Journal of the Nigerian Association of Mathematical Physics
***Volume 53* (November *2019 Issue*), pp11 – 14**

A PROOF OF THE DIVERGENCE CRITERION

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Abstract

A statement of the Divergence Criterion (4.1.9(b) of [1]), so labeled by Bartle and Sherbert, is difficult to locate in the literature, to talk less of locating a proof. This short note furnishes a proof.

Keywords: Sequential convergence, point of accumulation, function limit.
2010 AMS Subject Classification 26 *Real Functions*

1 SEQUENTIAL CRITERION

Our language and notation shall be pretty standard, as found, for example in [1]. $\mathbb{R}$ denotes the *real numbers*, and by $\mathbb{N}$ we denote the collection $\{1, 2, 3, \ldots\}$ of the natural numbers, and $\emptyset \neq A \subseteq \mathbb{R}$. We indicate the end or absence of a proof by ///.

Let $x_o, \delta \in \mathbb{R}$ and $\delta > 0$. The open interval $(x_o - \delta, x_o + \delta)$ usually denoted $N_\delta(x_o)$, i.e.,
$N_\delta(x_o) \equiv (x_o - \delta, x_o + \delta)$,
is called the δ-*neighbourhood* of x_o. And the set difference
$(x_o - \delta, x_o + \delta) - \{x_o\} \equiv N_\delta(x_o) - \{x_o\}$,
usually denoted $N'_\delta(x_o)$, is called the *deleted* δ-*neighbourhood of* x_o.

Let A be a non-empty set. The sequence $(x_n)_{n=1}^{\infty}$ is called a *sequence in A* if $x_n \in A$ for all $n \in \mathbb{N}$. If $A = \mathbb{R}$, we call $(x_n)_{n=1}^{\infty}$ a *real sequence*.

Let $x_o \in \mathbb{R}$ and suppose

$$(x_n)_{n=1}^{\infty} \qquad \ldots\text{(RealSeq)}$$

is a real sequence. (RealSeq) is said to *converge to* x_o if for every $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$, such that
$x_n \in N_\varepsilon(x_o) \equiv (x_o - \varepsilon, x_o + \varepsilon)$,
for all $n \in \mathbb{N}$, $n \geq N(\varepsilon)$.
If (RealSeq) converges to x_o, then (RealSeq) is said to *converge* and called a *convergent sequence*, and we write
$x_n \to x_o$ as $n \to \infty$
read "x_n goes to x_o as n goes to ∞." And x_o is called the *limit* of $(x_n)_{n=1}^{\infty}$.
In what follows, by a *sequence* we shall mean a real sequence.
A sequence that does not converge is called a *divergent sequence* and said to *diverge*.

Let $x_o \in \mathbb{R}$ and $\emptyset \neq A \subseteq \mathbb{R}$. The point x_o, which does not necessarily belong to A, is called a *point of accumulation of A* provided for *every* $\delta > 0$,
$A \cap N'_\delta(x_o) \neq \emptyset$.

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THEOREM 1 Let $\emptyset \neq A \subseteq \mathbb{R}$, $x_o \in \mathbb{R}$ a point of accumulation of A if and only if, there exists a sequence $(x_n)_{n=1}^{\infty}$ in A, $x_n \neq x_o$ for all n, and $x_n \to x_o$ as $n \to \infty$. ///

Definition of Function Limit 2 Consider the *real* function $f : A \to \mathbb{R}$, $\emptyset \neq A \subseteq \mathbb{R}$, the real number L, and $x_o \in \mathbb{R}$ a point of accumulation of A. The function $f : A \to \mathbb{R}$ is said to have the *number L as limit at the point* x_o provided:
Whenever given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{l} x \in A \\ \text{and} \\ 0 < |x - x_o| < \delta(\varepsilon) \end{array}\right\} \Rightarrow |f(x) - L| < \varepsilon$$

And we write $\lim_{x \to x_o} f(x) = L$.

Note 3 A real function $f : A \to \mathbb{R}$ may or may not have a limit at a point of accumulation x_o of A.

We have

Sequential Criterion 4 Let $\emptyset \neq A \subseteq \mathbb{R}$ and $x_o \in \mathbb{R}$ a point of accumulation of A. The real function $f : A \to \mathbb{R}$ has $L \in \mathbb{R}$ as limit at x_o (i.e., $\lim_{x \to x_o} f(x) = L$)

$\Leftrightarrow$ For every sequence $(x_n)_{n=1}^{\infty}$ in A, $x_n \neq x_o$ for all n, and $x_n \to x_o$ as $n \to \infty$, the sequence of values of f, $(f(x_n))_{n=1}^{\infty}$, converges to L. ///

2 THE DIVERGENCE CRITERION

According to Note 3 of the preceding section, a real function $f : A \to \mathbb{R}$, $\emptyset \neq A \subseteq \mathbb{R}$, may have a limit or may not have a limit at a point $x_o \in \mathbb{R}$ of accumulation of A. We consider sequential characterizations of the following statements.

(1) Let $L \in \mathbb{R}$. The real function $f : A \to \mathbb{R}$ does not have L as limit at the point $x_o \in \mathbb{R}$ of accumulation of A.

