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ON MAXIMUM WORKS AND STRETCHES PERFORMED BY TRANSFORMATIONS OF A FINITE SET

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Abstract

Let X_n be the finite set $\{1,2,3,\dots,n\}$. Wedenote by S_n, T_n, P_n and SP_n the sets of all permutations, full transformation, partial transformation and strictly partial transformation on X_n respectively. We define the stretch $s^+(\pi)$ of a permutation π to be the arithmetic average of$\{|\pi(i)-\pi(i+1)|: 1 \le i < n\}$. A (partial) transformation αmoves an element $i \in dom(\alpha)$ a distance of $|i - i\alpha|$ units. The work $w(\alpha)$ performed by α is the sum of all these distances. In this paper, we characterise elements of T_n, P_n and SP_nwhich attain maximum work, elements of T_n with maximum stretch and calculate the number of permutations attaining maximum works and maximum stretches. Equally, explicit formulas for these maximums are derived.

Keywords: work, stretch, maximum work, maximum stretch, permutations, full transformation, partial transformation, strictly partial transformation.

1. Introduction

A partial transformationα of a finite set $X_n = \{1,2,3,\dots,n\}$ is a map whose domain and codomain are subsets ofX_n. A partial transformation α is said to be full if its domain is the whole of X_n and is said to be strictly partial if it is not a full transformation. The sets of all partial full and strictly partial transformations of X_n are denoted by T_n, P_n and SP_n respectively. The set of partial transformations and its various subsets have been objects of study among various researchers over the course of the years. Aspects of transformations that have been explored are always in terms of their algebraic or combinatorial properties.

The concept of 'work' in transformations first appeared in [1].A (partial) transformationα of X_n moves an element $i \in dom(\alpha)$ a distance of $|i - i\alpha|$ units. The work $w(\alpha)$ performed by α is the sum of all these distances. Although [1] documented that their motivation for the study stems after attending a talk in Sydney in 2004 delivered by Tim laver, where the work performed by semigroup of order-preserving transformation on the finite set $X_n = \{1,2,\dots,n\}$ was conjectured as $(n-1)2^{2n-3}$. This conjecture was proved in the paper of the duo, and alongside they further studied the works and average works performed by the semigroup of partial transformation and some of its subsemigroups. They derived explicit formulas for the work and average work of these various subsemigroups of the partial transformations.

Prior to this, what appears to have the same definition as work has been studied under the name displacement. This concept of displacement appears in literature under various names with subtle variations and its usage has been with respect to permutations. The concept of total displacement of a permutation π on X_n whose expression first appear in [2] was defined as$\sum_{i=1}^{n} |i - \pi(i)|$. The idea was further considered by other authors although with slight differences and names depending on its usage in their various researches. To consider some instance, the term spearman's measure of disarray was used as substitute for displacement in [3]; delay, total relative displacement and shift factor were used [4], [5]and [6] respectively.

With respect to permutations, [7] considered stretch and displacement. The stretch $s^+(\pi)$(in terms of addition) of a permutation $\pi$is defined to be the arithmetic average of $\{|\pi(i)-\pi(i+1)|: 1 \le i < n\}$. Permutations that attain maximum value of stretch was obtained and their description was given, and with respect to displacement, they equally found

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permutations that attain maximum displacement and characterised them. The ideas of displacement in permutations have found applications in the areas of turbo coding [7], interleavers of turbo codes [4] and [8], speech scrambling [6] and many more.
In this paper, we extend the work of [7] with respect to transformations, this we did along the line of [1]. We subdivide this paper into three sections. In the second section, wepresent preliminary definitions and existing results relating to stretch and displacement in permutations. The last section however deals with the main findings of this paper.

