer up to the time $t = M_{32}$ is

$$p\left(\int_0^\mu (a+bt)dt + \int_\mu^{M_{32}} D'dt\right) + pI_e\left(\int_0^\mu (a+bt)tdt + \int_\mu^{M_{32}} D'tdt\right) = p\left(a\mu\left(1 + \frac{1}{2}I_e\mu\right) + b\mu^2\left(\frac{1}{2} + \frac{1}{3}I_e\mu\right) + D'(M_{32} - \mu)\left(1 + \right.\right.$$

$12IeM32+\mu$ (74)

The amount payable to the supplier should be equals to the amount available to the retailer at the time $t = M_{32}$. By using (73 & 74) we have

$$c(W+Q)\left(1 + I_p(M_{32} - M)\right) = p\left(a\mu\left(1 + \frac{1}{2}I_e\mu\right) + b\mu^2\left(\frac{1}{2} + \frac{1}{3}I_e\mu\right) + D'(M_{32} - \mu)\left(1 + \frac{1}{2}I_e(M_{32} + \mu)\right)\right)$$

After some algebraic simplification, we get

$$\frac{1}{2}pI_eD'M_{32}^2 + \left(pD' - I_pc(W+Q)\right)M_{32} + p\left(a\mu\left(1 + \frac{1}{2}I_e\mu\right) + b\mu^2\left(\frac{1}{2} + \frac{1}{3}I_e\mu\right)\right) - c(W+Q)(1 - I_pM) - pD'\mu\left(1 + \frac{1}{2}I_e\mu\right) = 0 \tag{75}$$

**Lemma 6**: The solution to equation (75) determines the value of $M_{32}$

Now, the retailer will continue to make sales and generate revenue and interest on the revenue generated during the period $[M_{32}\ \ T]$ is

$$p\int_{M_{32}}^T D'dt + pI_e\int_{M_{32}}^T tD'dt = pD'\left(1 + \frac{1}{2}I_e(M_{32} + T)\right)(T - M_{32}) \tag{76}$$

Therefore, the total profit $TP_9$ using (21), (22), (25) and (76) is given by

$$TP_9 = pD'\left(1 + \frac{1}{2}I_e(M_{32} + T)\right)(T - M_{32}) - OC - HC1 - DC1 \tag{77}$$

Sub – case 4: $\mu < t_w < M < T$ when the demand stabilized before the inventory at RW drops to zero.

At time $t = M$, the amount payable to the supplier by retailer is $c(W+Q)$

The generated amount of revenue by the retailer due to sales during the period $[0\ , M]$ is

$$p\int_0^M D(t)dt = p\int_0^\mu (a+bt)dt + p\int_\mu^M D'dt = p\left(a\mu + \frac{1}{2}b\mu^2 + D'(M - \mu)\right) \tag{78}$$

The interest generated/earned during the period $[0\ , M]$ on the sales revenue by the retailer is

$$pI_e\left(\int_0^{\mu} f(t)dt + \int_{\mu}^{M} f(\mu)dt\right) = pI_e\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \frac{1}{2}D'(M^2 - \mu^2)\right) \quad (79)$$

Therefore, letting $G_4$ to be the total amount generated by the retailer during the period $[0\ ,M]$
Using (78 & 79) we have

$$G_4 = p\left(a\mu\left(1 + \frac{1}{2}I_e\mu\right) + b\mu^2\left(\frac{1}{2} + \frac{1}{3}I_e\mu\right) + D'(M - \mu)\big(1 + I_e(M + \mu)\big)\right) \quad (80)$$

In this case, there are two cases to consider
Sub case 4.1: $G_4 \geq c(W + Q)$ when the amount of money at the retailer's disposal is larger than the amount to settle his account with the supplier at time $t = M$ and
Sub – case 4.2: $G_4 < c(W + Q)$ when the amount of money at the retailer's disposal is smaller than the amount to settle his account with the supplier at time $t = M$
Sub – case 4.1: $G_4 \geq c(W + Q)$
In this situation, the retailer has an excess amount after paying the supplier. The excess amount and the interest earned on the excess for the period $[M\ T]$is given by

$$G_4 - c(W + Q) + \big(G_4 - c(W + Q)\big)I_e(T - M) = \big(G_4 - c(W + Q)\big)\big(1 + I_e(T - M)\big) \quad (81)$$

