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TWO – WAREHOUSE INVENTORY MODEL FOR DETERIORATING ITEMS UNDER PARTIAL DOWNSTREAM TRADE CREDIT FINANCING

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Abstract

In the early literature of deteriorating items under trade credit financing, it was commonly assumed in most cases that the supplier offers the retailer a full trade credit period within which to settle the account. Later, the policy was extended to consider when the retailer also offers customers the benefit of the trade credit period. In this study, we develop two – warehouse inventory model for deteriorating items under two – level trade credit financing. We assume that the supplier offers the retailer full credit period (upstream) and the retailer passes it to his/her customer but requesting some portion of the consignment supplied be paid immediately they are received. A credit period is offered on the remaining portion of the goods supplied (partial downstream) in order to reduce the risk of failure in payments. We find the optimal replenishment cycle for the inventory system with the help of several cases and sub – cases. A numerical example is used to illustrate the application of the model and sensitivity analysis carried out to see the effect of parameter changes.

Keywords: Partial Downstream, Trade Credit Financing, Upstream, Deteriorating, Failure.

1. **Introduction**

The Economic Order Quantity (EOQ) model deals with the study of what to order, when to order and the right quantity to order of an item at a given time,[1]. In the past few decades, there has been so much study in the inventory control of deteriorating item(s) as reviewed in[2] and later updated by [3]. The study of deteriorating items was first carried out by[4] who developed an EOQ model with constant rate of deterioration which was laterextended by[5] considering variable rate of deterioration. Papers that considered deterioration items include [6 – 10]and many more others.

In the literature, it is commonly assumed that goods are paid for immediately they are delivered. However, this is not always the case. Some supplier give their retailer(s) grace period within which to settle their account(s), a phenomenon known as *permissible delay in payments*; as stated in[11]. The idea of permissible delay in payment has the advantage of making the retailer to order more quantity so as to make huge profit. Another importance of permissible delay is that it gives the supplier chance to make bulk supply to retailers. It also serves as alternative to price discount. It is a case whereby the retailer is given a grace period to settle the account within the allowed period without incurring any charges. After the allowed period, the retailer is charged an interest rate over the unsold items. In the literature of permissible delay in payment, [11] was the first, where he considered constant demand and fixed period within which the retailer is allowed to settle his/her account without incurring any charges. He mistakenly assumed the purchasing and selling prices to be the same as corrected by [12] which state the necessity of selling price to be higher than the purchasing cost. The study in [13] extended [11] to consider deteriorating items, while [9] extended [11] to consider different purchasing and selling prices.

In some cases, if the retailer wants to attract more customers, then he/she has to also offer/pass the permissible delay in payment grace to the customer(s). This is termed as two – level trade credit financing; [14 – 15]. Though it reduces the retailers profit fortune but it helps in stimulating the sales volume and that in turn increases the turnover fortune. The literature that considered two – level trade credit financing include [14] whereby the retailer offers the customers a period$[0\ ,N]$ within which to settle their individual account. In [15], the credit period was offered to the customer in form of N+t period from time t when the customer receives the consignment. The study in [16] considers trade credit financing for deteriorating items with maximum lifetime. On the other hand, apart from additional cost that trade credit period offers, it also provides additional dimension of default risk factor; Teng [15]. In [17], as an extension of[16]which considers partial

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downstream trade credit financing where the customer is assumed to be a risk and so he/she must pay some amount when placing order to cover some portion of the total quantity of the consignment ordered before benefitting the grace permissible delay in payment.

Due to permissible delay in payment or because of some other factors such as scarcity perception, seasonality of goods and so on, the retailer is motivated to order more quantity which may be in excess after storing the goods in his/her own warehouse. Thus, it will force the retailer to rent another warehouse where he/she will stock the remaining goods. As a result of this, researchers considered two – warehouse inventory policy to reflect the happenings and make proper replenishment policy. It is a case whereby the retailer has his/her own warehouse OW with limited capacity and rented another warehouse of unlimited capacity RW to keep the excess of the consignment ordered. It is commonly assumed in most of the papers that, the RW provides better preserving facilities than the OW, as a result of which deterioration rate in OW exceeds that in RW. The holding cost in RW is therefore assumed to be higher than that in OW. Due to economic reasons, it is assumed in most cases that, the dispatching policy is last – in first – out popularly known as *LIFO*. In the paper [18], the author looked at the two – warehouse model with constant demand and shortages under inflation. He/she compared the traditional model that starts with instant order and ends with shortages to a proposed model that starts with shortages and end without shortages. Yang found that, if the inflation rate is greater than zero, the proposed model is less expensive to run than the traditional model. If the inflation rate is equal to zero, both models have the same running costs. The study in[19] considers single deteriorating item under conditional permissible delay in payment with constant demand, constant deterioration and no shortages. Extension of the work in [19] to develop a two – warehouse model for deteriorating items partially backlogged, under permissible delay in payments and using time value of money approach was given in [20]. Also, [21] considered partial backlogged two – warehouse model for deteriorating items with permissible delay in payment under inflation/discounted cost.

In this work, we seek to extend [19] work to the case of deteriorating item under two – level trade credit financing considering risk customers. In this situation, the retailer is given full trade credit period and the retailer passes the grace to his/her customers if they make deposit of a substantive amount to cover some portion γ of the quantity ordered. The remaining balance is to be paid at the expiration of the permissible trade credit period. The demand and deterioration in both warehouses are assumed constant. The deterioration in RW is assumed to be less than that in OW but charges higher rate of holding cost which necessitates the last in – first out dispatching policy due to economic reasons.

The organization of the work is as follows: assumptions and notations are given in section 2, the model formulation in section 3. Section 4 gives analysis and optimization procedure whereas in section 5 a numerical example is given and sensitivity analysis carried out to see the effect of parameter changes and finally, in section 6 summary, conclusion and recommendations of area of further interest are given.

2. **Assumptions and Notations**

The following are assumptions made in building the model

a. The model considers single deteriorating item stocked in two – warehouses.

b. The demand and deterioration rates are constant in both warehouse.

c. Deterioration in RW is less than that in OW, i.e. $\alpha > \beta$ due to higher preserving facilities in RW and charges higher holding cost in RW than in OW, i.e. $h_r > h_o$, and therefore assumed $h_r - h_o > c(\alpha - \beta)$.

d. The demand rate in the warehouse is greater than the deterioration rate at the warehouse.

e. The dispatching policy is last – in first – out *LIFO* due to economic reasons.

f. The model considers full upstream trade credit, i.e. the retailer will make payment at the expiration of the allowed period without any charges and thereafter pays charges for the unsold items at higher rate up to the end of the period, T. The model also considers partial downstream (i.e. the customer must pay some amount to cover a fraction of the goods when making order) trade credit financing.

g. The downstream trade credit period considered in the model is Huang's [14] view, i.e. the retailer provides fixed period$[0\ \ N]$, where $N \leq t_w$(where t_w is the time when the inventory at RW finishes) for the customers within which they must settle their account or be charged interest rate over the unsold items.

h. Shortage is not allowed and lead time is assumed to be zero.

i. The condition of $M > N$ in Huang's [14] is relaxed.

j. $\beta M$is assumed to be sufficiently small, i.e. $\beta M \ll 1$.