2. Preliminaries

Consider the finite set$X_n = \{1,2,3,\dots,n\}$.
In this section, we present preliminary results and definitions. Basic concepts in semigroup can be found in [9]
Definition 2.1[7] Let $\pi \in S_n$. The displacement of π is defined as:
$d(\pi) = \sum_{i=1}^{n} \frac{|i-\pi(i)|}{n}$.
Definition 2.2[7]A permutation $\pi \in S_n$is called crossing if for every $i,j \in X_n$, the two closed interval $[i,\pi(i)]$, $[j,\pi(j)]$ intersect (possibly at a single point). Otherwise,π is said to be non-crossing.
In the next result and the one that follows, it is shown respectively that only crossing permutations can attain maximum displacement and such permutations are characterised.
Lemma 2.3[7]Let$\pi \in S_n$be a non-crossing permutation.Then there is$\rho \in S_n$with $d(\rho) > d(\pi)$.
Lemma 2.4[7]Let$\pi \in S_n$. If $n = 2m$, thenπ is crossing if and only if it maps$\{1,2,\dots,m\}$onto$\{m+1,m+2,\dots,n\}$. If $n = 2m+1$, then$\pi$is crossing if and only if it maps$\{1,2,\dots,m\}$ to$\{m+1,m+2,\dots,n\}$and$\{m+2,m+3,\dots,n\}$ to$\{1,2,\dots,m+1\}$.
The result that follows discusses on the value of this maximum displacement.
Theorem 2.5[7]Given $n \geq 1$, let$d_n = max\{d(\pi): \pi \in S_n\}$and $D_n = \{\pi \in S_n: d(\pi) = d_n\}$.Then$\pi \in D_n$ if and only if $\pi$is crossing. Moreover,$d_n = {}^{n}/_{2}$when$n$is even and $d_n = (n-1)(n+1)(2n)^{-1}$when$n$ is odd.
Definition 2.6[7] Let $\pi \in S_n$. Consider $\mathfrak{B} = \{\{i,i+1\}: 1 \leq i \leq n\}$ and $|\mathfrak{B}| = n-1$. The stretch of π (with respect to addition) is defined as $s_{\mathfrak{B}}^{+}(\pi) = \sum_{i=1}^{n-1} \frac{|\pi(i)-\pi(i+1)|}{n-1}$.
Definition 2.7[7]For two subsets A, B of X_n, we say that $\pi \in S_n$ oscillates between $A\ and\ B$ if for every$1 \leq i \leq n$, we have either $\pi(i) \in A, \pi(i+1) \in B$ or $\pi(i) \in B, \pi(i+1) \in A$.
Below is presented a result that shows for which $\pi \in S_n$ is $s_{\mathfrak{B}}^{+}(\pi)$ maximal, and the formula for this maximal value is also given.
Theorem 2.8[7]The maximum value of $s_{\mathfrak{B}}^{+}(\pi)$ among all $\pi \in S_n$ is ${(2m^2-1)}\big/{(2m-1)}$ when $n = 2m$ and ${(2m^2+2m-1)}\big/{(2m)}$ when $n = 2m+1$.When $n = 2m$, the maximum is attained by $\pi\ if\ and\ only\ if \pi$ oscillates between$\{1,2,\dots,m\}$, $\{m+1,m+2,\dots,n\}$ and $(\pi(1),\pi(n)) \in \{(m,m+1),(m+1,m)\}$. When $n = 2m+1$, the maximum is attained by $\pi\ if\ snd\ only\ if$ either π oscillates between$\{1,2,\dots,m\}$, $\{m+1,m+2,\dots,n\}$ and $(\pi(1),\pi(n)) \in \{(m+1,m+2),(m+2,m+1)\}$, or π oscillates between $\{1,2,\dots,m+1\}$, $\{m+2,m+3,\dots,n\}$ and $(\pi(1),\pi(n)) \in \{(m,m+1),(m+1,m)\}$.
Definition 2.9[1] the work performed by a partial transformation $\alpha \in P_n$ in moving a point$i \in \boldsymbol{n}$is defined to be:
$$w_i(\alpha) = \begin{cases} |i - i\alpha|\ if\ i \in dom(\alpha) \\ \quad 0 \qquad otherwise, \end{cases}$$
The (total) work performed by α is
$$w(\alpha) = \sum_{i \in \boldsymbol{n}} w_i(\alpha)$$
Notice that on multiplying n to the result of displacement by [7], we obtain a value equal to what was called work in [1]. Henceforth, we neglect the use of displacement and adopt work in the sense of its usage in [1]. Equally our usage of stretch shall be without consideration to the cardinality of the set $\mathfrak{B}$ as it were in [7]. This is purely for the purpose of consistency.

