Journal of the Nigerian Association of Mathematical Physics
Volume 53 **(November** ***2019 Issue*****), pp1 – 10**

A PROOF OF A SUPREMUM-INFIMUM PROPERTY OF ℝ

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Abstract

A property of ℝ that is rarely stated to talkless of being established in the literature of Elementary Real Analysis, is stated and proved. An application in the Theory of the Riemann Integral is pointed out.

***Keywords*:** supremum, infimum

1 LANGUAGE AND NOTATION

Our language and notation shall be pretty standard as found, for example, in [1]. $\varnothing$ denotes the empty set, ℝ is the collection of the real numbers. We shall indicate the end or absence of a proof by ///.

Let $a, b \in \mathbb{R}$. If $a \leq b$ (also written $b \geq a$), a is said to *precede* b and b said to *dominate* a. Let $\varnothing \neq S \subseteq \mathbb{R}$. If $\mu \in \mathbb{R}$ precedes all the elements of S, μ is called a *lower bound of S*. And, a lower bound, μ^*, say, of S, dominating all other lower bounds of S, is called the *infimum* of S, and denoted *inf S*. Similarly $\lambda \in \mathbb{R}$ that dominates all the elements of S is called an *upper bound* of S, and, an upper bound, λ^*, say, of S, preceding all other upper bounds, is called the *supremum* of S, and denoted *sup s*. If S has a lower bound, it is said to be *bounded below*; similarly, if S has an upper bound it is said to be *bounded above*. If S is bounded above and below, it is simply said to be *bounded* and called a *bounded set*.

Comparability Property 1 Let $x, y \in \mathbb{R}$. Then,
$x < y$, or $x = y$, or $x > y$ ….(C.P)
and one and *only one* of (CP) must be true. In particular, for $a \in \mathbb{R}$, one and only one of
$a < 0$, or $a = 0$, or $a > 0$
must be true. ///

Employing the above Comparability Property 1 of ℝ, if $a \in \mathbb{R}$, its *absolute value*, denoted $|\,a\,|$ is defined as follows.

$$|\,a\,| \equiv \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Let $\varnothing \neq S \subseteq \mathbb{R}$. We define the set $|\,S\,|$ as follows.
$S\,| \equiv \{|\,s\,| : s \in S\}$.

We can now state the *Supremum-Infimum Property* of ℝ, advertised in the *Abstract*.

The *Sup-Inf* Property 2 Let $\varnothing \neq S \subseteq \mathbb{R}$.. Suppose S is a bounded set. Then,
(i) $|S|$ is a bounded set, and
(ii) $\sup |S| - \inf |S|$
$\leq \sup\{|x - y| : x, y \in S\}$

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$= sup \{x - y : x, y \in S\}$
$= sup\, S - inf\, S$. ///

See also the statement of **The *Sup-Inf Propert*y 14:**
The Sup-Inf Property has applications in *Elementary Real Analysis,* but a proof is *difficult to locate* in the literature. This paper furnishes a proof of this property, and also points out one application. The notation $| S |$ and similar others given, presently, are the author's

2 PROOF OF THE SUP -INF PROPERTY

We reel out some properties of $\mathbb{R}$, and intermittently give some definitions and *observations* about the bounded non-empty subset S of $\mathbb{R}$.

Property 1 For $x, y \in \mathbb{R}$,
(i) $x < y \Rightarrow -y < -x$,
and
(ii) $x \leq y \Rightarrow -y \leq -x$. ///

Let $\varnothing \neq S \subseteq \mathbb{R}$. Define $-S \equiv \{-s : s \in S\}$.

Observation 1 Let $\varnothing \neq S \subseteq \mathbb{R}$. If S is bounded, so is $-S$.
Proof *Hypothesis* S is bounded.

So, let $\mu, \lambda \in \mathbb{R}$ be a lower bound and an upper bound of S, respectively. Hence,
$\mu \leq s \leq \lambda$ for all $s \in S$.
By Property 1, therefore,
$-\lambda \leq -s \leq -\mu$ for all $s \in S$.
And so, $-\lambda$ is a lower bound for $-S$, and $-\mu$ is an upper bound for $-S$. Hence $-S$ is bounded. ///

Property 2 Let $x, y, p, q \in \mathbb{R}$. Then,

(i) $\left.\begin{array}{r} x < y \\ and\ p < q \end{array}\right\} \Rightarrow x + p < y + q$

and

(ii) $\left.\begin{array}{r} x \leq y \\ \text{and } p \leq q \end{array}\right\} \Rightarrow x + p \leq y + q$. ///

Now let $\varnothing \neq S_1, S_2 \subseteq \mathbb{R}$, and define
$S_1 + S_2 \equiv \{x + y : x \in S_1, y \in S_2\}$.