The following are the notations used in the work:

$I_r(t)$,$I_o(t)$re the inventory levels of the RW and OW respectively at time *t*.

D is the constant demand rate of the item.

$W$is the maximum quantity that can be stored in OW.

$t_w$is time at which inventory at RW drops to zero.

T is the replenishment cycle for the model.

α, β are the deterioration rates at OW and RW respectively with $\alpha > \beta$.
h_r, h_o are the holding costs (excluding interest charges) per unit per unit time of RW and OW respectively.
A is the ordering cost per order.
I_p , I_e are the interest payable and interest earned return rates respectively.
c, p are the purchasing cost and selling price of the item respectively.
M the trade credit period offered to the retailer by the supplier (Upstream).
N the trade credit period offered to the customer by the retailer (Downstream).
γ the portion/fraction of the quantity ordered by the customer to be paid before giving the trade credit period
TRC is the total relevant costs per unit of the model to be minimized.
All other notations not defined here will be defined appropriately.

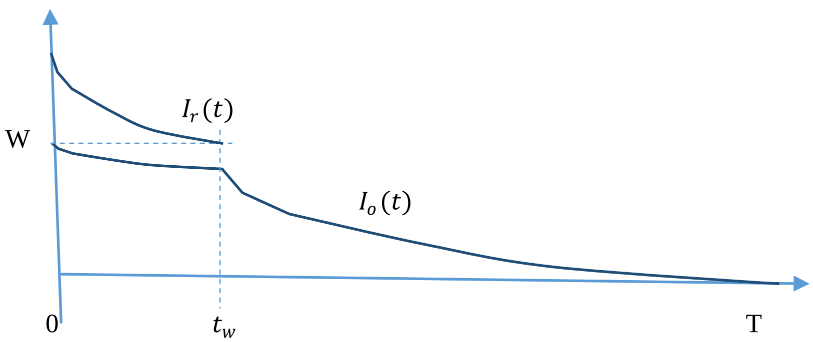

Fig 1: graphical representation of the model

3. Mathematical Model Formulation

Based on the model setup, we assume initially that at $t = 0$, the two warehouses are stocked. Then the replenishment cycle begins, such that during the time period $t \in (0, t_w)$, depletion of inventory at RW is due to the combined effect of demand and deterioration. At $t = t_w$, the inventory at RW drops to zero. For the OW, the depletion of inventory is due to the effect of deterioration only. These are represented by the following differential equations:

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -D , \qquad 0 \le t \le t_w \qquad (1)$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = 0 \quad , \qquad 0 \le t \le t_w \qquad (2)$$

The boundary and initial conditions for RW and OW are respectively given as follows:

$$at\ t = t_w,\ I_r(t) = 0 \qquad (3)$$

$$at\ t = 0,\ I_o(t) = W \qquad (4)$$

During the time period $t \in (t_w ,T)$, no activity in RW as the items stocked are finished; depletion of inventory at OW is due to demand by the customers combined with the effect of deterioration. At $t = T$, the inventory at OW drops to zero. From t_w to T, the situation in OW is represented by the following differential equation:

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -D \quad , \qquad t_w \le t \le T \qquad (5)$$

The boundary condition is given as

$$at\ t = T,\ I_o(t) = 0 \qquad (6)$$

The solutions to equations (1), (2) and (5) using (3), (4) and (6) respectively are

$$I_r(t) = \frac{D}{\beta}\left(e^{\beta(t_w - t)} - 1\right) \qquad 0 \le t \le t_w \qquad (7)$$

$$I_o(t) = We^{-\alpha t} \qquad 0 \le t \le t_w \qquad (8)$$

$$I_o(t) = \frac{D}{\alpha}\left(e^{\alpha(T-t)} - 1\right) t_w \le t \le T \qquad (9)$$

Due to physical deterioration and continuity of the model when $t = t_w$,using (8) and (9) we get

$$T = t_w + \frac{1}{\alpha}\ln\left(\frac{\alpha W}{D}e^{-\alpha t_w} + 1\right) \qquad (10)$$

For us to get the Total Relevant Costs per unit time (TRC), we calculate the following elements

1. Ordering Cost Per Unit Time, OC

The ordering cost per unit time, i.e. annual ordering cost is given by

$$OC = \frac{A}{T} \qquad (11)$$

2. Holding Cost Per Unit Time, HC

Thetotal annual holding cost, HC,using (7), (8) and (9) is given by

$$HC = \frac{Dh_r}{\beta^2 T}\left(e^{\beta t_w} - \beta t_w - 1\right) + \frac{Wh_o}{\alpha T}\left(1 - e^{-\alpha t_w}\right) + \frac{Dh_o}{\alpha^2 T}\left(e^{\alpha(T-t_w)} - \alpha(T - t_w) - 1\right) \qquad (12)$$

3. Deteriorating Cost Per Unit Per Unit Time, DC

Thetotal annual deterioration cost,DC , using (7), (8) and (9) is

$$DC = \frac{c}{T}\left(\frac{D}{\beta}\left(e^{\beta t_w} - \beta t_w - 1\right) + W(1 - e^{-\alpha t_w}) + \frac{D}{\alpha}\left(e^{\alpha(T-t_w)} - \alpha(T - t_w) - 1\right)\right) \qquad (13)$$

4. Interest Payable and Interest Earned Opportunity Costs I_P&I_E

The supplier offers a permissible delay in payment period M for the retailer to settle account, and the retailer also offers the customers a delay in payment permissible period $N \leq t_w$ for $1-\gamma$ fraction of the total placed order within which to settle their accounts. This is by first depositing an amount when placing order to cover a fraction, γ, of the placed order.

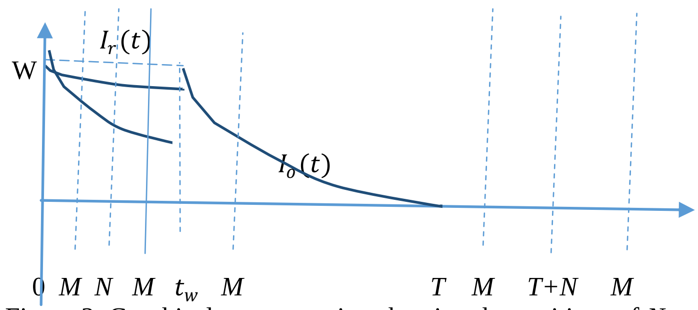

Figure 2: Graphical representation showing the positions of *N* and *M* in the model.

