Journal of the Nigerian Association of Mathematical Physics
Volume 53 **(November *2019 Issue*), pp15 – 20**

ON MULTISET RELATIONS AND MULTISET BASIS FOR MULTISET TOPOLOGY

A. J. Alkali and H. Dauda

Department of Mathematics, Ahmadu Bello University, Zaria – Nigeria

Abstract

In this paper, we investigated and studied some results of multiset relations and its properties. Similarly, we studied and investigated the results of multiset basis, sub multiset basis for multiset topology, and also finer and strictly finer multiset topologies. The paper extends some results of multiset relations and its properties, multiset basis and sub multiset basis for multiset topology, and also finer and strictly finer multiset topologies.

Keywords: multiset, multiset relation, multiset basis, multiset topology.

1. Introduction

The theory of set relations has been studied extensively in mathematics along with it applications in diverse fields. Especially in solving many real world problems; for instance, in solving a complex transportation problem where one needs to see how a city A is related to city B in [1].

In cantorian set theory usually called a standard or crisp set theory, a set is considered as any collection in to a definite and distinguished object (called elements) by [2] and [3]. One unavoidable consequence of cantor's definition is that no element can occur more than once in a classical set. Blizard in [2] argues that cantor's assertion of excluding repeated elements does not go hand in hand with many situations arising in solving real world problem. For example, the repeated roots of x^2 - 2x + 1 = 0, repeated hydrogen atom in a water molecule (H_2O), the repeated prime factors of an integers n > 0, etc, need to be considered significant. Once we admit repetition of elements, we have multiset (for short mset)

In view of the recent developments taking place in the study of multiset a number of important areas of research requiring of multiset relations have come to the fore, especially from a practical point of view, multisets are found useful in providing structures as they arise quite naturally in certain areas of mathematics, computer science, physics and philosophy seen in [2, 3, 4]. In particular multiset relations have been found to be natural representation for many important combinatorial optimization problems in [5] and [6].

A wide application of multiset can be found in various branches of mathematics. Algebraic structures for multiset space have been constructed by [7]. Application of multiset theory in decision making can be seen in [2]. Multiset topology induced by multiset relation was introduced by [8] in 2012. The same author further studied the notion of open set, closed set basis, sub-basis, closure, interior and related properties in multiset topological space.

Relation on a multiset is a simple mathematical model to which many real-life data can be connected by [6]. Topological structures on multiset are generalized methods for measuring similarity and dissimilarity between objects in the universes seen in [8] and [9]. Multiset relations are used in the construction of multiset topological structures in many fields such as dynamics, rough set theory and approximation spaces by [8].

The topological structure is usually represented by describing its part which is sufficient to recover the whole structure by [1]. A collection $\mathcal{B}$ of open mset is a multiset base for multiset topology if each nonempty multiset is a union of multisets belonging to $\mathcal{B}$. For instance, all intervals form a base for real line.

This paper extends some results of multiset relation and its properties by [10], multiset basis and multiset sub-basis for multiset topology, and also finer and strictly finer multiset topologies by [8] and [9].

Corresponding Author**:** Dauda H., Email: Daudah@gmail.com, Tel: +2349052332863

We subdivide this paper into three sections. In the second section, we present preliminary definitions and existing results relating to multiset relation and its properties, multiset basis and multiset sub basis for multiset topology and also finer and strictly finer multiset topologies. The last section however deals with the main findings of this paper.

2. Preliminaries

In this section, we present preliminary results and definitions

Definition 2.1 [2]
A multiset (mset) can be defined as an mset M drawn from the set X is represented by function count M or C_M defined as $C_M : X \rightarrow N$ where N represents the set of non – negative integers.

Definition 2.2 [2]
A domain X is defined as a set of elements from which msets are constructed. For any positive n, the mset space $[X]^n$ is the set of all msets whose elements are in X such that no element in the mset occurs more than n times.

Definition 2.3 [2]
The cardinality of mset M denoted by $C(M)$ is the sum of multiplicities of all its objects.

Definition 2.4 [2]
Let M and N be two msets drawn from a set X. Then N is called submultiset (submset) of M written as $N \subseteq M$ if $C_N(x) \leq C_M(x), \forall\, x \in X$. Clearly, $\emptyset$ is a submset of every mset

Definition 2.5 [10]
Let M and N be two msets drawn from set X then the cartesian product of M and N is define as
$M \times N = \{(m/x, n/y)/mn \colon x \in^m M$ and $y \in^n N\}$.

