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EUCLIDEAN NULL CONTROLLABILITY OF DYNAMICAL SYSTEM WITH LUMPED, MULTIPLE CONSTANT DELAYS IN THE STATE.

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Abstract

In the study of dynamical systems with lumps, multiple delay in the state, some basic definitions of admissible control, attainable set, asymptotic stability and a system being proper in the Euclidean space were stated and went further to establish theorems concerning Euclidean Null controllability of the system under study. Finally, numerical examples were used as a means of illustration to analyze the theorem.

1. INTRODUCTION.

Control theory is the area of application-oriented mathematics that deals with the basic principles of , analysis and design of control systems. Controlling an object has to do with influencing the object's behaviour so as to achieve a desired goal. This theory was first developed to satisfy the design needs of servomechanism known as "automatic control theory". Controllability of dynamical systems is one of the key areas in control theory. Controllability of delay dynamical systems affect large areas in control theory as it is not farfetched that most physical systems comes with delay especially in the fields of science.

In this paper, we study dynamical systems with delay in the state, linear systems with lumps multiple constant delay. However we concentrate on the Euclidean null controllability of the dynamical system (1). Chukwu [1] introduced the notion of proper control systems of linear delay systems with limited controls noting that controllability is equivalent to the system being proper for delay systems with unlimited power. Beata [2] in his study of linear stationary dynamical systems with multiple constant delay in the state, discoursed relative and approximate controllability properties with constrained controls. Several authors [3-4] and [9-12] analyzed criteria for relative and approximate controllability for linear stationary dynamical systems with single delay in the state concerned mainly on unconstrained controls.

Finally, numerical examples were shown following the theorems in the paper to show Euclidean null controllability of system (1).

2. MATHEMATICAL MODEL AND PRELIMINARIES.

Considering the dynamical system with lumped, multiple constant delay in the state by an ordinary differential equation:

$$\dot{x} = \sum_{i=0}^{M} A_i x(t-h_i) + Bu(t) \qquad (1)$$

$$x(0) = x^0, \qquad u(0) = u^0 \quad t \geq 0$$

$$x(t) = \varphi(t), \quad t\epsilon[-h,0], h > 0$$

Where $x(t) \in \mathbf{R}^n$ stand for the instantaneous n-dimensional state vector, $u(t) \in L^2_{loc}([0,\infty), \boldsymbol{R}^m)$ is the control, $A_i, i = 0,1,2,3,\ldots\ldots, M$ are $(n\times n)$-dimensional matrices with real elements, B is an $(n\times m)$-dimensional matrices with real elements and $h_i, i = 0,1,2,3,\ldots.., M$ denote the constant delays satisfying the inequalities

$0 = h_0 < h_1 < h_2 < h_3 \ldots < h_{M-1} < h_M$,

with initial conditions $\varphi = (x(0), x_0) \in \mathbf{R}^n \times L^2([-h_M, 0), \mathbf{R}^n)$, where $x(0) \in \mathbf{R}^n$ is the instantaneous state vector at $t = 0$, and x_0 is a function given in the time interval $[-h_M, 0), i.e., x_0(t) = x(t)$ for $t \in [-h_M, 0)$.

The Hilbert space $\mathbf{R}^n \times L^2([-h_M, 0), \mathbf{R}^n)$ endowed with the scalar product defined by:

$\langle \{x(t), x_t\}, \{y(t), y_t\} \quad \rangle = \sum_{i=1}^{n} x_i(t) y_i(t) + \int_{-h_M}^{0} \langle x_t(\tau), y_t(\tau)\rangle_{\mathbf{R}^n} d\tau$, is denoted by $M_2([-h_M, 0], \mathbf{R}^n)$ where $x_i(\tau) = x(t+\tau)$ for $\tau \in [-h_m, 0)$ is the segment of the trajectory of length h_m which is defined in the time interval $[t-h_m, 0)$.

