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ANALYTICAL SOLUTION OF A FLUID DYNAMIC TRAFFIC FLOW MODEL EQUATION WITH AN INITIAL CONDITION

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Abstract

The Greenberg Traffic flow model was modified and used to obtain a characteristics curve. We obtained the density when the flow will be maximized and also an equation for the maximum speed and flux, using our modified model.

Keywords: Traffic flow model, Characteristics curve, Nonlinear velocity, Density function, Macroscopic models, Characteristics speed and Maximum flow.

1.0 INTRODUCTION

The well celebrated macroscopic models introduced in the 1950s by the work of Lighthill and Witham and Richard, popularly abbreviated by LWR, have seen an extensive attention over the years [1,2]. These models are first-order nonlinear partial differential equations that describe traffic flow by the use of the relationship between the average quantities, like density and speed. Accordingly, the LWR model is mainly used for its simplicity since the density is the only dependend variable. Like other dynamic flow models, the LWR model is based on mass conservation, i.e traffic conservation, [3]. However, it was discovered that the LWR model cannot explain the amplifications of small disturbances on heavy traffic. This is due to the fact that stop and go characteristics of traffic would require the relative vehicle velocities to allow some fluctuation. Realistically, not all vehicles travel at equilibrium velocity[4].

The Payne-Whitham model (P-W) is a two equation model which is based on the properties of the flow of gas particles[5,6]. This model is an improvement over the LWR model because it can describe the amplifications of small disturbances in heavy traffic and allow fluctuations of speed around the equilibrium values. The forcing term of the model, is an acceleration term that forces vehicle velocity towards equilibrium. This happens because when the traffic flow velocity is greater than the equilibrium velocity, the forcing term forces the traffic flow to decelerate. Likewise, when the traffic flow velocity is less than the equilibrium velocity, the forcing term forces the flow to accelerate. Thus the PW model is capable of modelling the formation of vehicle clusters. However, in the PW model, there exist a characteristic speed that is greater than the macroscopic flow velocity. This means that the future traffic conditions of a traffic flow will be affected by the traffic conditions behind the flow. This violates a fundamental principle of traffic flow, in that, vehicles only travel in one direction and respond only to frontal stimuli [4].

2.0 FORMULATION OF THE MODEL EQUATION

Greenberg [7], proposed a single traffic flow model equation in which velocity is set to be inversely related to density. The model assumes that velocity of the flow can be very large for a low density with a traffic flow rate (flux) q(x,t), traffic speed v(x,t) and traffic density ρ(x,t) all of which are functions of space, $x \in R$ and time, $t \in R^+$. The model is expressed as;

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0 \qquad (1)$$

where the flux is given by

$$q(\rho) = \rho V(\rho) \qquad (2)$$

with velocity as

$$V(\rho) = v_{max} \ln\left(\frac{\rho_{max}}{\rho}\right), \qquad 0 \le \rho \le \rho_{max} \qquad (3)$$

where v_{max} and and ρ_{max} are constants

In our investigation, we obtained a nonlinear velocity-density relationship as

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$$V(\rho) = v_{max}\ \ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right], \qquad 0 \le \rho \le \rho_{max} \tag{4}$$

If in (4), we let

$$\alpha_1 = \frac{\rho_{max}}{\rho} \tag{5}$$

It implies that $\left(\frac{\rho_{max}}{\rho}\right)^2 = {\alpha_1}^2$, therefore we have

$$\alpha_2 = \frac{1}{2}\alpha_1^2 \tag{6}$$

and differentiating (6) with respect to α_1, we obtain

$$\frac{d\alpha_2}{d\alpha_1} = \alpha_1 \tag{7}$$

Equation (7) shows the following

(i) a refinement of the Greenberg's natural logarithm speed-density relation (3),

(ii) a rate of change of the Greenberg's speed-density function and it is also the slope of equation (3) i.e. $\frac{d\alpha_2}{d\alpha_1} = \alpha_1$.

Now from (2) and (4), we have

$$q(\rho) = \rho\left[v_{max}\ \ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right]\right] \qquad 0 \le \rho \le \rho_{max} \tag{8}$$

Therefore equation (1) becomes a nonlinear first order partial differential equation written as

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}\left[\rho v_{max}\ \ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right)\right] = 0 \tag{9}$$

Where the velocity-density relationship (4) satisfies the following qualitative physical properties

$$V(\rho = 0) = V_{max} \tag{10}$$

$$V(\rho_{max}) = 0 \tag{11}$$

In the same vain, the flux-density relationship (2) satisfies the following qualitative properties

$$q(\rho = 0) = 0 \tag{12}$$

$$q(\rho_{max}) = \rho_{max} V(\rho_{max}) = 0 \tag{13}$$

The solution of the non-linear PDE (9), can be obtained if we know the traffic density at a given initial time, i.e. given the traffic density at initial time t_0, we can predict the traffic density for all future time $t \ge t_0$, hypothetically.

Hence, we consider the non-linear PDE (9) as an initial value problem (IVP) of the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}\left[\rho v_{max}\ \ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right)\right] = 0 \tag{14}$$

With initial condition

$$\rho(t_0, x) = \rho_0(x) \tag{15}$$

where $\rho(x,t)$ represents the traffic density which we are interested in, v_{max} is the maximum traffic speed in the positive x direction, and $\frac{\partial\rho}{\partial t}$ represents the time dependence of the traffic density ρ.