3. Main Results

In this section, we present the findings of this work. We begin by presenting respectively the characterisations for mappings with maximum work and maximum stretch in T_n, together with their associated combinatorial results.
Consider the finite set $X_n = \{1,2,\dots,n\}$.

Theorem 3.1 Let $\alpha \in T_n$. Then,

a. If n is even, α performs maximum work in T_n if and only if for each $i \in X_n$,

$$i\alpha = \begin{cases} \mathrm{n} & \text{if } 1 \le \mathrm{i} \le \frac{\mathrm{n}}{2}, \\ 1 & \text{if } (\frac{\mathrm{n}}{2}) + 1 \le \mathrm{i} \le \mathrm{n}. \end{cases} \tag{1}$$

b. If n is odd, then α performs the maximum work in T_n if and only if for each $i \in X_n$,

$$i\alpha = \begin{cases} \mathrm{n} & \text{if } 1 \le \mathrm{i} \le \frac{\mathrm{n}-1}{2}, \\ \mathrm{n} \text{ or } 1 & \text{if } \mathrm{i} = (\frac{\mathrm{n}-1}{2}) + 1, \\ 1 & \text{if } (\frac{\mathrm{n}-1}{2}) + 2 \le \mathrm{i} \le \mathrm{n}. \end{cases} \tag{2}$$

Moreover,

$$\max_{\alpha \in \mathrm{T_n}}\{ \mathrm{w}(\alpha)\} = \begin{cases} \frac{\mathrm{n}}{4}(3\mathrm{n}-2) & \text{if n is even,} \\ \frac{1}{4}(\mathrm{n}-1)(3\mathrm{n}+1) & \text{if n is odd.} \end{cases} \tag{3}$$

Finally, if

$\Delta(\mathrm{S}) = |\{\alpha \in \mathrm{S} : \mathrm{w}(\alpha) \text{is maximum}\}|$. Then

$$\Delta(T_n) = \begin{cases} 1 & \text{if n is even,} \\ 2 & \text{if n is odd.} \end{cases} \tag{4}$$

Proof:

a. Let n be even. Suppose α performs maximum work in T_n, then by definition of work performed by $\alpha \in T_n$, we have

$w(\alpha) = \sum_{i=1}^{n} |i - i\alpha|$

Now, notice that $w(\alpha)$ can be maximum only when $|i - i\alpha|$ is made sufficiently large for each $i \in X_n$. Thus,

$$max|i - i\alpha| = \begin{cases} n - i & if\ 1 \le i \le \frac{n}{2}, \\ i - 1 & if\ (\frac{n}{2}) + 1 \le i \le n. \end{cases}$$

hence the map in (1).

Conversely, suppose $\alpha \in T_n$ is as in the even case, then clearly, $|i - i\alpha|$ is at maximum for each $i \in X_n$ and so the value of $w(\alpha)$ will be maximum. And the result follows.

b. For n odd, and suppose α performs maximum work in T_n, then by definition,

$$w(\alpha) = \sum_{i=1}^{n} |i - i\alpha|$$

Using similar argument as above, the maximum value of $w(\alpha)$ can be attained by maximizing $|i - i\alpha|$ for each $i \in X_n$. Now,

$$max|i - i\alpha| = \begin{cases} n - i & \text{if } 1 \le i \le \frac{n-1}{2}, \\ \frac{n-1}{2} & \text{if } i = (\frac{n-1}{2}) + 1, \\ i - 1 & \text{if } (\frac{n-1}{2}) + 2 \le i \le n. \end{cases}$$

Hence, α must be the map described in (2).