After paying the supplier at $t = M$, the retailer will continue to make sales and earned interest on the sales revenue generated for the period $[M\ ,T]$ and is given as

$$p\int_M^T D'dt + pI_e\int_M^T tD'dt = pD'\left(1 + \frac{1}{2}I_e(M + T)\right)(T - M) \quad (82)$$

The total amount with the retailer using equation (81) and (82) is

$$\big(G_4 - c(W + Q)\big)\big(1 + I_e(T - M)\big) + pD'\left(1 + \frac{1}{2}I_e(M + T)\right)(T - M) \quad (83)$$

The total profit of the retailer $TP_{10}$ using (23), (24), (25) and (83) is given by

$$TP_{10} = \big(G_4 - c(W + Q)\big)\big(1 + I_e(T - M)\big) + pD'\big(1 + I_e(M + T)\big)(T - M) - OC - HC2 - DC2 \quad (84)$$

Sub – case 4.2: $G_4 < c(W + Q)$ when the amount generated by the retailer at time $t = M$ is not up to the amount needed to settle his/her account with the supplier.
In this sub – case, there arise two scenarios
Sub – case 4.2.1: when the supplier is willing to accept part payment from the retailer at time $t = M$ and the balance (including interest charge) at $t = M_5$ where $M_5 > M > \mu$
After depositing the part payment $G_4$ with the supplier at time $t = M$, the remaining balance to be paid to the supplier at $t = M_5$ is his/her money and the interest on the money for the period $[M\ M_5]$ and is given by

$$\big(c(W + Q) - G_4\big)\left(1 + I_p(M_5 - M)\right) \quad (85)$$

Therefore, the total amount available to the retailer at time $t = M_5$ is the amount generated due to sales and interest on the sales by the retailer during the period $[M\ ,M_5]$ is

$$p\int_M^{M_5} D'dt + pI_e\int_M^{M_5} tD'dt = pD'\big(1 + I_e(M_5 + M)\big)(M_5 - M) \quad (86)$$

Therefore, using (85) and (86), at time $t = M_5$, the amount payable to the supplier = amount available to the retailer

$$\big(c(W + Q) - G_4\big)\left(1 + I_p(M_5 - M)\right) = pD'\big(1 + I_e(M_5 + M)\big)(M_5 - M)$$

After some algebraic simplification, we get

$$pD'I_eM_5^2 + \big(pD' - I_p(c(W + Q) - G_4\big)M_5 - pD'M(1 + I_eM) - \big(c(W + Q) - G_4\big)\big(1 - I_pM\big) = 0 \quad (87)$$

**Lemma 7**: The solution to (87) determine the value of $M_5$
Now the total amount for the retailer will be the selling revenue during $[M_5, T]$ + the interest earned on the sales for the period $[M_5, T]$

$$p\int_{M_5}^T D'dt + pI_e\int_{M_5}^T tD'dt = pD'\left(1 + \frac{1}{2}I_e(T + M_5)\right)(T - M_5) \quad (88)$$

The total profit $TP_{11}$ by the retailer using (23), (24), (25) and (88) is given by

$$TP_{11} = pD'\left(1 + \frac{1}{2}I_e(T + M_5)\right)(T - M_5) - OC - HC2 - DC2 \quad (89)$$

Sub – case 4.2.2: when the supplier reject part payment but full payment at a time $t = M_6$ where $M_6 > M > \mu$
At the time $t = M_6$, the retailer will pay $c(W + Q)$ and interest of $I_pc(W + Q)\int_M^{M_6} dt$
The total amount payable by the retailer to the supplier is

$$c(W + Q) + I_pc(W + Q)\int_M^{M_6} dt = c(W + Q)\left(1 + I_p(M_6 - M)\right) \quad (90)$$

The amount generated due to sales and interest earned over the sales revenue by the retailer up to the time $t = M_6$ is

$$p\left(\int_0^{\mu}(a + bt)dt + \int_{\mu}^{M_6} D'dt\right) + pI_e\left(\int_0^{\mu}(a + bt)tdt + \int_{\mu}^{M_6} D'tdt\right) = p\left(a\mu\left(1 + \frac{1}{2}I_e\mu\right) + b\mu^2\left(\frac{1}{2} + \frac{1}{3}I_e\mu\right) + D'(M_6 - \mu)\left(1 + \frac{1}{2}I_e(M_6 + \mu)\right)\right) \quad (91)$$