Based on the values of *N*, *M*, t_w,*T* and *T*+*N*, we have two main cases to consider (1) $N \leq M$ and (2) $N > M$. From case 1 however, we have four sub – cases as follows:

(1.1) $0 \leq M \leq t_w$ (1.2) $t_w \leq M \leq T$(1.3) $T \leq M \leq T + N$and (1.4) $T + N \leq M$.

In all these sub-cases, $N \leq M$ and $N \leq t_w$

Case 1: $N \leq M$ (when upstream permissible period is longer than the downstream permissible period).

Sub – case 1.1: $0 \leq M \leq t_w$

At time *N*, the retailer had obtained cash payments from the customers and also start to receive credit payment from them. The retailer must finance all items sold after *M* on cash and credit. Therefore the annual interest payable by the retailer is given by

$$I_P = \frac{cI_p}{T}\left(\gamma\left(\int_M^{t_w} I_r(t)dt + \int_M^T I_o(t)dt\right) + (1-\gamma)\left(\int_M^{t_w+N} I_r(t)dt + \int_M^{T+N} I_o(t)dt\right)\right)$$

Using Equations (7), (8) and (9) we get

$I_P =$

$\frac{cI_p}{T}\left(\frac{D}{\beta^2}\left(\gamma\left(e^{\beta(t_w-M)} - \beta(t_w - M) - 1\right) + (1-\gamma)\left(e^{\beta(t_w-M)} - \beta(t_w + N - M) - e^{-\beta N}\right)\right) + \frac{W}{\alpha}(e^{-\alpha M} - e^{-\alpha t_w}) + \frac{D}{\alpha^2}\Big(\gamma(e^{\alpha(T-t_w)} -\right.$

αT−tw−1)+1−γeαT−tw−αT+N−tw−e−αN (14)

On the other hand, the retailer accumulates revenue and earns interest from the cash payments made from the beginning time 0 through to *M* and credit payment beginning from time *N* through to *M*. Therefore the annual interest earned is given by

$$I_E = \frac{pI_e}{T}\left(\gamma\int_0^M Dtdt + (1-\gamma)\int_N^M D(t-N)dt\right) = \frac{DpI_e}{2T}(\gamma M^2 + (1-\gamma)(M-N)^2) \qquad (15)$$

The Total Annual Relevant Costs per Unit Time *TRC* is

$$TRC_{1.i} = OC + HC + DC + I_P - I_E \qquad i = 1,2,3,4 \qquad (16)$$

Therefore, putting Equations (11), (12), (13), (14) and (15) into equation (16) and simplifying, we get for $i = 1$ i.e. for sub – case 1.1,

$TRC_{1.1} = \frac{1}{T}\Big(A + \frac{D}{\beta^2}\Big[(h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right) + cI_p\left(\gamma\left(e^{\beta(t_w-M)} - \beta(t_w - M) - 1\right) + (1-\gamma)\left(e^{\beta(t_w-M)} - \beta(t_w + N - M) -\right.\right.$

e−βN+Wαho+αc1−e−αtw+cIpe−αM−e−αtw+Dα2ho+αc+γcIpeαT−tw−αT−tw−1+1−γcIpeαT−tw−αT+N−tw−e−αN−12pI eDγM2+1−γM−N2 (17)

Sub – case 1.2: $t_w \leq M \leq T$

No unsold items in RW as at the time M, but the retailer will pay interest on unsettled payments from the customer for the credit period$(t_w,\ t_w+N)$ only when $t_w+N\geq M$ otherwise no interest payable for the unsettled payments of goods sold in RW. For the items in OW, the retailer will pay interest for all items sold after M on cash and credit basis up to the time the last customer will settle the account. Hence, the annual interest payable is given by

$$I_P=\frac{cI_p}{T}\left(\gamma\left(\int_M^T I_o(t)dt\right)+(1-\gamma)\left(\int_M^{t_w+N} I_r(t)dt+\int_M^{T+N} I_o(t)dt\right)\right)$$

Using Equations (7), (8) and (9), we get

$$I_P=\frac{cI_p}{T}\left(\gamma\frac{D}{\alpha^2}\left(e^{\alpha(T-M)}-\alpha(T-M)-1\right)+(1-\gamma)\left(\frac{D}{\beta^2}\left(e^{\beta(t_w-M)}-\beta(t_w+N-M)-e^{-\beta N}\right)\right)+\frac{D}{\alpha^2}\left(e^{\alpha(T-t_w)}-\alpha(T+N-t_w)-\right.\right.$$

e−αN) (18)

Also, for the interest earned, the retailer will accumulate revenue and obtain interest from the cash payment starting from 0 to M and credit payment starting from N to M. Therefore the annual interest earned by the retailer is given by

$$I_E=\frac{pI_e}{T}\left(\gamma\int_0^M Dtdt+(1-\gamma)\int_N^M D(t-N)dt\right)=\frac{DpI_e}{2T}(\gamma M^2+(1-\gamma)(M-N)^2) \qquad (19)$$

Therefore, substituting (11), (12), (13), (18) and (19) into (16)and simplifying, we get for $i=2$

$$TRC_{1.2}=\frac{1}{T}\left(A+\frac{D}{\beta^2}\left[(h_r+c\beta)\left(e^{\beta t_w}-\beta t_w-1\right)+cI_p(1-\gamma)\left(e^{\beta(t_w-M)}-\beta(t_w+N-M)-e^{-\beta N}\right)\right]+\frac{W}{\alpha}(h_o+\alpha c)(1-e^{-\alpha t_w})+\right.$$

Dα2ho+αceαT−tw−αT−tw−1+cIpγeαT−M−αT−M−1+1−γeαT−tw−αT+N−tw−e−αN−12γpIeDM2−121−γpIeDM−N2

(20)

Sub – case 1.3: $T\leq M\leq T+N$

For all the items sold on cash payments by the retailer up to the time T, there is no interest charged. But the retailer will pay interest for the items sold on credit up to the time the last customer will settle the account, and is given by

$$I_P=\frac{cI_p}{T}\left((1-\gamma)\left(\int_M^{T+N} I_o(t)dt\right)\right)$$

Using Equation (9) we see that

$$I_P=(1-\gamma)\frac{DcI_p}{\alpha^2T}\left(e^{\alpha(T-M)}-\alpha(T+N-M)-e^{-\alpha N}\right) \qquad (21)$$

Likewise, the retailer accumulates revenue and earn interest from the cash payment starting from 0 to M and credit payment starting from N to M. Therefore the annual interest earned is given by

$$I_E=\frac{pI_e}{T}\left(\gamma\left(\int_0^T Dtdt+DT(M-T)\right)+(1-\gamma)\left(\int_N^T D(t-N)dt+DT(M-T)\right)\right)=\frac{DpI_e}{2T}\left(\gamma T^2+(1-\gamma)(T-N)^2+2T(M-T)\right)$$

(22)