Conversely, suppose $\alpha \in T_n$ is in the odd case, then clearly, $|i - i\alpha|$ is at maximum for each $i \in X_n$. Thus, $w(\alpha)$ will be at maximum and so we have our desired result.

Now, for an even n and α as in(1), we have,

$$w(\alpha) = \sum_{i=1}^{\frac{n}{2}} (n - i) + \sum_{i=\frac{n}{2}+1}^{n} (i - 1)$$

$$= \sum_{i=1}^{\frac{n}{2}} n - \sum_{i=1}^{\frac{n}{2}} i + \sum_{i=\frac{n}{2}+1}^{n} i - \sum_{i=\frac{n}{2}+1}^{n} 1$$
$$= \sum_{i=1}^{\frac{n}{2}} n - 2\sum_{i=1}^{\frac{n}{2}} i + \sum_{i=1}^{n} i - \sum_{i=\frac{n}{2}+1}^{n} 1$$
$$= \frac{n^2}{2} - \frac{n}{2}\left(\frac{n}{2}+1\right) + \frac{n}{2}(n+1) - (n - \frac{n}{2})$$
$$= n^2 - \frac{n^2}{4} + \frac{n}{2} - n$$
$$= \frac{n}{4}(3n-2).$$

Also, for an odd$n$ and α as in (2), we have,

$$w(\alpha) = \sum_{i=1}^{(\frac{n-1}{2})} (n-i) + \sum_{i=\left(\frac{n-1}{2}\right)+2}^{n} (i-1) + (\frac{n-1}{2})$$
$$= \sum_{i=1}^{(\frac{n-1}{2})} (n-i) + \sum_{i=1}^{n} (i-1) - \sum_{i=1}^{\left(\frac{n-1}{2}\right)+1} (i-1) + (\frac{n-1}{2})$$
$$= \sum_{i=1}^{(\frac{n-1}{2})} (n-i) + \sum_{i=1}^{n} (i-1) - \sum_{i=1}^{\left(\frac{n-1}{2}\right)+1} (i-1)$$
$$= \sum_{i=1}^{n} (i-1) + \sum_{i=1}^{(\frac{n-1}{2})} (n-2i+1)$$
$$= \frac{n(n+1)}{2} - n + \frac{n(n-1)}{2} - \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right) + \frac{n-1}{2}$$
$$= \frac{2n(n-1)}{2} - \frac{(n-1)(n+1)}{4} + \frac{(n-1)}{2}$$
$$= \frac{(n-1)}{4}[4n-(n+1)+2]$$
$$= \frac{1}{4}(n-1)(3n+1).$$

Finally, when$n$is even, it is clear from the map (1) that in $T_n$only 1 such map will attain the maximum.
On the other hand, for an odd n, 2 such maps will obviously exist since the central element in the domain is mapped to either n or 1.
Next, we present a result that characterises mappings in T_n that attain maximum stretch. But note that we consider the stretch only in the addition case.

Theorem 3.2 For $n \geq 2$, a mapping $\alpha \in T_n$attains maximum additive stretch in $T_n$if and only if α oscillates between1 and n. This maximum value is $(n-1)^2$. Moreover, there are only 2 elements of T_n that attain this maximum.
Proof:
By definition, the additive stretch$s^+(\alpha)$ of a map $\alpha \in T_n$is given by,
$s^+(\alpha) = \sum_{i=1}^{n-1} |\alpha(i) - \alpha(i+1)|$.
Obviously, the maximum value of $s^+(\alpha)$is attained by maximizing $|\alpha(i) - \alpha(i+1)|$ for each $i \in \{1,2,3,\ldots,n-1\}$. Now, notice that, for each
$i \in \{1,2,3,\ldots,n-1\}$, the maximum value of $|\alpha(i) - \alpha(i+1)|$ is $n-1$. This
is clearly possible only when α oscillates between 1 and n.
Conversely, suppose $\alpha \in T_n$ oscillates between1 and n. Then clearly,
$|\alpha(i) - \alpha(i+1)|$ is at maximum for each $i \in \{1,2,3,\ldots,n-1\}$. And so the additive stretch of α is also at maximum.
Now, if $\alpha \in T_n$ attains maximum additive stretch, then by the above description, we have