Observation 2 Let $\varnothing \neq S_1, S_2 \subseteq \mathbb{R}$. If S_1 and S_2 are bounded sets, so is the set $S_1 + S_2$.
Proof *Hypothesis* S_1 and S_2 are bounded sets.
So, let λ_1, λ_2 be respective upper bounds for S_1 and S_2, and so for $x \in S_1$ and $y \in S_2$,
$x \leq \lambda_1$
and
$y \leq \lambda_2$.
By Property 2, therefore,
$x + y \leq \lambda_1 + \lambda_2$ for $x \in S_1$, $y \in S_2$ …(Δ)
Since x and y were arbitrary, it follows from (Δ) that $\lambda_1 + \lambda_2$ is an upper bound for $S_1 + S_2$, and hence, $S_1 + S_2$ is bounded above. Similarly, $S_1 + S_2$ is bounded below. ///

Let $\varnothing \neq S_1, S_2 \subseteq \mathbb{R}$. Define
$S_1 - S_2 \equiv \{x - y : x \in S_1, y \in S_2\}$

Observation 3 Let $\varnothing \neq S_1, S_2 \subseteq \mathbb{R}$. Then,
(i) $S_1 - S_2 = S_1 + (-S_2)$,
(ii) If S_1 and S_2 are bounded sets, so is $S_1 - S_2$.

Proof Immediate from Observation 1 and Observation 2. ///

Observation 4 Let $\varnothing \neq S \subseteq \mathbb{R}$. If S is bounded, so is $S - S = \{x - y : x, y \in S\}$.
Proof Immediate from Observation 3. ///.
Comparability Property 1 of Section 1 says:

For $x, y \in \mathbb{R}$, one and only one of
$x < y$ or $x = y$ or $x > y$
must be true.

Therefore, for $x, y \in \mathbb{R}$, define

$$x \vee y = \max\{x, y\} = \begin{cases} y, & \text{if } x < y \\ x, & \text{if } x = y \\ x, & \text{if } x > y \end{cases}$$

Property 3 For $x \in \mathbb{R}$,
$|x| = \max\{x, -x\} = x \vee -x$. ///

Let $\varnothing \neq S \subseteq \mathbb{R}$. Define
$|S| = \{|s| : s \in S\}$.

Observation 5 Let $\varnothing \neq S \subseteq \mathbb{R}$. If S is a bounded set, so is $|S|$.
Proof *Hypothesis* S is a bounded set.

By Observation 1, therefore, $-S$ is a bounded set. By the *Hypothesis* there exist $\mu, \lambda \in \mathbb{R}$ such that μ is a lower bound for S and λ is an upper bound for S. Because $-S$ is also bounded, $\mu^-, \lambda^- \in \mathbb{R}$ exist such that μ^- is a lower bound for $-S$ and λ^- is an upper bound for $-S$. Hence, $\mu \wedge \mu^-$ is a lower bound for *both* S and $-S$, and $\lambda \vee \lambda^-$ is an upper bound for both S and $-S$. And so,
$\mu \wedge \mu^- \leq s, -s \leq \lambda \vee \lambda^-$ for all $s \in S$.
By Property 3, therefore,
$\mu \wedge \mu^- \leq |s| \leq \lambda \vee \lambda^-$ for all $s \in S$.
And from this follows that $|S|$ is a bounded set. ///

Observation 6 Let $\varnothing \neq S \subseteq \mathbb{R}$, and suppose that S is a bounded set. Then.
(i) $-S$,
(ii) $S - S$
(iii) $|S|$
and
(iv) $|S - S|$
are all bonded sets.