Therefore, putting (11), (12), (13), (21) and (22) into (16) we get for $i=3$

$$TRC_{1.3}=\frac{1}{T}\left(A+\frac{D}{\beta^2}\left[(h_r+c\beta)\left(e^{\beta t_w}-\beta t_w-1\right)\right]+\frac{W}{\alpha}\left((h_o+\alpha c)(1-e^{-\alpha t_w})\right)+\frac{D}{\alpha^2}\left((h_o+\alpha c)\left(e^{\alpha(T-t_w)}-\alpha(T-t_w)-1\right)+(1-\gamma)cI_p\left(e^{\alpha(T-M)}-\right.\right.\right.$$

αT+N−M−e−αN−12pIeDγT2+1−γT−N2+2TM−T (23)

Sub – case 1.4: $T+N\leq M$

In this case, the retailer had received all cash and credit payments before the allowed period given by the supplier, hence incurs no interest. Therefore, annual interest payable is given by

$$I_P=0 \qquad (24)$$

For the interest earned, the retailer accumulates revenue and earn interest from the cash payment starting from time 0 to M and credit payment beginning from N to $T+N$. therefore the annual interest earned is given by

$$I_E=\frac{pI_e}{T}\left(\gamma\left(\int_0^T Dtdt+DT(M-T)\right)+(1-\gamma)\left(\int_N^T D(t-N)dt+DT(T+N-T)+D(T+N)(M-T-N)\right)\right)=$$

$$\frac{pI_eD}{T}\left(\gamma\left(MT-\frac{1}{2}T^2\right)+(1-\gamma)\left(\frac{1}{2}T^2+\frac{1}{2}N^2+(T+N)(M-T-N)\right)\right) \qquad (25)$$

Substituting (11), (12), (13), (24) and (25) into (16), we get for $i=4$

$$TRC_{1.4}=\frac{1}{T}\left(A+\frac{D}{\beta^2}(h_r+c\beta)\left(e^{\beta t_w}-\beta t_w-1\right)+\frac{W}{\alpha}(h_o+\alpha c)(1-e^{-\alpha t_w})+\frac{D}{\alpha^2}\left((h_o+\alpha c)\left(e^{\alpha(T-t_w)}-\alpha(T-t_w)-1\right)\right)-pI_eD\left(\gamma\left(MT-\right.\right.\right.$$

12T2+(1−γ)12T2+12N2+T+NM−T−N (26)

Case 2: $N > M$ (when upstream permissible period is shorter than the downstream permissible period)
In this scenario, the only case that will occur is when $M < N \leq t_w$ due to the restriction of the period N in the assumptions. In this case, the retailer must finance all items sold in cash from time M to T and all items sold on credit payments from M to $T+N$. Therefore the annual interest payable is given by

$$I_P = \frac{cI_p}{T}\left(\gamma\left(\int_M^{t_w} I_r(t)dt + \int_M^{T} I_o(t)dt\right) + (1-\gamma)\left(\int_M^{t_w+N} I_r(t)dt + \int_M^{T+N} I_o(t)dt\right)\right)$$

Using Equations (7), (8) and (9) we see that

$I_P = \frac{cI_p}{T}\left(\frac{D}{\beta^2}\left(\gamma\left(e^{\beta(t_w-M)} - \beta(t_w - M) - 1\right) + (1-\gamma)\left(e^{\beta(t_w-M)} - \beta(t_w + N - M) - e^{-\beta N}\right)\right) + \frac{D}{\alpha^2}\left(\gamma\left(e^{\alpha(T-t_w)} - \alpha(T - t_w) -\right.\right.\right.$
1+1−γeαT−tw−αT+N−tw−e−αN+Wαe−αM−e−αtw (27)

Likewise, since the retailer gives the customers trade credit period more than what was given by the supplier, then no interest earned on the $(1-\gamma)$ proportion of the quantity given to the customers but will generate revenue from the γ proportion of the quantity for the period, 0 to M, therefore the annual interest earned is given by

$$I_E = \frac{pI_e}{T}\gamma\int_0^M Dtdt = \frac{1}{2T}\gamma DpI_eM^2 \qquad (28)$$

Substituting (11), (12), (13), (27) and (28) into (16), we get

$TRC_2 = \frac{1}{T}\left(A + \frac{D}{\beta^2}\left[(h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right) + cI_p\left(\gamma\left(e^{\beta(t_w-M)} - \beta(t_w - M) - 1\right) + (1-\gamma)\left(e^{\beta(t_w-M)} - \beta(t_w + N - M) -\right.\right.\right.\right.$
e−βN+Wαho+αc1−e−αtw+cIpe−αM−e−αtw+Dα2ho+αceαT−tw−αT−tw−1+cIpeαT−tw−γeαT−tw−αT−tw−1+1−γ eαT−tw−αT+N−tw−e−αN−12γDpIeM2 (29)

4. Optimization and Analysis

The necessary conditions for $TRC_{1.1}$ to be minimized are $\frac{\partial TRC_{1.1}}{\partial t_w} = 0$ and $\frac{\partial TRC_{1.1}}{\partial T} = 0$

Using Equation (17), we get

$$\frac{\partial TRC_{1.1}}{\partial t_w} = \frac{1}{T}\left[\frac{D}{\beta}\left((h_r + c\beta)\left(e^{\beta t_w} - 1\right) + cI_p\left(e^{\beta(t_w-M)} - 1\right)\right) + W\left(h_o + \alpha c + cI_p\right)e^{-\alpha t_w} - \frac{D}{\alpha}\left(\left(h_o + \alpha c + cI_p\right)\left(e^{\alpha(T-t_w)} - 1\right)\right)\right] = 0 \qquad (30)$$

So also

$$\frac{\partial TRC_{1.1}}{\partial T} = \frac{1}{T}\left(\frac{D}{\alpha}\left(h_o + \alpha c + cI_p\right)(e^{\alpha(T-t_w)} - 1) - TRC_{1.1}\right) = 0 \qquad (31)$$

To obtain the solutions t_w^{11*} and T_{11}^* of the highly non - linear equations (30) and (31) respectively, we use the optimization technique, Newton – Raphson method.
To prove that t_w^{11*} and T_{11}^*exist and unique, we show it satisfy the sufficient condition for a minimum, i.e. the determinant of Hessian matrix H evaluated at the point (t_w^{11*}, T_{11}^*) is positive definite. Hence, evaluating the gradients of the H at the point(t_w^{11*}, T_{11}^*), we have;
Taking the second derivative of (17) with respect tot_w, and evaluating at the point (t_w^{11*}, T_{11}^*)we see that

$$\left.\frac{\partial^2 TRC_{1.1}}{\partial t_w^2}\right|_{(t_w^{11*}, T_{11}^*)} = \frac{1}{T}\left.\left(D\left((h_r + c\beta)e^{\beta t_w} + cI_pe^{\beta(t_w-M)}\right) - \alpha W\left(h_o + \alpha c + cI_p\right)e^{-\alpha t_w} + D\left(h_o + \alpha c + cI_p\right)e^{\alpha(T-t_w)}\right)\right|_{(t_w^{11*}, T_{11}^*)} > \frac{1}{T}\left.\left(D\left(h_o + \alpha c + cI_p\right)e^{\alpha(T-t_w)}\right)\right|_{(t_w^{11*}, T_{11}^*)} > 0 \qquad (32)$$