Definition 2.5 [10]
A submset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has a count product of $C_1(x, y)$ and $C_2(x, y)$. m/x related to n/y is denoted by $(m/x)\ R\ (n/y)$

Definition 2.6 [10]
Let R be mset relation defined on M and $x \in^m M$. We defined $R(m/x)$, the R- relative mset of m/x, as the mset of all n/y in M such that there exist some k such that $k/x\ R\ n/y$. ie.,
$R(m/x) = \{n/y \colon \exists$ some k with $k/x\ Rn/y\}$.

Definition 2.7 [10]
The inverse of an mset relation R denoted by R^{-1} is defined as
$R^{-1} = \{(n/y, m/x)/m\,n \colon (m/x, n/y) \in^{m\,n} R\}$

Definition 2.8 [10]
The identity mset relation in any mset M is the set of all pairs in $M \times M$ with equal co-ordinates and it's denoted by I_M.

Definition 2.9 [10]
Let M, N and P be three msets, R be an mset relation from M to N and S be an mset relation from N to P. The composition of R and S denoted by $S\ o\ R$ is an mset relation from M to P and is defined as follows:
If m/x is in M and k/z is in P, then $m/x\ (S\ o\ R)\ k/z$ if and only if there is some $m/x\ R\ n/y$ and $n/y\ S\ k/z$. Such that $C_1(x, z) = max\,\{min\{C_1(x, y), C_1(y, z)\}\}$ and
$C_2(x, z) = max\,\{min\,\{C_2(x, y), C_2(y, z)\}\}$, i.e.,
$m = max_i\,\{min_j\,\{m_i, n_j\}\}$ and $k = max_j\,\{min_i\,\{n_i, k_j\}\}$
The next results and the one that follow respectively are obtained from multiset relation R-relative multiset relation, inverse of multiset relation and composition of multiset relation. And shown how many data can be connected.

Theorem 2.10 [10] Let R be an mset relation from M to N and S be an mset relation from N to P. If M_1 is any submset of M, then $(S\ o\ R)(M_1) = S(R(M_1))$.

Theorem 2.11 [10] Let R be an mset relation from M to N and S be an mset relation from N to P. Then $(S\ o\ R)^{-I} = R^{-I}\ o\ S^{-I}$.

Definition 2.12 [10]
a. An mset relation R on an mset M is reflexive if $m/x\ R\ m/x$ for all m/x in M.
b. An mset relation R on an mset M is symmetric if $m/x\ R\ n/y$ implies $n/y\ R\ m/x$ for all $m/x, n/y$ in M.

c. An mset relation R on an mset M is transitive if $m/x\ R\ n/y$, $n/y\ R\ k/z$, then $m/x\ R\ k/z$ for all $m/x, n/y$ and k/z in M.

An mset relation R on an mset M is an equivalence mset relation if it is reflexive, symmetric and transitive

Definition 2.13 [10]
A partition of a non empty mset M is a collection P of non empty sub msets of M such that

a. Each element of M belongs to one of the msets in P.

b. If M_1 and M_2 are distinct elements of P, then $M_1 \cap M_2 = \emptyset$

The msets in P are called the blocks or cells of the partition
Below are results that shown equivalence and partition of multiset have many things in common.

Theorem 2.14 [10] Let R be an equivalence mset relation on an mset M. If $x \in^m M$ and $y \in^n M$. Then $m/x\ R\ n/y$ if and only if $R(m/x) = R(n/y)$.

Theorem 2.15 [10] Let R be an equivalence mset relation on M and let P be the collection of all distinct R-relative msets $R(m/x)$ for every m/x in M. Then P is a partition of M and R is an equivalence mset relation determine by P.

Definition 2.16 [8] Let $M \in [X]^W$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology (for short M-topology) of M if τ satisfies the following properties;

i. The multiset M and empty multiset $\emptyset$ are in τ.

ii. The multiset union of the element of any sub collection of τ is in τ.

iii. The multiset intersection of the elements of any finite sub collection of τ is in τ.

A multiset topological space is an order pair (M, τ) consisting of a multiset space $[X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M.

Definition 2.17 [8] If M is an mset, an m-basis for an M-topology on M is a collection $\mathcal{B}$ of partial whole submsets of M (called M-basis element) such that

(i) For each $x \in^m M$, for some $m > 0$, there is at least one M-basis element $B \in \mathcal{B}$ containing m/x, i.e., for each mset in B, there is at least one element with full multicity as in M.

(ii) If m/x belongs to the intersection of two M-basis elements B_1 and B_2 then there is an M-basis element B_3 containing m/x such that $B_3 \subseteq B_1 \cap B_2$ i.e., there is an M-basis element B_3 containing an element with full multicity as in M and that element must be in B_1 and B_2.