$$s^{+}(\alpha)=\sum_{i=1}^{n-1}|\alpha(i)-\alpha(i+1)|$$
$= (n-1)+(n-1)+(n-1)+\cdots+(n-1)$
$= (n-1)^2$.

Finally, by the definition of oscillation, $\alpha \in T_n$ oscillates between 1 and n if and only if, for any $i \in \{1,2,3,\dots,n-1\}$, either $\alpha(i)=1,\ \alpha(i+1)=n$ or $\alpha(i)=n,\ \alpha(i+1)=1$. Thus, only 2 such $\alpha \in T_n$ will exist.

In what follows, we count the number of elements in S_n that attain maximum work and maximum stretch.

Theorem 3.3 Let $\pi \in S_n$. Then, the total number of permutations in S_n

i. that attain maximum value of additive stretch is

$$\begin{cases} 2\left[\left(\frac{n-2}{2}\right)!\right]^2 & \text{if n is even} \\ 2\left[\left(\frac{n-1}{2}\right)!\left(\frac{n-3}{2}\right)!\right) & \text{if n is odd.} \end{cases}$$

for $n \geq 2$.

ii. that attain maximum work is

$$\begin{cases} [\left(\frac{n}{2}\right)!]^2 & \text{if n is even} \\ n[\left(\frac{n-1}{2}\right)!]^2, & \text{if n is odd} \end{cases}$$

Proof:

i. Let n be even. Then by Theorem 2.8[7], $\pi \in S_n$ oscillates between $A=\left\{1,2,\dots,\frac{n}{2}\right\}$ and $B=\left\{\frac{n}{2}+1,\dots,n\right\}$ and $(\pi(1),\pi(n)) \in \left\{\left(\frac{n}{2},\frac{n}{2}+1\right),\left(\frac{n}{2}+1,\frac{n}{2}\right)\right\}$. If $(\pi(1),\pi(n))=\left(\frac{n}{2},\frac{n}{2}+1\right)$, it follow that $\pi|_{\{2,3,4,\dots,n-1\}}$ oscillates between $A'=(A\backslash\{\frac{n}{2}\})$ and $B'=(B\backslash\{\frac{n}{2}+1\})$. Thus, $\pi|_{\{2,3,4,\dots,n-1\}}$ is a union $\alpha \cup \beta$, where α is a bijection from $\{2,4,6,\dots,n-2\}$ onto B', and β is a bijection from $\{3,5,7,\dots,n-1\}$ onto A'. Now, it is clear from simple combinatorial argument that, there are $\left(\frac{n-2}{2}\right)!$ such α and $\left(\frac{n-2}{2}\right)!$ such β. But then, there are $[\left(\frac{n-2}{2}\right)!]^2$ possible maps of the form $\alpha \cup \beta = \pi|_{\{2,3,4,\dots,n-1\}}$. And so, in this case, this is exactly the number of permutations $\pi \in S_n$ that are of maximum additive stretch.

If $(\pi(1),\pi(n))=\left(\frac{n}{2}+1,\frac{n}{2}\right)$, then a similar observation as above shows that $\pi|_{\{2,3,4,\dots,n-1\}}$ is a union $\gamma \cup \tau$, where γ is a bijection from $\{2,4,6,\dots,n-2\}$ onto A' and τ is a bijection from $\{3,5,7,\dots,n-1\}$ onto B'. And so, there are $\left(\frac{n-2}{2}\right)!$ such γ and $\left(\frac{n-2}{2}\right)!$ such τ. Thus here too, we have exactly $[\left(\frac{n-2}{2}\right)!]^2 \pi \in S_n$ that are of maximum additive stretch. Hence when n even, the total number of permutations in S_n that attain the maximum additive stretch is $2([\left(\frac{n-2}{2}\right)!]^2)$.