If and only if

$$D\left((h_r + c\beta)e^{\beta t_w} + cI_pe^{\beta(t_w-M)}\right) - \alpha W\left(h_o + \alpha c + cI_p\right)e^{-\alpha t_w} > 0$$

Lemma 1:- Given that $D > \alpha W$ and $h_r - h_o > c(\alpha - \beta)$ as in assumption 2(c), then

$$D\left(h_r + c\beta + cI_pe^{-\beta M}\right)e^{\beta t} > \alpha W\left(h_o + \alpha c + cI_p\right)e^{-\alpha t} \quad for\ t > 0$$

Proof

Let $f(t) = D\left(h_r + c\beta + cI_pe^{-\beta M}\right)e^{\beta t} - \alpha W\left(h_r + c\beta + cI_p\right)e^{-\alpha t} \quad t \geq 0$

$$f(0) = D\left(h_r + c\beta + cI_pe^{-\beta M}\right) - \alpha W\left(h_r + c\beta + cI_p\right) = (D - \alpha W)(h_r + c\beta) + cI_p\left(De^{-\beta M} - \alpha W\right)$$

But from series expansion at origin, i.e. McLaurin series, we get

$$e^{-\beta M} = 1 - \beta M + \frac{(\beta M)^2}{2!} - \cdots$$

$$\Longrightarrow 1 - \beta M < e^{-\beta M}$$

Therefore, $f(0) = (D - \alpha W)(h_r + c\beta) + cI_p\left(De^{-\beta M} - \alpha W\right) > (D - \alpha W)(h_r + c\beta) + cI_p(D[1 - \beta M] - \alpha W) > 0$
Since from assumption 2(d) and 2(j) $D - \alpha W > 0$ and βM is assumed to be sufficiently small respectively.
$f'(t) = \beta D\left(h_r + c\beta + cI_p e^{-\beta M}\right)e^{\beta t} + \alpha^2 W\left(h_r + c\beta + cI_p\right)e^{-\alpha t} > 0$
Since $f'(t) > 0$ for all $t > 0$, implies $f(t)$ is an increasing function of t.
Also, from assumption 2(c), $h_r + c\beta > h_o + \alpha c$ since $h_r > h_o$ and $0 \le \beta \le \alpha \le 1$
$\Rightarrow D\left(h_r + c\beta + cI_p e^{-\beta M}\right)e^{\beta t} > \alpha W\left(h_o + \alpha c + cI_p\right)e^{-\alpha t} \quad for\ all\ t > 0$ proved

Taking the second derivative of (17) with respect to T, and evaluating at the point (t_w^{11*}, T_{11}^*)we see that

$$\left.\frac{\partial^2 TRC_{1.1}}{\partial T^2}\right|_{(t_w^{11*}, T_{11}^*)} = \frac{1}{T}\left(D\left(\left(h_o + \alpha c + cI_p\right)e^{\alpha(T - t_w)}\right)\right)\Big|_{(t_w^{11*}, T_{11}^*)} > 0, \quad (33)$$

Also using Equation (17), after simplification and evaluation at the point(t_w^{11*}, T_{11}^*)we find that

$$\left.\frac{\partial^2 TRC_{1.1}}{\partial t_w \partial T}\right|_{(t_w^{11*}, T_{11}^*)} = -\frac{1}{T}\left(D\left(\left(h_o + \alpha c + cI_p\right)e^{\alpha(T - t_w)}\right)\right)\Big|_{(t_w^{11*}, T_{11}^*)} = \left.\frac{\partial^2 TRC_{1.1}}{\partial T \partial t_w}\right|_{(t_w^{11*}, T_{11}^*)} \quad (34)$$

Using (32), (33) and (34), we find that the Hessian matrix has determinant greater than zero, i.e.
$\left\{\frac{\partial^2 TRC_{1.1}}{\partial t_w^2}\frac{\partial^2 TRC_{1.1}}{\partial T^2} - \frac{\partial^2 TRC_{1.1}}{\partial t_w \partial T}\frac{\partial^2 TRC_{1.1}}{\partial T \partial t_w}\right\}\Big|_{(t_w^{11*}, T_{11}^*)} > 0$ implying that it is positive definite. Proved

Theorem 1: The cost function, $TRC_{1\cdot1}$, represented by Equation (17) is a convex function.

<u>Proof.</u>

The proof follows from the fact that the solutions to the cost function, $TRC_{1\cdot1,}$ satisfy the sufficiency condition as shown in Lemma 1 and the explanation before it.

The necessary conditions for $TRC_{1\cdot2}$ to be minimized are $\frac{\partial TRC_{1.2}}{\partial t_w} = 0$ and $\frac{\partial TRC_{1.2}}{\partial T} = 0$

Using Equation (20) we get

$$\frac{\partial TRC_{1.2}}{\partial t_w} = \frac{1}{T}\Big[\frac{D}{\beta}\left((h_r + c\beta)\left(e^{\beta t_w} - 1\right) + (1 - \gamma)cI_p\left(e^{\beta(t_w - M)} - 1\right)\right) + W(h_o + c\alpha)e^{-\alpha t_w} + \frac{D}{\alpha}\left(h_o + \alpha c + (1 - \gamma)cI_p\right)(1 - e\alpha(T - tw) = 0 \quad (35)$$

Also,

$$\frac{\partial TRC_{1.2}}{\partial T} = \frac{1}{T}\left[\frac{D}{\alpha}\left(\left(h_o + \alpha c + (1 - \gamma)cI_p\right)\left(e^{\alpha(T - t_w)} - 1\right) + \gamma cI_p\left(e^{\alpha(T - M)} - 1\right)\right) - TRC_{1.2}\right] = 0 \quad (36)$$

To obtain t_w^{12*} andT_{12}^*, the optimal solutions of equations (35) and (36) respectively, we use the optimization technique Newton – Raphson method.

To prove that t_w^{12*} andT_{12}^* satisfy the sufficient condition for minimum, we show the Hessian matrix H evaluated at the point (t_w^{12*}, T_{12}^*) is positive definite.