Proof Immediate. ///

Observation 7 Let $\varnothing \neq S \subseteq \mathbb{R}$. Then,
$-S \subseteq S \Rightarrow |S| \subseteq S$.
Proof *Hypothesis* $-S \subseteq S$.
Hence,
$s \in S \Rightarrow -s \in S$
and so
$s \in S \Rightarrow s, -s \in S$.
Hence,
$s \in S \Rightarrow \max\{s, -s\} \in S$
By Property 3, therefore,
$s \in S \Rightarrow |s| \in S$.
Hence, since S was arbitrary, we have shown that

$-S \subseteq S \Rightarrow |S| \subseteq S$. ///

Remark In a forthcoming book of the author, *The Real Numbers*, the set $|S|$ is called *absolute S*|].

Property 4 *The LUB Axiom* Every non-empty subset of $\mathbb{R}$, bounded above, has a supremum. ///

FACT 5 Let $\emptyset \neq A \subseteq S \subseteq \mathbb{R}$, and suppose that S is bounded above. Then,
(i) A is also bounded above
(ii) $\sup S$ and $\sup A$ exist, and
(iii) $\sup A \leq \sup S$.

Proof Immediate from Property 4 above, and the definition of the supremum as an upper bound preceding all other upper bounds. ///

Observation 8 Let $\emptyset \neq S \subseteq \mathbb{R}$, and suppose that S is bounded. Then,
(i) $|S|$ is bounded, and

$$-S \subseteq S \Rightarrow \begin{cases} |S| \subseteq S \\ \text{and} \\ \sup |S| \leq \sup S. \end{cases}$$

Proof (i) is Observation 5, and so by Property 4, both $\sup S$ and $\sup |S|$ exist.
(ii): That $|S| \subseteq S$ is Observation 7. And, that $\sup |S| \leq \sup S$ is now immediate from FACT 5. ///

Property 6 For $x \in \mathbb{R}$, $x \leq |x|$. ///

Observation 9 Let $\emptyset \neq S \subseteq \mathbb{R}$. Then,

$$\left.\begin{matrix} S \text{ bounded} \\ \text{and} \\ -S \subseteq S \end{matrix}\right\} \Rightarrow \sup |S| = \sup S \qquad \ldots(\rho)$$

Proof That

$$\sup |S| \leq \sup S \qquad \ldots(1)$$

is Observation 8. It suffices, therefore, to reverse the inequality in (1) to prove (ρ). By Property 6,
$s \leq |s|$, for all $s \in S$.
And so,
$s \leq |s| \leq \sup |S|$ for all $s \in S$.
That is,

$$s \leq \sup |S| \text{ for all } s \in S. \qquad \ldots(\nabla)$$

Hence, $\sup |S|$ is an upper bound for S, and so by the definition of the supremum as an upper bound preceding all other upper bounds, it follows from (∇) that

$$\sup S \leq \sup |S| \qquad \ldots(2)$$

Clearly, (1) and (2) gives (Δ). ///

Observation 10 Let $\emptyset \neq S \subseteq \mathbb{R}$, and suppose that S is a bounded set. Then
(i) $S - S$ and $|S - S|$ are bounded sets,
and
(ii) $\sup(S - S) = \sup|S - S|$

Proof That $S - S$ and $|S - S|$ are bounded sets are claims of Observation 6. To prove (ii), simply observe that $-(S - S) \subseteq S - S$, and so invoke Observation 9. ///
We recast Observation 10(ii) as follows.

Observation 10(ii) Let $\emptyset \neq S \subseteq \mathbb{R}$, and suppose S is bounded. Then
$\sup \{x - y : x, y \in S\}$
$= \sup \{|x - y| : x, y \in S$. ///

To establish our next Observation, we reel out some six Properties of $\mathbb{R}$; the superscripts identify those properties. First,

Property 7[1] *The EQUI-LUB Axiom* Every non-empty subset of $\mathbb{R}$ bounded below has an infimum. ///

Property 8[2] Let $a, b \in \mathbb{R}$. Then,
$a \le b \le a \Leftrightarrow a = b$. ///

Property 9[3] Let $x, y, p, q \in \mathbb{R}$. Then,

(i) $\left.\begin{array}{l} x < y \\ \text{and} \\ p < q \end{array}\right\} \Rightarrow x - q < y - p$

and

(ii) $\left.\begin{array}{l} x \le y \\ \text{and} \\ p \le q \end{array}\right\} \Rightarrow x - q \le y - p$. ///

