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TWO HYBRID POINTS BLOCK METHODS FOR THE SOLUTION OF INITIAL VALUE PROBLEMS

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Abstract

Two hybrid points methods are developed for solving initial value problems using collocation and interpolation technique. Polynomial approximate solution is considered and the resulted methods implemented in block. The stability properties of the new block methods were established and its efficiency tested on some numerical examples with a written program using MATLAB 8.5. The results show that, the new block methods are efficient for the solution of initial value problems of first order ordinary differential equations.

Keywords**:** Hybrid point, Block method, Collocation, Interpolation, Initial Value Problems *AMS Subject Classification*: 65L04, 65L05

1.0 INTRODUCTION

This paper considers the numerical solution to initial value problem for ordinary differential equations (ODEs) of first order of the form

$$y' = f(x, y), \quad y(x_0) = y_0, \; x \in [x_n, x_N] \qquad (1.1)$$

where $f : [x_n, x_N] \times \mathrm{R}^m \to \mathrm{R}^m$, $y_0 \in \mathrm{R}^m$ is continuous and differentiable. However, f is assumed to satisfy the existence and uniqueness theorem [1].

Differential equation (1.1) arises in wide of fields of sciences and engineering, such as physics, economics, medicine and engineering. Equation (1.1) can be solved using hybrid block methods which allow the numerical approximation to be computed at more than one point at the same time while avoiding the computational burden and the zero stability barrier [2].

Hybrid one step second derivative method considered in this paper, for the solution of (1.1) can be written in the form

$$\sum_{j=0}^{1} \alpha_j y_{n+j} + \alpha_k y_{n+k} = h\left[\sum_{j=0}^{1} \beta_j f_{n+j} + \beta_k f_{n+k}\right] + h^2\left[\sum_{j=0}^{1} \gamma_j g_{n+j} + \gamma_k g_{n+k}\right] \qquad (1.2)$$

where $k \in [0,1]$ is a rational number.

An ordinary differential equation problem is stiff if the solution being sought is varying slowly, but there are nearby solutions that vary rapidly, so the numerical method must take small steps to obtain satisfactory results [3].

Block method was introduced by Milne in 1953 to generate starting values for predictor corrector Linear Multistep Method (LMM) [4] and since then, several block methods have been developed by researchers such as [2-7], etc.

According to [4], hybrid technique was used independently by [8-10]. The beauty of this method, which was named Hybrid methods" by Gear in 1964 is that, while retaining certain characteristics of CLMMs, hybrid methods share with Runge-Kutta methods, the property of utilizing data at off-step points and the flexibility of changing step length.

This paper therefore, is concerned with the development of block methods while considering two hybrid points v, u for the solution of stiff IVPs ODEs where $v = 3u$.

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2.0 METHODOLOGY

2.1 Mathematical Background [11]

We consider approximating the exact solution $y(x)$ to (1.1) on the partition $\pi[a,b]=[a=x_0<x_1<...<x_n<x_{n+1}<...<x_N=b]$ of the integration interval $[a,b]$ by polynomial approximate solution of the form

$$y(x)=\sum_{n=0}^{k}a_n x^n \tag{2.1}$$

where $x\in[a,b]$, $a_n\in\mathrm{R}$ are the unknown parameters to be determined. The first and second derivative of (2.1) respectively gives

$$y'(x)=\sum_{n=1}^{k}na_n x^{n-1} \tag{2.2}$$

$$y''(x)=\sum_{n=2}^{k}n(n-1)a_n x^{n-2} \tag{2.3}$$

Evaluating (2.2) and (2.3) at $x=x_{n+j}$, $j=0,1,...,k$ gives a system of nonlinear equation in the form

$$XA=U \tag{2.4}$$

where

$$A=[a_0,\ a_1,\ ...,\ a_k],\ U=\left[y_n,\ y_{n+v_1},\ ...,\ y_{n+v_m},\ \ y'_n,\ y'_{n+v_1}...,\ y'_{n+v_m},\ \ y''_n,\ y''_{n+v_1}...,\ y''_{n+v_m}\right]^T$$