3.0 SOLUTION OF THE MODEL EQUATION BY CHARACTERISTIC MTHOD

We now solve the IVP (14) and (15).

Let $q(\rho) = \rho v_{max}\ \ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right]$

$$= v_{max}\left\{\rho\ \ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right]\right\} \tag{16}$$

If m= $\ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right]$ (17)

and u= $\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2$ (18)

then $m = lnu$ (19)

and $\frac{dm}{du} = \frac{1}{u}$ (20)

$$\frac{du}{d\rho} = -\frac{\rho_{max}^2}{\rho^3} \tag{21}$$

$$\Rightarrow \frac{dm}{d\rho} = -\frac{2}{\rho} \tag{22}$$

therefore, $\frac{dq}{d\rho} = v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right]$ (23)

which is the rate of change of the flux and is called the characteristics speed.
Differentiating $q(\rho) = q[\rho(x,t)]$ partially with respect to x and substitute the result into (1), we get

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0 \tag{24}$$

Substituting equation (23) into (24), we obtain

$$\frac{\partial \rho}{\partial t} + v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right]\frac{\partial \rho}{\partial x} = 0 \tag{25}$$

Again, differentiating $q(\rho) = q[\rho(x,t)]$ partially with respect to t, we get

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} = 0 \tag{26}$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} = 0 \tag{27}$$

and

$$\frac{dx}{dt} = -\frac{\partial \rho}{\partial t}\frac{\partial x}{\partial \rho} \tag{28}$$

From equation (25), we have

$$v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right] = -\frac{\partial \rho}{\partial t}\frac{\partial x}{\partial \rho} \tag{29}$$

Equating equation (28) and (29), we have

$$\frac{dx}{dt} = v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right] \tag{30}$$

Integrating equation (30) with respect to t, we get

$$x(t) = v_{max}\int\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right]\mathrm{dt}$$

$$\Rightarrow \quad x(t) = v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right]t + A(x) \tag{31}$$

where A(x) is the constant of integration.
Since the problem under consideration is an IVP, we now consider when $t = 0$ in equation (31), then we shall have

$$x(0) = x_0 = A(x) \tag{32}$$

$$\therefore x(t) = v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right]t + x_0, \text{ where } A(x) = x_0 \tag{33}$$

Equation (33) is called the characteristics curve of the IVP (14) and (15)
Next, we shall solve for the density, $\rho(x,t)$.
From equation (26),

$$\frac{d\rho}{dt} = 0$$

$$\therefore \rho(x,t) = c, \text{ where c = constant} \tag{34}$$

Since the characteristics $x(t)$, passes through any point (x,t), it therefore passes through $(x_0, 0)$ and $\rho(x,t) = c$ is constant on this curve, then

$$c = \rho(x,t) = \rho(x_0, 0) = \rho_0(x_0) \tag{35}$$

Making x_0 the subject from equation (33), we have

$$x_0 = x(t) - v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right]t \tag{36}$$

and also from equation (35), we have

$$\rho(x,t) = \rho_0(x_0) \tag{37}$$

Substituting equation (36) into (37), we have

$$\rho(x,t) = \rho_0\left(x(t) - v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right]t\right) \tag{38}$$

This is the analytic solution of the IVP (14) and (15) which is in its implicit form.
From our modified model,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[\rho v_{max}\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right)\right] = 0$$

Recall that

$$q(\rho) = \rho V(\rho) \tag{39}$$

$$\text{where } V(\rho) = v_{max}\ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right] \tag{40}$$

therefore $q(\rho) = \rho v_{max} \ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right]$ (41)

when the flow is at maximum, $\frac{dq}{d\rho} = 0.$

$$\frac{dq}{d\rho} = v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right] = 0$$

$$\Rightarrow \ v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right) - 2\right] = 0$$

which reduces to $\rho = \frac{\rho_{max}}{\sqrt{2e^2}}$ (42)

Equation (42) is the density when the flow will be maximized. Substituting (42) into (40) yields the speed at which the flow is maximum, i.e

$$V(\rho) = v_{max} \ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\frac{\rho_{max}}{\sqrt{2e^2}}}\right)^2\right]$$

$$= v_{max} \ln\left[\frac{1}{2}\left(\rho_{max} \times \frac{\sqrt{2e^2}}{\rho_{max}}\right)^2\right] \quad (43)$$

$$V(\rho) = v_{max} \ln(e^2) \quad (44)$$

This indicates that the maximum flow occurs when traffic is flowing at twice its maximum speed (v_{max}). Substituting the maximum speed and density into speed-flow-density relation (39), yields the maximum flow.

$$q(\rho) = \frac{\rho_{max}\, v_{max} \ln(e^2)}{\sqrt{2e^2}} \quad (45)$$

4.0 Conclusion

We extended the Greenberg model by modifying the speed-density function. The analytical solution of the extended model was given using the method of characteristics, which gives the solution in its implicit form.

We recommend the use of numerical method, for the numerical solution of the IVP (14) and (15). We shall therefore take up this challenge in our next paper.

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