Then, at time $t = M_6$ the amount payable to the supplier = the amount available to the retailer. Using (90) and (91), we have

$$c(W + Q)\left(1 + I_p(M_6 - M)\right) = p\left(a\mu\left(1 + \frac{1}{2}I_e\mu\right) + b\mu^2\left(\frac{1}{2} + \frac{1}{3}I_e\mu\right) + D'(M_6 - \mu)\left(1 + \frac{1}{2}I_e(M_6 + \mu)\right)\right)$$

After some algebraic simplification, we get

$$\frac{1}{2}pI_eD'M_6^2 + \left(pD' - I_pc(W + Q)\right)M_6 + p\left(a\mu\left(1 + \frac{1}{2}I_e\mu\right) + b\mu^2\left(\frac{1}{2} + \frac{1}{3}I_e\mu\right)\right) - c(W + Q)\big(1 - I_pM\big) - pD'\mu\left(1 + \frac{1}{2}I_e\mu\right) = 0 \quad (92)$$

**Lemma 8**: The solution to (92) determine the value of $M_6$
Now, the retailer will continue to make sales and generate revenue and interest on the sales revenue generated for the period$[M_6, T]$. This is given by

$$p\int_{M_6}^{T} D^{'}dt + pI_e\int_{M_6}^{T} tD^{'}dt = pD^{'}\left(1+\frac{1}{2}I_e(T+M_6)\right)(T-M_6) \quad (93)$$

The total profit $TP_{12}$ of the retailer by using (23), (24), (25) and (93) is given by

$$TP_{12} = pD^{'}\left(1+\frac{1}{2}I_e(T+M_6)\right)(T-M_6) - OC - HC2 - DC2 \quad (94)$$

Sub – case 5: $t_w \le \mu < T < M$ when the demands stabilized after the inventory at RW drops to zero

In this situation, the retailer will generate revenue due to sales and earned interest over the sales revenue for the time period$[0\,, M]$. Letting it to be$G_5$, then it is given by

$$G_5 = p\left(\int_0^{\mu}(a+bt)dt + \int_{\mu}^{M} D^{'}dt\right) + pI_e\left(\int_0^{\mu}(a+bt)tdt + \int_{\mu}^{M} D^{'}tdt\right) = p\left(a\mu + \frac{1}{2}b\mu^2 + D^{'}(M-\mu) + I_e\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \frac{1}{2}D^{'}(M^2-\mu^2)\right)\right) \quad (95)$$

The total amount payable to the supplier by the retailer at the time $t = M$ is $c(W+Q)$

Therefore, in this situation, the total amount generated by the retailer must be greater than the amount payable to the supplier, hence using (87) we have the retailer's profit to be

$$p\left(a\mu + \frac{1}{2}b\mu^2 + D^{'}(M-\mu) + I_e\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \frac{1}{2}D^{'}(M^2-\mu^2)\right)\right) - c(W+Q) \quad (96)$$

Therefore, the total profit of the retailer $TP_{13}$using (21), (22), (25) and (96) is given by

$$TP_{13} = p\left(a\mu + \frac{1}{2}b\mu^2 + D^{'}(M-\mu) + I_e\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \frac{1}{2}D^{'}(M^2-\mu^2)\right)\right) - c(W+Q) - OC - HC1 - DC1 \quad (97)$$

Sub – case 6: $\mu < t_w < T < M$ when the demands stabilized before the inventory at RW drops to zero.

In this situation, the retailer will generate revenue due to sales and earned interest over the sales revenue for the time period $[0\,, M]$ given by $G_5$ in (87).

The total amount payable to the supplier by the retailer at the time $t = M$ is $c(W+Q)$

Therefore, in this situation also, the total amount generated by the retailer must be greater than the amount payable to the supplier i.e. $G_5 \ge c(W+Q)$

Therefore, the total profit of the retailer $TP_{14}$using (23), (24), (25) and (97) is given by

$$TP_{14} = p\left(a\mu + \frac{1}{2}b\mu^2 + D^{'}(M-\mu) + I_e\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \frac{1}{2}D^{'}(M^2-\mu^2)\right)\right) - c(W+Q) - OC - HC1 - DC1 \quad (98)$$

## 7. **Conclusion and future direction**

The model for the two – warehouse that considers demand as ramp type with upstream permissible delay in payment and time value of money has been developed to reflect the realities of the market in the present dispensation. It is therefore recommended that a numerical example for the model be given and sensitivity analyses carried out to see the effect of parameter changes on the model. The price and time dependent demand can also be look at.

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