Property 10[4] Let $\alpha, p \in \mathbb{R}$. Then,

(i) $\left.\begin{array}{l} p < \alpha \\ \text{and} \\ -p < \alpha \end{array}\right\} \Rightarrow |p| < \alpha$,

and

(ii) $\left.\begin{array}{l} p \le \alpha \\ \text{and} \\ -p \le \alpha \end{array}\right\} \Rightarrow |p| \le \alpha$. ///

Property 11[5] *The Great Characterizations of the Supremum & the Infimum* Let $\emptyset \ne S \subseteq \mathbb{R}$, and suppose that S is bounded above. Then,

(i) $\lambda = \sup S$
$\Leftrightarrow$
λ is an upper bound of S, and if $\varepsilon > 0$, then $\lambda - \varepsilon$ is not an upper bound of S,

(ii) $\lambda = \sup S$
$\Leftrightarrow$
λ is an upper bound of S, and if $\varepsilon > 0$, there exists $x \in S$ such that
$\lambda - \varepsilon < x \le \lambda$,

(iii) $\lambda = \sup S$
$\Leftrightarrow$
λ is an upper bound of S, and if $\lambda^* \in \mathbb{R}$ and $\lambda^* < \lambda$, then there exist $x \in S$ such that
$\lambda^* < x \le \lambda$

Suppose S is bounded below. Then.

(i)′ $\mu = \inf S$
$\Leftrightarrow$
μ is a lower bound of S, and if $\varepsilon > 0$, then $\mu + \varepsilon$ is not a lower bound of S,

(ii)′ $\mu = \inf S$
$\Leftrightarrow$
μ is a lower bounded of S, and if $\varepsilon > 0$, then there exists $x \in S$ such that
$\mu \le x < \mu + \varepsilon$,

(iii)′ $\mu = \inf S$
$\Leftrightarrow$
μ is a lower bound of S, and if $\mu^* \in \mathbb{R}$ and $\mu < \mu^*$, then there exists $x \in S$ such that
$\mu \le x < \mu^*$. ///

Property 12[6] Let $a, b \in \mathbb{R}$. Then,
$a \le b + \varepsilon$ for every $\varepsilon > 0 \Rightarrow a \le b$. ///

Now to our next

Observation 11 Let $\varnothing \neq S \subseteq \mathbb{R}$ and suppose S is a bounded set. Then,

$\sup S - \inf S = \sup\{x - y : x, y \in S\}$

$= \sup\{|x - y| : x, y \in S\}$ …($\wedge$)

Proof By Observation 10(ii),

$\sup\{x - y : x, y \in S\} = \sup\{|x - y| : x, y \in \mathbb{R}\}$.

And so, to prove ($\wedge$), it suffices to show that

$$\left.\begin{aligned} &\sup\{|x - y| : x, y \in S \\ &\leq \sup S - \inf S \\ &\leq \sup\{x - y : x, y \in S\} \end{aligned}\right\} \qquad \ldots(\vee)$$

And so, to obtain ($\vee$), we shall separately show that

$\sup\{|x - y| : x, y \in S\} \leq \sup S - \inf S$ …(3)

and

$\sup S - \inf S \leq \sup\{x - y : x, y \in S\}$ …(4)

***Proof of* (3)**: Let $x, y \in S$. Then, clearly,

$x, y \leq \sup S$

and

$\inf S \leq x, y$.

By a repeated application of Property 9[3], therefore, we have

$x - y \leq \sup S - \inf S$

and

$-(x - y) = y - x \leq \sup S - \inf S$.

And so, by Property 10[4],

$|x - y| \leq \sup S - \inf S$, for all $x, y \in S$ …{*}

Since x and y were arbitrary, it follows from (*) that $\sup S - \inf S$ is an upper bound for the set $\{|x - y| : x, y \in S\}$. And so, by the definition of the supremum as an upper bound preceding all other upper bounds it follows that

$\sup\{|x - y| : x, y \in S\} \leq \sup S - \inf S$,

which is (3) that we set out to prove.