Definition 2.18 [8] A sub collection S of τ on M is called a sub M-basis for τ, if the collection of all finite mset intersection of element of S is an M-basis for τ. The M-topology generated by sub M-basis S is defined to be the collection τ of mset union of all finite mset intersection of elements of S.

The results follows discussed how τ equals the collection of all msets union of elements in multiset basis for multiset topology.

Theorem 2.19 [8] Let M be an mset in $[X]^w$ and $\mathcal{B}$ be an M–basis for an M–topology τ on M. Then τ equals the collection of all mset unions of elements of the M–basis $\mathcal{B}$.

Definition 2.20 [9] Suppose τ and τ^1 are two M-topologies on a given mset M in $[X]^w$. If $\tau^1 \subseteq \tau$, then we say τ^1 is finer than τ. If $\tau^1 \subset \tau$, then τ^1 is strictly finer than τ.

Theorem 2.21 [9] Let $\mathcal{B}$ and $\mathcal{B}^1$ be two M-basis for the M-topologies τ and τ^1 on M in $[X]^w$respectively. Then the following are equivalent.

i. τ^1 is finer than τ

ii. For each $x \in^m M$ and each M-basis element $B \in \mathcal{B}$ containing m/x, there is an M-basis element $B^1 \in \mathcal{B}^1$ containing m/x such that $C_{B^1}(x) \leq C_B(x)$

3. Main Results

In this section, we present the findings of this work. We begin by presenting the results obtained from multiset relations and its properties.

Theorem 3.1 Let R be an mset relation from M to N and S be an mset relation from N to P. Then if M_1 is any submset of M, we have $(S\ o\ R)^{-1}(M_1) = R^{-1}(S^{-1}(M_1))$

Proof:

Let $m/x \in M$ and $k/z \in P$. Suppose $k/z \in (S\ o\ R)^{-1}(M_1)$, then $k/z\ (S\ o\ R)^{-1}\ m/x$ if and only if $m/x\ (S\ o\ R)\ k/z$, i.e., if and only if there is some $m_i/x\ R\ n_i/y$ and $n_j/y\ S\ k_j/z$ such that $m= max_i\{min_j\{m_i,n_j\}\}$ and $k= max_j\{min_i\{n_i,k_j\}\}$. Thus $n_i/y \in R^{-1}(m_i/x)$ and

$k_j/z \in S^{-1}(R^{-1}(m_i/x))$ since $\{m_i/x\} \subseteq M_1$ and

$S^{-1}(R^{-1}(m_i/x)) \subseteq S^{-1}(R^{-1}(M_1))$. Hence $k_j/z \in S^{-1}(R^{-1}(M_1))$.

Therefore $(S\ o\ R)^{-1}(M_1) \subseteq (S^{-1}(R^{-1}(M_1)$ (1)

Similarly, suppose that $m/x \in S^{-1}(R^{-1}(M_1))$, then $k_j/z \in S^{-1}(n_j/y)$ so $n_i/y \in R\ (m_i/x)$. By the definition of composition $k_j/z\ S^{-1}\ n_j/y$ and $n_i/y\ R^{-1}\ m_i/x$ i.e., $k/z\ (R\ o\ S)^{-1}\ m/x$ where $k=max_j\{min_i\{n_i,k_j\}\}$ and $m= max_i\{min_j\{m_i,n_j\}\}$. Thus $m/x \in (R\ o\ S)^{-1}(k/z)$ Since $\{k_j/z\} \subseteq M_1$ and $(S\ o\ R)^{-1}(k_j/z) \subseteq (S\ o\ R)^{-1}(M_1))$. Hence $m/x \in (S\ o\ R)^{-1}(M_1))$.

Therefore, $S^{-1}(R^{-1}(M_1)) \subseteq (S\ o\ R)^{-1}(M_1))$ (2)

Hence from (1) and (2) we have $(S\ o\ R)^{-1}(M_1) = S^{-1}(R^{-1}(M_1))$

Theorem 3.2 Let R be an mset relation from M to N Then $R^{-1}\ o\ R = I_M$ and $R\ o\ R^{-1} = I_N$

Proof:

Let R be an mset relation from M to N and also R^{-1} be an mset relation from N to M. Then $n/y = R(m/x)$ which is equivalent to $m/x = R^{-1}(n/y)$. Since R and R^{-1} are mset relations for all $m/x \in M$ and $n/y \in N$. Thus $R(R^{-1}(n/y)) = n/y$ and $R(R^{-1}(m/x)) = m/x$. Then