φ is continuous. The control u is a measurable m-vector-valued function with values $u(t)$ constrianed to lie in an m-dimensional unit cube B^m where $B^m = \{u \in E^m : |u_j| \leq 1, j = 1,2,3, \ldots m\}$. u of this form is called admissible.

Let $W_2^{(1)}$ denote the sobolev space, $W_2^{(1)}([-h, 0], E^n)$ of function

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$\varphi: [-h, 0] \to E^n$ whose derivatives are square integrable.
If $x: [-h, t_1] \to E^n$, then for $t \in [o, t_1]$, the symbol x_t denotes the function on $[-h, 0)$ defined by
$x_t(s) = x(t + s),\ s \in [-h, 0]$.

3. BASIC DEFINITIONS

Definition 1: The system (1) is Euclidean controllable if for each $\varphi \in W_2^{(1)}, x_1 \in E^n$, there exist a $t_i > 0$, and an admissible control u such that the solution $x(t, \varphi, u)$ of system (1) satisfies $x_0(\varphi, u) = \varphi$ and $x(t_1, \varphi, u) = x_1$.
The system (1) is Euclidean null-controllable if $x_1 = 0$ in the above definition.

Let $U \subset \mathbf{R}^m$ be a non-empty, convex and compact set such that $0 \in U$. Any control $u \in L^2_{loc}([0, \infty)U)$ is called an admissible control for system (1).
The pair $\varphi_1 = (x(t), x_t) \in \mathbf{R}^n \times L^2([-h_M, 0), \mathbf{R}^n) = M_2([-h_M, 0], \mathbf{R}^n)$, where $x(t) \in \mathbf{R}^n$ is the vector of the current state and $x_i(\tau) = x(t + \tau)$ for $\tau \in [-h_M, 0)$ is the segment of the trajectory of lengthh_M, that is defined in the time interval $[t - h_M, t)$, is called the complete state of system (1) for $t \geq 0$.
For a given initial condition $\varphi_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbf{R}^n)$ and an admissible control $u \in L^2([0, t], U)$, for every $t \geq 0$, there exist a unique, absolutely continuous solution $x(t, \varphi_0,, u)$ of system (1). [6] and [5].

$$x(t, \varphi_0, u) = x(t, \varphi_0, 0) + \int_0^t X(t - \tau) B u(\tau) d\tau \qquad (2)$$

Where $X(t)$ is the $(n \times n)$-dimensional transition matrix is the solution of the following linear matrix integral equation:

$$\dot{x} = \sum_{i=0}^{M} A_i x(t - h_i), \qquad (3)$$

$$X(t) = I + \sum_{i=0}^{M} \int_0^{t - h_1} X(\tau) A_i d\tau \qquad (4)$$

for $t > 0$,

$$X(t) = \begin{cases} I, & t = 0 \\ 0, & t < 0 \end{cases}$$

And $x(t, \varphi_0, 0)$ is the free solution of system (1) with zero control $u(t) = 0$ for $t \geq 0$, given by

$$x(t, \varphi_0, 0) = X(t) x(0) + \sum_{i=0}^{M} \int_{-h_i}^{0} X(t - \tau - h_i) A_i x_0 d\tau \qquad (5)$$

The solution $x(t, \varphi_0, 0)$ depends only on the initial complete state $\varphi_0 = (x(0), x_0)$.
The Set of solutions of system (1) at $t_1 > 0$ with initial conditions $\varphi_0 = (x(0), x_0) \in M_2([-h_M, 0], \mathbf{R}^n)$ with admissible control $u \in L^2([0, t_1], U)$ is called the attainable set in time $t_1 > 0$ of the dynamical system (1) from the initial complete state φ_0 with constrained control. The set is denoted by $K_U([0, t_1], \varphi_0)$.