Taking the second derivative of Equation (20) with respect tot_w, and evaluating at the point (t_w^{12*}, T_{12}^*)we see that

$$\left.\frac{\partial^2 TRC_{1.2}}{\partial t_w^2}\right|_{(t_w^{12*}, T_{12}^*)} =$$

$$\frac{1}{T}\left(D\left((h_r + c\beta)e^{\beta t_w} + (1 - \gamma)cI_p e^{\beta(t_w - M)}\right) - \alpha W(h_o + \alpha c)e^{-\alpha t_w} + D\left(h_o + \alpha c + (1 - \gamma)cI_p\right)e^{\alpha(T - t_w)}\right)\Big|_{(t_w^{12*}, T_{12}^*)} > 0 \quad (37)$$

If and only if

$D\left((h_r + c\beta)e^{\beta t_w} + (1 - \gamma)cI_p e^{\beta(t_w - M)}\right) - \alpha W(h_o + \alpha c)e^{-\alpha t_w} > 0 \ \ \forall\ t_w$

<u>Lemma 2:</u> Given $D > \alpha W$ then

$D\left((h_r + c\beta)e^{\beta t_w} + (1 - \gamma)cI_p e^{\beta(t_w - M)}\right) - \alpha W(h_o + \alpha c)e^{-\alpha t_w} > 0 \ \ \forall\ t_w$

<u>Proof</u>

The proof follows from Lemma 1, since $h_r + c\beta > h_o + \alpha c$ and $\beta \le \alpha \Longrightarrow -\alpha \le -\beta < \beta$ because β is positive.

Taking the second derivative of equation (20) with respect to T, and evaluating at the point (t_w^{12*}, T_{12}^*)we see that

$$\left.\frac{\partial^2 TRC_{1.2}}{\partial T^2}\right|_{(t_w^{12*}, T_{12}^*)} = \frac{D}{T}\left(\left(h_o + \alpha c + (1 - \gamma)cI_p\right)e^{\alpha(T - t_w)} + \gamma cI_p e^{\alpha(T - M)}\right)\Big|_{(t_w^{12*}, T_{12}^*)} > 0 \quad (38)$$

Also from Equation (20), after simplification we find that

$$\left.\frac{\partial^2 TRC_{1.2}}{\partial t_w \partial T}\right|_{(t_w^{12*}, T_{12}^*)} = -\frac{D}{T}\left(\left(h_o + \alpha c + (1 - \gamma)cI_p\right)e^{\alpha(T - t_w)}\right)\Big|_{(t_w^{12*}, T_{12}^*)} = \left.\frac{\partial^2 TRC_{1.2}}{\partial T \partial t_w}\right|_{(t_w^{12*}, T_{12}^*)} \quad (39)$$

Using (37), (38) and (39), we find that the Hessian matrix has determinant greater than zero, i.e. $\left\{\frac{\partial^2 TRC_{1.2}}{\partial t_w^2}\frac{\partial^2 TRC_{1.2}}{\partial T^2} - \frac{\partial^2 TRC_{1.2}}{\partial t_w \partial T}\frac{\partial^2 TRC_{1.2}}{\partial T \partial t_w}\right\}\Big|_{(t_w^{12*}, T_{12}^*)} > 0$ and so it is positive definite. Proved□

Theorem 2: The cost function, $TRC_{1\cdot 2}$, represented by Equation (20) is a convex function.
Proof.
The proof follows from the fact that the solutions to the cost function, $TRC_{1\cdot 2}$, satisfy the sufficient condition for minimum as shown in Lemma 2 and the explanation before it.

The necessary conditions for $TRC_{1\cdot 3}$ to be minimized are $\frac{\partial TRC_{1.3}}{\partial t_w} = 0$ and $\frac{\partial TRC_{1.3}}{\partial T} = 0$
Using Equation (23), we have

$$\frac{\partial TRC_{1.3}}{\partial t_w} = \frac{1}{T}\left[\frac{D}{\beta}(h_r + c\beta)\left(e^{\beta t_w} - 1\right) + W(h_o + \alpha c)e^{-\alpha t_w} + \frac{D}{\alpha}(h_o + \alpha c)\left(1 - e^{\alpha(T - t_w)}\right)\right] = 0 \quad (40)$$

Also,

$$\frac{\partial TRC_{1.3}}{\partial T} = \frac{1}{T}\left[\frac{D}{\alpha}\left((h_o + \alpha c)\left(e^{\alpha(T-t_w)} - 1\right) + (1-\gamma)cI_p\left(e^{\alpha(T-M)} - 1\right)\right) - pI_e D((\gamma - 1)N + M - T) - TRC_{1.3}\right] = 0 \quad (41)$$

To obtain t_w^{13*} and T_{13}^* the optimal solutions of equations (40) and (41) respectively, we use the optimization technique, Newton – Raphson method.
To prove that t_w^{13*} and T_{13}^* exist and unique, we show they satisfy the sufficient condition for minimum, that is, we show the determinant of Hessian matrix H evaluated at the point (t_w^{13*}, T_{13}^*) is positive definite.
Differentiating Equation (23) twice with respect to t_w and evaluating at the point (t_w^{13*}, T_{13}^*) we see that

$$\left.\frac{\partial^2 TRC_{1.3}}{\partial t_w^2}\right|_{(t_w^{13*}, T_{13}^*)} = \frac{1}{T}\left[D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + \alpha c)e^{-\alpha t_w} + D(h_o + \alpha c)e^{\alpha(T - t_w)}\right]\Big|_{(t_w^{13*}, T_{13}^*)} >$$

$$\frac{1}{T}D(h_o + \alpha c)e^{\alpha(T-t_w)}\Big|_{(t_w^{13*}, T_{13}^*)} > 0 \quad (42)$$

If and only if

$$D\left((h_r + \beta c)e^{\beta t_w}\right) - \alpha W(h_o + \alpha c)e^{-\alpha t_w} > 0 \ \ \forall t_w$$

Lemma 3: Given $D > \alpha W$ then
$D(h_r + c\beta)e^{\beta t} - \alpha W(h_o + \alpha c)e^{-\alpha t} > 0 \ \ if \ t > 0$
Proof
Let $g(t) = D(h_r + c\beta)e^{\beta t} - \alpha W(h_r + c\beta)e^{-\alpha t}$ for all $t > 0$
$g(0) = (D - \alpha W)(h_r + c\beta) > 0$ From assumption in 2(d)

$$g'(t) = \beta D(h_r + c\beta)e^{\beta t} + \alpha^2 W(h_r + \alpha c)e^{-\alpha t} > 0$$

Since $g'(t) > 0$, for all $t > 0$, we can conclude that $g(t)$ is an increasing function of t.
Using assumption 2(c),
$\Longrightarrow D(h_r + c\beta)e^{\beta t} - \alpha W(h_o + \alpha c)e^{-\alpha t} > 0$ proved□

Differentiating Equation (23) twice with respect to T, and evaluating at the point (t_w^{13*}, T_{13}^*) we see that

$$\left.\frac{\partial^2 TRC_{1.3}}{\partial T^2}\right|_{(t_w^{13*}, T_{13}^*)} = \frac{1}{T}\left(D\left((h_o + \alpha c)e^{\alpha(T-t_w)} + (1-\gamma)cI_p e^{\alpha(T-M)} + pI_e\right)\right)\Big|_{(t_w^{13*}, T_{13}^*)} > 0 \quad (43)$$