For all $m/x \in M$, $I_M(m/x) = m/x = R^{-1}(R(m/x)) = (R^{-1}\ o\ R)(m/x)$, therefore $R^{-1}\ o\ R = I_M$. Also

For all $n/y \in N$, $I_N(n/y) = n/y = R(R^{-1}(n/y)) = (R\ o\ R^{-1})n/y$. Therefore $R\ o\ R^{-1} = I_N$

Theorem 3.3 Let R be an mset relation from M to N. Then

(a) $R\ o\ I_M = R$ and $I_M\ o\ R = R$

(b) $R\ o\ I_N =$ R and $I_N\ o\ R = R$

Proof:

Let R be an mset relation from M to N, if $R^{-1}\ o\ R = I_M$ and $R\ o\ R^{-1} = I_N$ for all $m/x \in M$ and $n/y \in N$. Then

(a) $(R\ o\ I_M)(m/x) = R(I_M(m/x))$ for all $m/x \in M$. So $R\ o\ I_M = R$ and
$(I_M\ o\ R)(m/x) = I_M(R(m/x))$ for all $m/x \in M$. So $I_M\ o\ R = R$

(b) $(R\ oI_N)(n/y) = R(I_N(n/y))$ for all $n/y \in N$. So $R\ o\ I_N = R$ and
$(I_N\ o\ R)(n/y) = I_N(R(n/y))$ for all $n/y \in N$. So $I_N\ o\ R = R$

Theorem 3.4 Let R be an mset relation from M to N. Then if M_1 is any submset of M we have $(R^{-1}\ o\ R)(M_1) = R^{-1}(R(M_1))$

Proof:

Let R be an mset relation from M to N and R^{-1} is an mset relation from N to M for all $m/x \in M$ and $n/y \in N$.

Let $m/x \in (R^{-1}\ o\ R)(M_1)$. Then from the definition of composition we have $m/x\ (R^{-1}\ o\ R)\ m/x$. Thus $m/x = R(m/x)$ is the same as $m/x = R^{-1}(m/x)$. Since R and R^{-1} are mset relations. Also $m_i/x \in R(m_i/x)$ and so $m_i/x \in R^{-1}(R(m_i/x))$

Since $\{m_i/x\} \subseteq M_1$ and $R^{-1}(R(m_i/x)) \subseteq R^{-1}(R(M_1))$. Hence $m/x \in R^{-1}(R(M_1))$

Therefore $(R^{-1}\ o\ R)(M_1) \subseteq R^{-1}(R(M_1))$……..(1)

Similarly, let $m/x \in R^{-1}(R(M_1))$. Then $m_i/x \in R^{-1}(R(m_i/x))$ for some $m_i/x \in R^{-1}(R(M_1))$. From the definition of composition we have $m/x\ (R^{-1}\ o\ R)\ m/x$.

Thus $m_i/x \in (R^{-1}\ o\ R)(m_i/x)$ since $\{m_i/x\} \subseteq M_1$ and $(R^{-1}\ o\ R)(m_i/x)) \subseteq R^{-1}(R(M_1))$. Hence $m/x \in R^{-1}(R(M_1))$, therefore $R^{-1}(R(M_1) \subseteq (R^{-1}\ o\ R)(M_1)$………(2).

Hence from (1) and (2) we have $(R^{-1}\ o\ R)(M_1) = R^{-1}(R(M_1))$

Next, we present result that shown the inverse of relation is also an equivalence relation

Theorem 3.5 Let $R^{-`}$ be an equivalence inverse mset relation on an mset $M$. If $x \in^m M$ and $y \in^n M$. Then $n/y\ R^{-`}\ m/x$ if and only if $R^{-`}(n/y) = R^{-`}(m/x)$.

Proof:

Let $R^{-`}(n/y) = R^{-`}(m/x)$. Since $R^{-`}$ is reflexive, $y \in^n R^{-`}(n/y)$. Therefore $y \in^n R^{-`}(m/x)$. Thus $n/y\ R^{-`}\ m/x$

Conversely, suppose that $n/y\ R^{-`}\ m/x$ . Then by definition of equivalent mset relation $y \in^n R^{-`}(m/x)$ and since R symmetric, $x \in^m R^{-`}(n/y)$. To prove $R^{-`}(n/y) = R^{-`}(m/x)$, let $z \in^k R^{-\prime}(n/y)$.