Definition 2: The Attainable Set $K_U([0, t_1], \varphi_0)$ of system (1) from the initial complete state $\varphi_0 = (x(0), x_0)$ in time $t > 0$, $u(t) \in U$ is the set

$$K_U([0, t_1], \varphi_0) = \{x(t) \in \mathbf{R}^n : x(t) = x(t, \varphi_0, 0) + \int_0^t X(t - \tau) B u(\tau) d\tau,\ \ u \in L^2([0, t], U\} \qquad (6)$$

Note: The attainable set is convex, closed and $0 \in K_U([0, t], 0)$ for all $t \geq 0$. [1]
In equation (3), we set the matrix function $X(t - \tau)B = Y(s)$, $t \geq \tau \geq 0$ and define the reachable set of system (1) at time t by

$$IR(t) = \{\int_0^t Y(s) u(s) ds;\ u \text{ is measurable}, u(.) \in B^m\} \qquad (7)$$

Lema 1: The reachable set $IR(t)$ is symmetric, convex, and closed. Also, $0 \in IR(s)$ for each $0 \leq s$.
Definition 3: The system (1) is said to be proper in E^n on an interval $[t_0, t_1]$ if $B^* Y(s) = 0$, that is, $\tau \in [t_0, t_1]\ B \in E^n$, implies $B = 0$. If system (1) is proper on $[t_0, t_0 + \epsilon]$ for each $\varepsilon > 0$, we say that the system (1) is proper at t_0. If system (1) is proper on each interval, $[t_0, t_1],\ t_1 > t_0 \geq 0$, we say that the system is proper in E^n.
Definition 4: System (1) is said to be asymptotically stable if for any initial complete state $\varphi_0 \in M_2([-h_M, 0], \mathbf{R}^n)$ and $u = 0$, the complete state at time $t > 0$, i.e, $\varphi_t = (x(t), x_t)$ satisfies the condition
$\lim_{t \to \infty} \|\varphi_t\| M_2 = 0$, [6-7]
Theorem 1: The system (1) is asymptotically stable if and only if all the roots s_i of the quasi-characteristic equation
$\omega(s) = \det(sI - \sum_{i=0}^{M} A_i e^{-sh_i}) = 0$ of the autonomous system (1) $(u(t) \equiv 0)$ have negative real parts, i.e., $\Re[s_i] < 0$ for all $i = 1,2,3, \dots, M$. [7-8].

4. ECLUDIAN NULL CONTROLLABILITY RESULTS.

Generalising the results obtained in [1] and [5], we define the following matrix.

$$Q_n(s) = \sum_{i=0}^{M} A_i\, Q_{n-1}(s - h_i), \quad n = 1,2, \dots, s \in [0, \infty) \qquad (8)$$

$$Q_0(s) = \begin{cases} B, & s = 0 \\ 0, & s \neq 0 \end{cases} \qquad (9)$$

And define

$$\widehat{Q_n}(t_1) = \{Q_0(s), Q_1(s), Q_2(s), \dots, Q_{n-1}(s), s \in [0, t_1] \qquad (10)$$

For $s = h_i, 2h_i, 3h_i, \dots \dots,\quad i = 0,1,2,3, \dots \dots, M$ for system (1).
We define the rank of $\widehat{Q_n}(t_1)$ as the rank of block matrix composed of all matrices from the set $\widehat{Q_n}(t_1)$.

Lema 1: For every$t_1 \in (0,\infty)$, $rank\widehat{Q_n}(t_1) = n$. [1].
Theorem 2: The system (1) is proper in E^n on an interval $[0,t_1]$ if and only if