Imposing the following conditions on (2.1)

$$y(x_{n+v_i})=y_{n+v_i},\quad i=0,1,...m$$
$$y'(x_{n+v_i})=f_{n+v_i},\quad i=0,1,..,m$$
$$y''(x_{n+v_i})=g_{n+v_i},\quad i=0,1,..,m$$

$$X=\begin{bmatrix}1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^k\\ 1 & x_{n+v_1} & x_{n+v_1}^2 & x_{n+v_1}^3 & \ldots & x_{n+v_1}^k\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_{n+v_m} & x_{n+v_m}^2 & x_{n+v_m}^3 & \cdots & x_{n+v_m}^k\\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & kx_n^{k-1}\\ 0 & 1 & 2x_{n+v_1} & 3x_{n+v_1}^2 & \ldots & kx_{n+v_1}^{k-1}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 1 & 2x_{n+v_m} & 3x_{n+v_m}^2 & \ldots & kx_{n+v_m}^{k-1}\\ 0 & 0 & 2 & 6x_n & \ldots & k(k-1)x_n^{k-2}\\ 0 & 0 & 2 & 6x_{n+v_1} & \ldots & k(k-1)x_{n+v_1}^{k-2}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 2 & 6x_{n+v_m} & \ldots & k(k-1)x_{n+v_m}^{k-2}\end{bmatrix}$$

where $v_i\in(x_0,x_{n+m}),\ i=1,2,...,m.$ Equation (2.4) is solved for the unknown parameters using Crammer's rule, the results obtained is then substituted into (2.1), after some algebraic simplifications gives Continuous Block Method of the form

$$\begin{aligned}y_{n+t}&=\sum_{j=0}^{r}\alpha_j(t)y_{n+j}+\sum_{v_i}\alpha_{v_i}(t)y_{n+v_i}\\&+h\left[\sum_{j=0}^{s}\beta_j(t)f_{n+j}+\sum_{v_i}\beta_{v_i}(t)f_{n+v_i}\right]+h^2\left[\sum_{j=0}^{\tau}\sigma_j(t)g_{n+j}+\sum_{v_i}\sigma_{v_i}(t)g_{n+v_i}\right]\end{aligned} \tag{2.5}$$

where $\alpha_j(t),\ \beta_j(t),$ and $\sigma_j(t)$ are polynomial of degree $r+s+\tau-1$ and $t=\dfrac{x-x_{n+v_i}}{h}$.

2.2 Block Methods

Evaluate (2.1) at x_n, evaluate (2.2) at $[x_n,\ x_{n+v},\ x_{n+u},\ x_{n+1}]$ where $v=3u$ and evaluate (2.3) at $[x_{n+v},\ x_u,\ x_{n+1}]$ for the continuous scheme, evaluate the continuous scheme at $t=v,u,1$ to give the discrete schemes which is written in block form as;

$$A^{(1)}Y_{m+1} = A^{(0)}Y_m + hB^{(0)}F_m + hB^{(1)}F_{m+1} + h^2C^{(1)}G_{m+1} \tag{2.6}$$

where

$Y_{m+1} = [y_{n+v} \;\; y_{n+u} \;\; y_{n+1}]^T, Y_m = [y_{n-1} \;\; y_{n-2} \;\; y_n]^T,$

$F_{m+1} = [f_{n+v} \;\; f_{n+u} \;\; f_{n+1}]^T, F_m = [f_{n-1} \;\; f_{n-2} \;\; f_n]^T, G_{m+1} = [g_{n+v} \; g_{n+u} \; g_{n+1}]^T,$