(2) The function $f : A \to \mathbb{R}$ does **not** have a limit at the point $x_o \in \mathbb{R}$ of accumulation of A.

It is easy to answer (1). Immediate from the *Sequential Criterion* 1.4 is that :

$f : A \to \mathbb{R}$ does not have L as limit at $x_o \Leftrightarrow$ there exists a sequence $(x_n)_{n=1}^{\infty}$ in A, $x_n \neq x_o$ for all n, $x_n \to x_o$ as $n \to \infty$, but $(f(x_n))_{n=1}^{\infty}$ does **not** converge to L.

One may suspect from the preceding that an answer to (2) is :

$f : A \to \mathbb{R}$ does not have a limit at $x_o \Leftrightarrow$ for every given L there exists a sequence $(x_n^L)_{n=1}^{\infty}$ in A, $x_n^L \neq x_o$ for all n, $x_n^L \to x_0$ as $n \to \infty$,

but $(f(x_n^L))_{n=1}^{\infty}$ does not converge to L.

Clearly, if this answer to (2) is true, it is certainly not very simple. The answer to (2) given in [1], 4.1.9(b), p.102, **without proof**, is there labeled.

The Divergence Criterion The real function $f : A \to \mathbb{R}$, $\emptyset \neq A \subseteq \mathbb{R}$, does not have a limit at the point $x_o \in \mathbb{R}$ of accumulation of $A \Leftrightarrow$ there exists a sequence $(x_n)_{n=1}^{\infty}$ in A, $x_n \neq x_o$ for all n, $x_n \to x_o$ as $n \to \infty$, but $(f(x_n))_{n=1}^{\infty}$ diverges. ///

The Divergence Criterion stated above is what this paper sets out to prove. We need some three lemmas.

LEMMA 1 Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be real sequences converging to same point $z \in \mathbb{R}$. Then, the *mixed sequence*
$(x_1,\ y_1,\ x_2,\ y_2,\ x_3,\ y_3,\ \ldots\ldots)$
also converges to z.

Proof Employ the characterization:
$a_n \to a$ as $n \to \infty \Leftrightarrow$ for any given $\varepsilon > 0$, all terms of
$(a_n)_{n=1}^{\infty}$, except perhaps finitely many,
lie in the ε-neighbourhood of a. ///

LEMMA 2 *The Subsequence Theorem* If the real sequence
$(x_1, x_2, \ldots)$ …(RealSeq)
converges to $x_o \in \mathbb{R}$, then, every subsequence
$(x_{n_1},\ x_{n_2},\ \ldots)$ …(SubSeq)
of (RealSeq) converges to x_o as well.

Proof This is a folklore in *Elementary Real Analysis*. ///

LEMMA 3 Let $\varnothing \neq A \subseteq \mathbb{R}$, and suppose $x_o \in \mathbb{R}$ is a point of accumulation of A. Let $f : A \to \mathbb{R}$, and that for every sequence, $(x_n)_{n=1}^{\infty}$, in A, having the properties
(i) $x_n \neq x_o$ for all n,
and
(ii) $x_n \to x_o$ as $n \to \infty$,
the sequence of values of f, $\left(f(x_n)\right)_{n=1}^{\infty}$, converges.
Then, all such sequences converge to the same limit.

Proof Let $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ be sequences in A meeting the conditions (i) and (ii), and by the hypotheses, $\left(f(y_n)\right)_{n=1}^{\infty}$ and $\left(f(z_n)\right)_{n=1}^{\infty}$ converge to L_1 and L_2, respectively, say. Then, by LEMMA 1, the mixed sequence $(y_1,\ z_1,\ y_2,\ z_2,\ y_3,\ z_3,\ \ldots)$ converges to x_o, and by the hypotheses the sequence
$(\ f(y_1),\ f(z_1),\ f(y_2),\ f(z_2),\ f(y_3),\ f(z_3),\ \ldots)$ …(Δ)
converges to L_3, say. Clearly, the sequences $\left(f(y_n)\right)_{n=1}^{\infty}$ and $\left(f(z_n)\right)_{n=1}^{\infty}$ are subsequences of (Δ). Hence, by LEMMA 2, therefore,
$L_1 = L_3$ and $L_2 = L_3$
Hence, $L_1 = L_2 = L_3$. ///
Now to the

Proof of *the Divergence Criterion* The reverse implication $\Leftarrow$ is immediate from the Sequential Criterion 1.4.
$\Rightarrow$: Assume that the statement "there exists a sequence $(x_n)_{n=1}^{\infty}$ in A, $x_n \neq x_o$ for all n, $x_n \to x_o$ as $n \to \infty$, but $\left(f(x_n)\right)_{n=1}^{\infty}$ diverges" is false. Then, this implies that for every sequence $(x_n)_{n=1}^{\infty}$ in A, $x_n \neq x_o$ for all n, $x_n \to x_o$ as $n \to \infty$, the sequence $\left(f(x_n)\right)_{n=1}^{\infty}$ converges. Then, this, LEMMA 3 and the Sequential Criterion 1.4 force f to have a limit at x_o. ///

REFERENCES

[1] Robert G. Bartle and Donald R. Sherbert, *INTRODUCTION TO REAL ANALYSIS*, 3rd Edition, John Wiley & Sons, New York, 2000.