$$rank\ \widehat{Q_n}(t_1) = n \quad (11)$$

See proof in [8], [Theorem 2.3].
Theorem 3: The dynamical system (1) is proper on $[0,t_1], t_1 > 0$ if and only if the origin is an interion point of $IR(t_1)$.
Proof:
Since $IR(t_1)$ is a closed convex subset of E^n through all the points of the boundary, there exist a support plane..
Let the point q be on the boundary of the reachable set $IR(t_1)$, such that $\eta^*(p-q) < 0$ for all $p \in IR(t)$, where α is an outward normal to the support plane of $IR(t)$ through q.
If $q = y(t,u+)$, $u+$ will be of the form
$u+(s) = sgn(\eta^* Y(s))$, such that
$\eta^* \int_0^{t_1} Y(s)u(s)ds \le \int_0^{t_1} |\eta^* Y(s)|ds$ for all $u \in B^m$.
But since $0 \in IR(t)$, if $0 \notin intIR(t_1)$, then 0 is in the boundary. Hence, this is equivalent to $\int_0^{t_1}|\eta^*Y(s)|ds = 0$; meaning that $\eta^*Y(s) = 0$ and $s \in ([0,t_1])\eta \neq 0$.
Therefore, the dynamical system (1) is not proper on $[0,t_1]$.

Definition 5: The domain Φ of Euclidian null-controllability is the set of initial functions in $W_2^{(1)}$ which can be steered to the origin $0 \in E^n$ in finite time, using admissible controls.
Corollary 1: If the dynamical system (1) is proper in E^n, then Φ, the domain of Euclidean null-controllability, contains the zero in its interior.
Proof:
Since $0 \in \Phi$, and $x(t) = 0$ is a solution of system (1) with $u = 0$. We assume that system (1) is proper which means that 0 is in the interior of the reachable set $IR(t)$ for every t. On the other hand, supposing that 0 is not in the interior of Φ. Then there exist a sequence $\{x_m\}_1^\infty \subseteq B$ such that $x_m \to 0$ as $m \to \infty$ and no $x_m$is in Φ, that is $x_m \neq 0$.
From variation of parameter,
$0 \neq x(t_1, x_m, u) = x(t_1, x_m, 0) + \int_0^{t_1} Y(s)u(s)ds$ for all $t_1 \ge 0$ and any $u \in B^m$.
Therefore, $j_m \stackrel{\text{def}}{=} x(t_1, x_m, 0)$ is not in $IR(t_1)$ for any $t_1 \ge 0$. Hence the sequence $\{j_m\}_1^\infty \subseteq E^n$, $j_m \in IR(t_1) j_m \neq 0$ is such that as $j_m \to 0, m \to \infty$. This implies that 0 is not in the interior of $IR(t_1)$ for any t_1. Hence, $0 \in int\Phi$.

Main Theorem:
Assume that the dynamical system (1) is proper in E^n, and that the trivial solution of (3) is uniformly asymptotically stable. Then system (1) is Euclidean null-controllable.
Proof:
Supposing that system (1) is proper on E^n, by corollary (1), the domain Φ of null controllability contains an open ball P of finite radius around the zero function φ_0. Given the initial function $\varphi_1 \in W_2^{(1)}$. Using the null control $u(t) = 0$, the solution $x(t, \varphi_1, 0)$ of (4a) satisfies
$x_t(\varphi_1, 0) \to \varphi_0 \equiv 0$ as $t \to \infty$, and $x_{t_1}(\varphi_1, 0) \in P \subseteq \Phi$ for some finite time t_1.
Since $x_{t_1}(\varphi_1, 0)$ can be steered to $0 \in E^n$ in finite time, the dynamical system (1) is Euclidean null-Controllable.

5. EXAMPLES

Example 1. Consider a dynamical system with two delays in the state given by the equation;

$$\dot{x} = A_0 x(t-0) + A_1 x(t-h_1) + A_2 x(t-h_2) + Bu(t) \quad (12)$$

Let $A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C^* = [1\ \ 1]$, $D = 0$, $x(0) = 0$.
With control values in the m-dimensional unit hypercube $B^m = \{u(t) \in \mathbf{R}^n : |u_j| \le 1, j = 1,2,3,\dots m\}$, $t > 0$ and it is convex and compact set containing $0 \in intC^m$. Where $h_0 = 0, h_1 = 1, h_2 = 2, n = 2, m = 1, M = 2\ and\ U = C^m$.
In studying the stability of the dynamical system (12), the quasi-characteristic equation is of the form:

$$\varphi(s) = \det(sI - A_0 - \sum_{i=1}^{2} A_i e^{-sh_i}) = 0$$
$$= \det(sI - A_0 - e^{-s}A_1 - e^{-2s}A_2) = 0$$
$$= \det(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} - e^{-s}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} - e^{-2s}\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix})$$
$$= \det(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -e^{-s} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -2e^{-2s} & e^{-2s} \end{bmatrix})$$
$$= \det\begin{pmatrix} s+2 & 0 \\ e^{-s} + 2e^{-2s} & s+3-e^{-2s} \end{pmatrix}$$
$$= (s+2)(s+3-e^{-2s}) - (0)(e^{-s} + 2e^{-2s})$$