For Case I: $v = \frac{2}{9}, u = \frac{2}{3}$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A^{(0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; B^{(0)} = \begin{bmatrix} 0 & 0 & \frac{38984}{688905} \\ 0 & 0 & \frac{184}{2835} \\ 0 & 0 & \frac{113}{1680} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} \frac{124591}{864360} & \frac{337}{29160} & \frac{2347456}{236294415} \\ \frac{32967}{96040} & \frac{9}{40} & \frac{32576}{972405} \\ \frac{544563}{1536640} & \frac{261}{640} & \frac{6142}{36015} \end{bmatrix}, \; C^{(1)} = \begin{bmatrix} -\frac{28873}{1666980} & -\frac{6179}{918540} & -\frac{31456}{33756345} \\ -\frac{3}{6860} & -\frac{43}{1260} & -\frac{416}{138915} \\ \frac{243}{109760} & -\frac{33}{2240} & -\frac{89}{10290} \end{bmatrix}$$

Case II: $v = \frac{3}{11}, u = \frac{9}{11}$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A^{(0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; B^{(0)} = \begin{bmatrix} 0 & 0 & \frac{2587}{38115} \\ 0 & 0 & \frac{3489}{46585} \\ 0 & 0 & \frac{5737}{76545} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} \frac{132963}{788480} & -\frac{8537}{55440} & \frac{1649727}{8673280} \\ \frac{312579}{788480} & -\frac{1329}{6160} & \frac{53675541}{95406080} \\ \frac{767987}{1935360} & -\frac{142417}{1224720} & \frac{46199}{71680} \end{bmatrix}, \; C^{(1)} = \begin{bmatrix} -\frac{61497}{2168320} & -\frac{4427}{135520} & -\frac{26001}{2168320} \\ -\frac{16281}{2168320} & -\frac{16011}{135520} & -\frac{807003}{23851520} \\ -\frac{10769}{1451520} & -\frac{31097}{272160} & -\frac{1931}{53760} \end{bmatrix}$$

3.0 STABILITY ANALYSIS

3.1 Order and error constant of the block methods

Evaluating each row of (2.6) in a Taylor series about x_n gives

$$L[y(x);h] = y_{n+t} - \sum_{j=0}^{r} \alpha_j(t) y_{n+j} - h\sum_{j=0}^{s} \beta_j(t) f_{n+j} - h^2 \sum_{j=0}^{\tau} \sigma_j(t) g_{n+j} \tag{3.1}$$

$h^{p+n} \neq 0, \; p + n = 8$

where n is the order of the differential equation.

Therefore, the order of the Block Methods are $p = [7,7,7,7]^T$ and $p = [7,7,7,7]^T$ with error constants

$$\left[\frac{16276}{1423770297075}h^8, \frac{52}{1953045675}h^8, \frac{109}{3086294400}h^8\right]^T \tag{3.2}$$

and

$$\left[\frac{1261629}{33611472540800}h^8, \frac{220887}{3055588412800}h^8, \frac{4511}{61984137600}h^8\right]^T \tag{3.3}$$

for case I and case II respectively.

3.2 Consistency of the block methods

The order of the methods are $p = 7 > 1$, therefore, the block methods are consistent.

3.3 Zero stability of the block methods

The block methods are zero stable since the roots $z_s, s = 1,2,3,...n$ of the first characteristics polynomial $\rho(z)$ is defined by

$$\rho(\lambda) = \det\left[A^{(1)}\lambda - A^{(0)}\right] = 0 \tag{3.4}$$

$\lambda^3 - \lambda^2 = \lambda^2(\lambda - 1) = 0$

Solving for λ we have $\lambda = [0,0,1]^T$. Hence the block methods are zero stable.