If n be an odd integer. By Theorem 2.8[7], $\pi \in S_n$ oscillates between $C=\{1,2,3,\dots,\frac{n-1}{2}\}$ and $D=\{\frac{n+1}{2},\frac{n+1}{2}+1,\dots,n\}$ and $(\pi(1),\pi(n)) \in \{\left(\frac{n+1}{2},\frac{n+3}{2}\right),\left(\frac{n+3}{2},\frac{n+1}{2}\right)\}$. Now firstly, if $(\pi(1),\pi(n))=\left(\frac{n+1}{2},\frac{n+3}{2}\right)$, it then follows that the permutation $\pi|_{\{2,3,4,\dots,n-1\}}$ oscillates between C and $D'=(D\backslash\{\frac{n+1}{2},\frac{n+3}{2}\})$. Thus, $\pi|_{\{2,3,4,\dots,n-1\}}$ is a union $\delta \cup \sigma$, where δ is a bijection from $\{2,4,6,\dots,n-1\}$ onto C and σ is a bijection from $\{3,5,7,\dots,n-2\}$ onto D'. Since π is a permutation, there will be $\left(\frac{n-1}{2}\right)!$ such δ and $(\frac{n-3}{2})!$ Such σ. And so, we shall have $\left(\frac{n-1}{2}\right)!\left(\frac{n-3}{2}\right)!$ Possible maps $\pi|_{\{2,3,4,\dots,n-1\}}$. Now, since there are two possibilities for $(\pi(1),\pi(n))$, we have that the total number of permutations $\pi \in S_n$ that attain the maximum additive stretch is $2(\left(\frac{n-1}{2}\right)!\left(\frac{n-3}{2}\right)!)$. The second description of the permutation follows.

ii. Let $\pi \in S_n$ be a permutation whose work is maximum. If n is even. Then, by Lemma 2.4[7] the subset $A=\{1,2,\dots,\frac{n}{2}\}$ is mapped by π onto the subset $B=\left\{\frac{n}{2}+1,\dots,n\right\}$, obviously π will force B to be mapped onto A.It therefore follows that π will map both A onto B and B onto A in $\left(\frac{n}{2}\right)!$possible ways since$|A|=|B|$.Hence, we shall have a total of $[\left(\frac{n}{2}\right)!]^2$ permutations that attain maximum work.

If n is odd, again by the characterisation in Lemma 2.4[7], $\pi \in S_n$ imposes no condition to the image of the central element of X_n $(\frac{n+1}{2})th$ element, and thus allowing it to be mapped to any $i \in X_n$. The other two subsets of X_n $(A = \{1,2,3,\dots,\frac{n-1}{2}$ and $B = \{\frac{n-1}{2}+1,\dots,n\})$ on the left and right of $(\frac{n+1}{2})th$ elementrespectively have the same cardinality $\frac{n+1}{2}$. Thus, we shall have $\left(\frac{n+1}{2}\right)$!ways π will map A onto the elements of B and vice versa. This therefore results to a total of $[\left(\frac{n+1}{2}\right)!]^2$ possible arrangements of the two partitions. Lastly, since the $(\frac{n+1}{2})th$ element can be mapped to any $i \in X_n$, we shall have a total of $n[\left(\frac{n+1}{2}\right)!]^2$ permutations that attain the maximum work in S_n.

In the next result, we characterise elements of $P_n$that attain maximum work. But first, we write an important lemma that will aid us in the proof of the characterisation.