Using Equation (23), after simplification we find that

$$\left.\frac{\partial^2 TRC_{1.3}}{\partial t_w \partial T}\right|_{(t_w^{13*}, T_{13}^*)} = -\frac{1}{T}\left(D\left((h_o + \alpha c)e^{\alpha(T-t_w)}\right)\right)\Big|_{(t_w^{13*}, T_{13}^*)} = \left.\frac{\partial^2 TRC_{1.3}}{\partial T \partial t_w}\right|_{(t_w^{13*}, T_{13}^*)} \quad (44)$$

Using (42), (43) and (44), we find that the Hessian matrix has determinant greater than zero, i.e. $\left\{\frac{\partial^2 TRC_{1.3}}{\partial t_w^2}\frac{\partial^2 TRC_{1.3}}{\partial T^2} - \frac{\partial^2 TRC_{1.3}}{\partial t_w \partial T}\frac{\partial^2 TRC_{1.3}}{\partial T \partial t_w}\right\}\Big|_{(t_w^{13*}, T_{13}^*)} > 0$ and so is positive definite. Proved□

Theorem 3: The cost function, $TRC_{1\cdot 3}$, represented by Equation (23) is a convex function.
Proof.
The proof follows from the fact that the solutions to the cost function, $TRC_{1\cdot 3}$, satisfy the sufficient condition as shown in Lemma 3 and the explanation before it.
The necessary conditions for $TRC_{1\cdot 4}$ to be minimized are $\frac{\partial TRC_{1.4}}{\partial t_w} = 0$ and $\frac{\partial TRC_{1.4}}{\partial T} = 0$
Using Equation (26), we have

$$\frac{\partial TRC_{1.4}}{\partial t_w} = \frac{1}{T}\left[\frac{D}{\beta}\left((h_r + c\beta)\left(e^{\beta t_w} - 1\right)\right) + W\left((h_o + \alpha c)e^{-\alpha t_w}\right) + \frac{D}{\alpha}(h_o + \alpha c)(1 - e^{\alpha(T-t_w)})\right] = 0 \quad (45)$$

Also,

$$\frac{\partial TRC_{1.4}}{\partial T} = \frac{1}{T}\left[\frac{D}{\alpha}\left((h_o + \alpha c)(e^{\alpha(T-t_w)} - 1)\right) - pI_e D(M - T - 2(1-\gamma)N) - TRC_{1.4}\right] = 0 \quad (46)$$

To obtain solutions t_w^{14*} and T_{14}^* to equations (45) and (46) respectively which are non – linear, we use the optimization technique Newton – Raphson Method.

To prove that t_w^{14*} and T_{14}^* exist and unique, we show they satisfy the sufficient conditions for minimum, that is, we show the determinant of Hessian matrix *H* evaluated at the point (t_w^{14*}, T_{14}^*) is positive definite.
Taking the derivative of (26) twice with respect to t_w and evaluating at the point (t_w^{14*}, T_{14}^*) we see that

$$\left.\frac{\partial^2 TRC_{1.4}}{\partial t_w^2}\right|_{(t_w^{14*}, T_{14}^*)} = \frac{1}{T}\left(D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + \alpha c)e^{-\alpha t_w} + D(h_o + \alpha c)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{14*}, T_{14}^*)} > \frac{1}{T}D\big(h_o + \alpha c + 1 - \gamma cIpe\alpha T - twtw14*, T14*>0 \qquad (47)$$

If and only if
$D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + \alpha c)e^{-\alpha t_w} > 0, \ \forall\, t_w$

Lemma 4: Given that $D > \alpha W$ then
$D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + \alpha c)e^{-\alpha t_w} > 0, \ \forall\, t_w$
Proof
The proof is obvious from Lemma 3□

Differentiating (26) twice with respect to T, and evaluating at the point (t_w^{14*}, T_{14}^*) we see that

$$\left.\frac{\partial^2 TRC_{1.4}}{\partial T^2}\right|_{(t_w^{14*}, T_{14}^*)} = \frac{1}{T}\left(D\left((h_o + \alpha c)e^{\alpha(T-t_w)} + pI_e\right)\right)\Big|_{(t_w^{14*}, T_{14}^*)} > 0 \qquad (48)$$

Using Equations (26), and after simplification we find that

$$\left.\frac{\partial^2 TRC_{1.4}}{\partial t_w \partial T}\right|_{(t_w^{14*}, T_{14}^*)} = -\frac{1}{T}\left(D\left((h_o + \alpha c)e^{\alpha(T-t_w)}\right) + \frac{\partial TRC_{1.4}}{\partial t_w}\right)\Big|_{(t_w^{14*}, T_{14}^*)} = \left.\frac{\partial^2 TRC_{1.4}}{\partial T \partial t_w}\right|_{(t_w^{14*}, T_{14}^*)} \qquad (49)$$

Using (47), (48) and (49), we found that the Hessian matrix has determinant greater than zero, i.e. $\Big\{\frac{\partial^2 TRC_{1.4}}{\partial t_w^2}\frac{\partial^2 TRC_{1.4}}{\partial T^2} - \partial 2TRC1.4\partial tw\partial T\partial 2TRC1.4\partial T\partial twtw14*, T14*>0$ is positive definite. Proved□

Theorem 4: The cost function, $TRC_{1.4,}$ represented by Equation (26) is a convex function.
Proof.
The proof follows from the fact that the solution to the cost function, $TRC_{1.4,}$ satisfy the sufficient condition for minimum as shown in Lemma 4 and the explanation before it.

The necessary conditions for TRC_2 to be minimized are $\frac{\partial TRC_2}{\partial t_w} = 0$ and $\frac{\partial TRC_2}{\partial T} = 0$
Using Equation (29), we get

$$\frac{\partial TRC_2}{\partial t_w} = \frac{1}{T}\left[\frac{D}{\beta}\left((h_r + c\beta)\left(e^{\beta t_w} - 1\right) + cI_p\left(e^{\beta(t_w - M)} - 1\right)\right) + W\left(\left(h_o + \alpha c + cI_p\right)e^{-\alpha t_w}\right) + \frac{D}{\alpha}\left(\left(h_o + \alpha c + cI_p\right)(1 - e\alpha(T-tw)=0 \qquad (50)$$

Also,

$$\frac{\partial TRC_2}{\partial T} = \frac{1}{T}\left[\frac{D}{\alpha}\left(\left(h_o + \alpha c + cI_p\right)\left(e^{\alpha(T-t_w)} - 1\right)\right) - TRC_2\right] = 0 \qquad (51)$$