$= s^2 + 3s - se^{-2s} + 2s + 6 - 2e^{-2s} = s^2 + 5s - se^{-2s} - 2e^{-2s} + 6 = 0$
To show that the root of the above quasi-characteristic equation have a negative real part, we assume that contradiction that $\alpha \geq 0$, where $s = \alpha \pm \beta i$ as the root of the equation. Since $\beta \neq 0$, we have the imaginary part as: $\frac{\Im(s^2+5s-se^{-2s}-2e^{-2s}+6)}{\beta}$

$$= 2\alpha + 5 - 2e^{-2\alpha}\left(\frac{sin2\beta}{2\beta}\right) - 4e^{-2\alpha}\left(\frac{sin2\beta}{2\beta}\right)$$

$$> 2 + 5 - 2 - 4 > 0.$$

This is not in line with the assumption that the complex number $s = \alpha \pm \beta i,\ where\ \alpha > 0$ is a root of the above quasi-characteristic equation. Therefore, all the roots of the above quasi-characteristic equation have negative real parts. Hence by Theorem 1, we conclude that system (12) is asymptotically stable.

To show that system (12) is controllable, we use the method by Beata [2] showing that $\tilde{Q}_n(t_1) = n, n = 2$ for all $t_1 > 0$, hence we find all matrices that belong to the set $\tilde{Q}_2(t_1)$:

$Q_0(0) = B = \begin{bmatrix}0\\1\end{bmatrix}$ and $Q_0(s) = 0$ for $s \neq 0$.

Since

$Q_1(s) = \sum_{i=0}^{2} A_i Q_0(s - h_i) = A_0 Q_0(s) + A_1 Q_0(s - h_1) + A_2 Q_0(s - h_2)$,

for all $s = h_i, 2h_i, 3h_i, \dots \dots, i = 0, 1, 2$, we get

$Q_1(0) = A_0 B = \begin{bmatrix}-2 & 0\\0 & -3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix}$,

$Q_1(h_1) = A_1 Q_1(0) = \begin{bmatrix}0 & 0\\-1 & 0\end{bmatrix}\begin{bmatrix}0\\-3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$,

$Q_1(h_2) = A_2 Q_1(0) = \begin{bmatrix}0 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}0\\-3\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix}$,

$Q_1(2h_1) = A_2 Q_o(0) = Q_1(h_2)$, as $2h_1 = h_2$.

Hence, $\tilde{Q}_2(t_1) = \{Q_0(0), Q_1(0), Q_1(h_1), Q_1(h_2)\}, \quad n = 2$

and $rank\tilde{Q}_2(t_1) = rank\begin{bmatrix}-2 & 0 & 0 & 0\\0 & -3 & 0 & -3\end{bmatrix} = 2$

By Theorem 2, the dynamical system is proper in E^n on the interval $[0, t_1]$.

See Matlab code below on getting the rank.

```
>> Q = [-2 0 0 0;0 -3 0 -3]
Q =
   -2    0    0    0
    0   -3    0   -3
>> K = rank(Q)
K =
    2
>>
```

Therefore, by the Main Theorem, the dynamical system (12) is Euclidean Null controllable.

6. CONCLUSION: We have considered linear stationary dynamical systems with multiple delays in the state, introduced some definitions, and established theorems necessary and sufficient for the dynamical system (1) to be Euclidean Null Controllable by extending the work of [1-2].

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