Lemma 3.4 LetX_n be the finite set $\{1,2,3,\dots,n\}$. If $K = \{\alpha \in P_n: |dom(\alpha)| = m\}$ and $L = \{\beta \in P_n: |dom(\beta)| = m+1 < n\}$. Then there exists at least a $\gamma \in L$ such that $w(\gamma) > w(\delta)$ for all $\delta \in K$.

Proof:

Note first of all that for all $\alpha \in K$ and all $\beta \in L$ there exists at least one map in K and at least one map in L whose work is maximum in K and L respectively. For such $\alpha \in K$ and all $\beta \in L$, and for $i \in dom(\alpha)$ and $j \in dom(\beta)$,

$$\max|i - i\alpha| = \begin{cases} n - i & \text{if } 1 \le i \le \frac{n}{2}, \\ i - 1 & \text{if } (\frac{n}{2}) + 1 \le i \le n. \end{cases}$$

and

$$\max|j - j\beta| = \begin{cases} n - j & \text{if } 1 \le j \le \frac{n}{2}, \\ j - 1 & \text{if } (\frac{n}{2}) + 1 \le j \le n. \end{cases}$$

Suppose $\delta \in K$ and $\gamma \in L$ are such maps, that is $\max_{\alpha \in K} w(\alpha) = w(\delta)$ and $\max_{\beta \in L} w(\beta) = w(\gamma)$, then by definition,

$w(\delta) = \sum_{r=1}^{m} |i_r - i_r\delta|$ for $i_r \in dom(\delta)$

and

$w(\gamma) = \sum_{s=1}^{m+1} |i_s - i_s\gamma|$ for $i_s \in dom(\gamma)$

It follows obviously from the definition that since there are more summation of terms under γ, $w(\gamma) > w(\delta)$.

Theorem 3.5 An element $\alpha \in P_n$ attains maximum work if and only if $\alpha \in T_n$ and α is such that:

$$i\alpha = \begin{cases} n & \text{if } 1 \le i \le \frac{n}{2}, \\ 1 & \text{if } (\frac{n}{2}) + 1 \le i \le n. \end{cases}$$

if n is even,

and

$$i\alpha = \begin{cases} n & \text{if } 1 \le i \le \frac{n-1}{2}, \\ n \text{ or } 1 & \text{if } i = (\frac{n-1}{2}) + 1, \\ 1 & \text{if } (\frac{n-1}{2}) + 2 \le i \le n. \end{cases}$$

if $n$is odd.

Proof:

The proof follows from lemma 3.4 and the proof of theorem 3.1.

We consider below description of elements in the set of all strictly partial transformations(SP_n)that attain maximum work.

Theorem 3.6 Let $\alpha \in SP_n$. Then,

(a) If n is even, α performs maximum work in SP_n if and only if, for each $i \in X_n$,

(i) $dom(\alpha) = X_n \backslash \{\frac{n}{2} \; or \; \frac{n}{2} + 1\}$

(ii) either

$$i\alpha = \begin{cases} n & \text{if } 1 \leq i \leq \frac{n}{2}, \\ 1 & \text{if } (\frac{n}{2}) + 1 \leq i \leq n. \end{cases}$$

or

$$i\alpha = \begin{cases} n & \text{if } 1 \leq i \leq \frac{n}{2} - 1, \\ 1 & \text{if } (\frac{n}{2}) + 1 \leq i \leq n. \end{cases}$$

(b) if n is odd, α performs maximum work in SP_n if and only if for each $i \in X_n$,

(i) $dom(\alpha) = X_n \backslash \{\frac{n+1}{2}\}$

(ii) $$i\alpha = \begin{cases} n & \text{if } 1 \leq i \leq \frac{n-1}{2}, \\ 1 & \text{if } \frac{n+3}{2} \leq i \leq n. \end{cases}$$

Proof:

By lemma 3.4 and the proof of Theorem 3.1, the result follows.

Remark:It can be deduced fromTheorem 3.5 that for any n, no map will perform work greater than the map described in Theorem 3.1.

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