3.4 Convergence
Since the block methods are consistent and zero-stable, therefore, they are convergent.

3.5 Region of absolute stability
This is achieved by substituting the test equation

$$y' = \lambda y \tag{3.5}$$

in the block formula. This gives

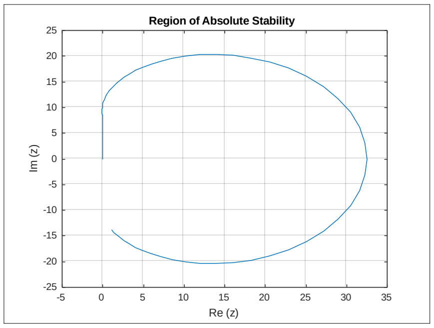

Fig. 1: Region of absolute stability of BMCI

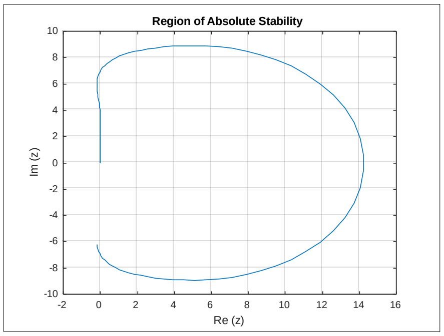

Fig. 2: Region of absolute stability of BMCII

4.0 NUMERICAL EXAMPLES
The following problems are considered to test the efficiency of the new integrators.

Problem 1 [5]

$$y' = -y \qquad y(0) = 1 \; [0, 20]$$

with exact solution

$$y(x) = e^{-x}$$

Table 1: Notations

x	Step Size
BMCI	Block Method Case I
BMCII	Block Method Case II
3PZ	3-Point implicit block method
MAXE	Maximum Error
TOL	Tolerance

Table 2: Result of problem 1

TOL	MAXEin BMC1	MAXEin BMC2	MAXEin 3PZ
10^{-2}	7.05965(-5)	7.05965(-5)	7.05966(-5)
10^{-4}	1.77717(-6)	1.77716(-6)	1.77719(-6)
10^{-6}	5.22501(-9)	5.22510(-9)	6.22500(-8)
10^{-8}	2.40196(-9)	2.40197(-9)	2.40197(-9)
10^{-10}	3.18639(-10)	2.18629(-11)	3.18638(-10)

Problem 2 [5]

$$y' = y \qquad y(0) = 1 \; [0, 20]$$

with exact solution

$$y(x) = e^{x}$$

Table 3: Result of problem 2

TOL	MAXEin BMC1	MAXEin BMC2	MAXEin 3PZ
10^{-2}	3.29766(-4)	3.29765(-4)	3.29774(-4)
10^{-4}	6.85459(-5)	6.85470(-5)	6.85574(-5)
10^{-6}	3.16301(-6)	3.16311(-6)	3.16305(-6)
10^{-8}	1.28788(-7)	1.28789(-7)	1.28837(-7)
10^{-10}	5.13399(-9)	5.14229(-9)	5.13356(-9)

Table 4: Result of problem 3

TOL	MAXEin BMC1	MAXEin BMC2	MAXEin 3PZ
10^{-2}	1.16503(-4)	1.16503(-4)	1.16502(-4)
10^{-4}	1.45174(-6)	1.45175(-6)	1.45165(-6)
10^{-6}	2.75068(-8)	2.75077(-8)	2.75066(-8)
10^{-8}	2.30887(-9)	2.30887(-9)	2.30879(-9)
10^{-10}	3.61444(-10)	3.61445(-10)	3.53662(-10)

Problem 3 [5]

$y_1' = y_2;\ y_2' = 2y_2 - y_1; \qquad y_1(0) = 0;\ \ y_2(0) = 1\ \ [0,20]$

with exact solution

$$y(x) = \begin{bmatrix} xe^x \\ (1+x)e^x \end{bmatrix}$$

5.0 CONCLUSION

Second derivative block methods are developed using collocation and interpolation technique by considering two off-step points v and u where v = 3u. Polynomial approximate solution is considered and the resulted methods implemented in block. The stability properties of the new block methods were established and found to be stable, consistent and convergent. The new block methods are of order 7 with large region of absolute stability as can be seen in Fig. 1 and Fig. 2. The efficiency of the new block methods is tested on some numerical examples with a written program using MATLAB 8.5 and the results of the new methods are shown in Table 2 to Table 4, which establish that the new methods compete effectively with the existing methods with less computational burden. The new block methods are efficient for the solution of stiff initial value problems of first order ordinary differential equations.

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