To obtain the optimal solutions t_w^{2*} and T_2^* to equations (50) and (51) respectively, we use the optimization technique Newton – Raphson method.
To prove that t_w^{2*} and T_2^* exist and unique, we show they satisfy the sufficient condition for minimum, that is, we show the determinant of Hessian matrix *H* evaluated at the point (t_w^{2*}, T_2^*) is positive definite.
Differentiating (29) twice with respect to t_w and evaluating at the point (t_w^{2*}, T_2^*) we see that

$$\left.\frac{\partial^2 TRC_2}{\partial t_w^2}\right|_{(t_w^{5*}, T_5^*)} =$$

$$\frac{1}{T}\left(D\left((h_r + c\beta)e^{\beta t_w} + cI_p e^{\beta(t_w - M)}\right) - \alpha W\left(h_o + \alpha c + cI_p\right)e^{-\alpha t_w} + D\left(h_o + \alpha c + cI_p\right)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{2*}, T_2^*)} > \frac{1}{T}D\big(h_o + \alpha c + cIpe\alpha T - twtw2*, T2*>0 \qquad (52)$$

If and only if
$D\left((h_r + c\beta)e^{\beta t_w} + cI_p e^{\beta(t_w - M)}\right) - \alpha W\left(h_o + \alpha c + cI_p\right)e^{-\alpha t_w} > 0, \qquad \forall t_w$

Lemma 5: given that $D > \alpha W$ and $h_r - h_o > c(\alpha - \beta)$ then

$$D\left((h_r + c\beta)e^{\beta t_w} + cI_p e^{\beta(t_w - M)}\right) - \alpha W\left(h_o + \alpha c + cI_p\right)e^{-\alpha t_w} > 0$$

Proof.

The proof follows from Lemma 1.

Taking second derivative of (29) with respect to T, and evaluating at the point (t_w^{2*}, T_2^*)we see that

$$\left.\frac{\partial^2 TRC_2}{\partial T^2}\right|_{(t_w^{2*}, T_2^*)} = \frac{1}{T}\left(D\left(\left(h_o + \alpha c + cI_p\right)e^{\alpha(T - t_w)}\right)\right)\Big|_{(t_w^{2*}, T_2^*)} > 0 \qquad (53)$$

Using Equation (29), after simplification we find that

$$\left.\frac{\partial^2 TRC_2}{\partial t_w \partial T}\right|_{(t_w^{2*}, T_2^*)} = -\frac{1}{T}\left(D\left(\left(h_o + \alpha c + cI_p\right)e^{\alpha(T - t_w)}\right)\right)\Big|_{(t_w^{2*}, T_2^*)} = \left.\frac{\partial^2 TRC_2}{\partial T \partial t_w}\right|_{(t_w^{2*}, T_2^*)} \qquad (54)$$

Using (52), (53) and (54), we found that the Hessian matrix has determinant greater than zero, i.e. $\Big\{\frac{\partial^2 TRC_2}{\partial t_w^2}\frac{\partial^2 TRC_2}{\partial T^2} -$ *∂2TRC2∂tw∂T∂2TRC2∂T∂twtw2∗, T2∗>0* implying that it is positive definite. Proved□

Theorem 5: The cost function, $TRC_{2,}$ represented by Equation (29) is a convex function.

Proof.

The proof follows from the fact that the solutions to the cost function, $TRC_{2,}$ satisfy the sufficient condition as in Lemma 5 and the explanation before it.

5. **Numerical Example**

Example: we consider the situation when we have the following inventory parameters,
$A = 1500$, $D = 2000$, $W = 100$, $c = 10$, $p = 15$, $h_r = 3$, $h_o = 1$, $\beta = 0.06$, $\alpha = 0.1$, $\gamma = 0.6$, $M = 0.5$, $N = 0.25$, $I_e = 0.12$, $I_p = 0.15$.

Using the Newton-Raphson iterative method for multivariable non-constraints optimization problem, we get

Empirical Solution Table 5.1

Cases	t_w	T	$f_i(t_w, T)$
Case 1.1	0.28	0.70	3023.1
Case 1.2	0.48	1.27	5711.4
Case 1.3	0.17	0.53	2942.8
Case 1.4	0.20	0.61	3116.4
Case 2	0.45	0.5	1859.1

Therefore, $t_w^* = 0.17, T^* = 0.53$ and $f_i(t_w, T) = 2942.8$

Sensitivity Analysis

The sensitivity analysis is given in Table 5.2 below by adjusting the values of the ordering cost, A, the demand, D, and the capacity of the own – warehouse, W. the values

Discussion of the result:

a. In all the cases, when the ordering cost, A, the demand, D and the capacity of the own warehouse are increased, the cost function also increase and vice versa except in case 1.3, when the cost function decrease while D and W are increased.

b. The effect of each of A, W and D is very significant in the study as any slight change will either result in increase or decrease of the cost function.

Sensitivity Analysis on the optimal solution Table 5.2

Cases	Parameters	Values of the major parameters	t_w	T	$f_i(t_w, T)$
Case 1.1	A	1000	0.58	0.63	3153.0
		1500	0.28	0.70	3023.1
		2000	0.33	0.83	3583.3
	D	1500	0.32	0.83	2709.8
		2000	0.28	0.70	3023.1
		2500	0.58	0.62	4350.1
	W	50	0.29	0.71	2982.0
		100	0.28	0.70	3023.1
		150	0.58	0.65	3918.2
Case 1.2	A	1000	0.48	1.26	5316.6
		1500	0.48	1.27	5711.4
		2000	0.48	1.27	6106.9
	D	1500	0.48	1.27	4598.6
		2000	0.48	1.27	5711.4
		2500	0.48	1.26	6825.6
	W	50	0.48	1.26	5676.3
		100	0.48	1.27	5711.4
		150	0.48	1.27	5750.1
Case 1.3	A	1000	0.17	0.53	2002.2
		1500	0.17	0.53	2942.8
		2000	0.25	0.73	3565.0
	D	1500	0.25	0.73	2689.3
		2000	0.17	0.53	2942.8
		2500	0.20	0.59	2932.9
	W	50	0.23	0.67	2814.4
		100	0.17	0.53	2942.8
		150	0.22	0.67	2881.0
Case 1.4	A	1000	0.19	0.56	2267.8
		1500	0.20	0.61	3116.4
		2000	0.20	0.61	3936.1
	D	1500	0.20	0.61	2968.3
		2000	0.20	0.61	3116.4
		2500	0.20	0.60	3261.6
	W	50	0.20	0.61	3083.9
		100	0.20	0.61	3116.4
		150	0.20	0.61	3148.9
Case 2	A	1000	0.49	0.97	1436.5
		1500	0.45	0.50	1859.1
		2000	0.49	0.97	2467.4
	D	1500	0.49	0.97	1877.8
		2000	0.45	0.50	1859.1
		2500	0.45	0.49	1546.4
	W	50	0.45	0.47	1842.5
		100	0.45	0.50	1859.1
		150	0.45	0.52	1881.9

5. Summary, Conclusion and Recommendation

In the study, partial downstream trade credit financing were considered when the customers are assumed to be not credit worthy to reduce the effect of failure in payments. The sensitivity analysis were carried out on the obtained result. It is recommended that this work be looks at when the retailers are also credit – risk. Ramp type demand and linear trend in demand can also be look at to observe the effect of demand especially for newly introduced items for this kind of models.

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