Since $R^{-'}$ is transitive and $y \in^n R^{-'}(n/y)$, it follows that $z \in^k R^{-'}(m/x)$. Thus $R^{-'}(n/y) \subseteq R^{-'}(m/x)$. So also $R^{-'}(m/x) \subseteq R^{-'}(n/y)$. Hence we can conclude that $R^{-'}(n/y) = R^{-'}(m/x)$.
The following remark indicates that $R(m/x)$ and $R(n/y)$ are identical if
$R(m/x) \cap R(n/y) \neq \emptyset$

Remark 3.6 If R is an equivalence mset relation on M and P is a collection of all distinct R-relative msets $R(m/x)$ for every m/x in M. Then P is a partition of M and R is an equivalence mset relation determine by P, if $R(m/x)$ and $R(n/y)$ are identical, then $R(m/x) \cap R(n/y) \neq \emptyset$
Our last findings is a result obtained from m-basis for an M-topology and it shown that τ can also be equals the collection of some mset intersection of elements of the m-basis

Theorem 3.7 Let M be an mset in $[X]^w$ and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two M–bases for an M–topology τ on M. Then τ equals the collection of some mset intersection of elements of the M–basis $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proof:
Given a collection of elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ which are also elements of τ, because τ is an M–topology and their intersection is in τ. Let $\mathcal{B}_3 \in \tau$, for each $m/x \in \mathcal{B}_3$, there is an element B of $\mathcal{B}_1$ and $\mathcal{B}_2$ containing m/x denoted by $B_{m/x}$ such that $B_{m/x} \subseteq \mathcal{B}_3$. Then
$\mathcal{B}_3 = \cap B_{m/x}$, so $\mathcal{B}_3$ equals some mset intersection of elements of $\mathcal{B}_1$ and $\mathcal{B}_2$.
The result below is a result obtained from finer and strictly finer M-topologies ant it shown that it is possible for m-bases to be strictly finer than the other.

Theorem 3.8 Let $\mathcal{B}$ and $\mathcal{B}'$ be two M-basis for the M-topologies τ and τ' on M respectively.
Then the following are equivalent.
1. τ' is strictly finer than τ
2. For each $x \in^m M$ and each M-basis element $B \in \mathcal{B}$ containing m/x, there is an M-basis element $B' \in \mathcal{B}'$containing m/x such that $C_{B'}(x) \leq C_B(x)$

Proof
(1) ⟹ (2). Given an element m/x in M and $B \in \mathcal{B}$ containing m/x, B belongs to τ by definition and $\tau' \subset \tau$ by (1). Therefore $B \in \tau'$. Since τ' is generated by $\mathcal{B}'$, there is an M- basis element $B' \in \mathcal{B}'$ containing m/x such that $C_{B'}(x) \leq C_B(x)$.
(2) ⟹ (1). Given an element G of τ, we show that $G \in \tau'$. Let $x \in^m G$, since $\mathcal{B}$ generates τ,
there is an M-basis element $B \in \mathcal{B}$ containing m/x such that $B \subset G$. From (2), there exist an M
basis element $B' \in \mathcal{B}'$containing m/x such that $B' \subseteq B$. Then $B' \subset G$ and $G \in \tau'$.

References

[1] Slapal, Josef (1993). Relations and topologies. *Czechoslovac Mathematical Journal* 43(1), 141-150.
[2] Blizard W. D. (1989*)*. Multiset Theory. *Notre Dame Journal of Formal Logic*. 30:36-66
[3] Singh, D. (1994). A Note on the Development of Multiset Theory. *Modern logic*, 4:405-406.
[4] Yager, R. R. (1986). On the theory of bags. *International Journal of General Systems* 13(1), 23-37.
[5] Singh D. Singh J. N. (2003). Some combinatorics of multisets. *International Journal of Mathematics Education in Science and Technology* 34:489-499
[6] Silvia G. Jovanka P. and Gradmir V. (2014). Binary Relations and Algebras on multisets. *Publications de L'Institute Mathematique, Nouvelle serie, tome* 95 (109):111-1173. Grish
[7] Singh D., A. M. Ibrahim, T. Yohana and J. N. Singh (2007). An over view of the applications of multisets. *Novi Sad J. Math* 37(2), 73-92.

[8] Grish K. P. and John S. J. (2012). Multiset Topology Induced by Multiset Relation. *Information Science*. 188:298-313.4.

[9] Grish K. P. and John S. J. (2012). On Multiset Topologies. *Theory and Applications of Mathematics and Computer Science*. 2: 37-52.

[10] Grish K. P. and John S. J. (2009). Relations and Functions in Multiset Context. *Information Science*.179: 758-768.