***Proof of* (4):** Let $\varepsilon > 0$. Then, $\frac{\varepsilon}{2} > 0$. By Property 11[5](ii) and (ii)′, there exists $x \in S$ such that

$\sup S - \frac{\varepsilon}{2} < x$ …(σ)

and there exists $y \in S$ such that

$y < \inf S + \frac{\varepsilon}{2}$ …($\sigma\sigma$)

By Property 9[3] applied to (σ) and ($\sigma\sigma$) therefore, we have

$(\sup S - \frac{\varepsilon}{2}) - (\inf S + \frac{\varepsilon}{2}) < x - y$.

That is,

$\sup S - \inf S - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} < x - y$

That, is,

$\sup S - \inf S < (x - y) + \varepsilon$

from which follows that

$\sup S - \inf S \leq (x - y) + \varepsilon \leq \sup\{p - q : p, q \in S\} + \varepsilon$.

That is,

$\sup S - \inf S \leq \sup\{p - q : p, q \in S\} + \varepsilon$, for every $\varepsilon > 0$.

And so, by Property 12[6], (4) follows. ///

Property 13 For $x, y \in \mathbb{R}$,
(i) $x \le |x|,\ -x \le |x|$,
(ii) $|x+y| \le |x| + |y|$,
and
(iii) $\big||x| - |y|\big| \le \left\{\begin{matrix} |x+y| \\ |x-y| \end{matrix}\right\} \le |x| + |y|$. ////

Observation 12 Let $\varnothing \ne S \subseteq \mathbb{R}$ and suppose that S is bounded. Then
(i) $|S|$,
(ii) $S - S$,
and
(iii) $|S - S|$
are bounded sets, and
(iv) $\sup|S| - \inf|S| \le \sup\{|x-y| : x, y \in S\}$
$(= \sup|S-S|)$.

Proof Clearly, (i), (ii) and (iii) are well-known by now, for boun- ded S, $\varnothing \ne S \subseteq \mathbb{R}$. Clearly, (iv) is
$\sup|S| - \inf|S| \le \sup\{|x-y| : x, y \in S\}$ …(δ)
We prove (δ). By the first equality in Observation 11, we have
$\sup|S| - \inf|S| = \sup\{|x| - |y| : x, y \in S\}$ …(μ)
Let $x, y \in S$. From the first claim of Property 13(i), we have
$|x| - |y| \le \big||x| - |y|\big|$ …(5)
And, by (iii) of same Property 13,
$\big||x| - |y|\big| \le |x - y|$ …(6)
(5) and (6), therefore, give
$|x| - |y| \le |x - y|$ …(7)
for any $x, y \in S$. And hence, by now familiar arguments,
$\sup\{|x| - |y| : x, y \in S\} \le \sup\{|x-y| : x, y \in S\}$ …(8)
Clearly, (μ) and (8) give (δ), which is what we set out to prove. ///
We have thus, completed the proof of

The Sup-Inf Property 14 Let $\varnothing \ne S \subseteq \mathbb{R}$ and suppose S is a bounded set. Then,
(i) $|S|$,
(ii) $S - S = \{x - y : x, y \in S\}$,
and
(iii) $|S - S| = \{|x-y| : x, y \in S\}$
are bounded sets, and
(iv) $\sup|S| - \inf|S| \le \sup\{|x-y| : x, y \in S\}$
$= \sup\{x - y : x, y \in S\} = \sup S - \inf S$.

In particular,
$\sup|S| - \inf|S| \le \sup S - \inf S$. ///

3 AN APPLICATION

Definition of the Integral

THROUGHOUT, $a, b \in \mathbb{R}$, $a < b$, $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$is a closed bounded interval of $\mathbb{R}$, and $f : [a, b] \to \mathbb{R}$. We do not rep- eat this standing rule.
Let $P = \{x_0, x_1, \dots, x_n\} \subseteq [a, b]$ be such that
$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$.

P is called a *partition* of $[a, b]$, the points, $x_0, x_1, \ldots, x_n$ of P called the *partition points* of P and the closed bounded intervals
$[x_0, x_1], [x_1, x_2], \ldots, [x_{k-1}, x_k], \ldots, [x_{n-2}, x_{n-1}], [x_{n-1}, x_n]$
called the *subintervals* of the partition P; In particular, $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, is called the kth *subinterval* of P.
By $\wp[a, b]$ we denote the collection of all the partitions of $[a, b]$; and so, by
$P \in \wp[a, b]$
we mean that P is a partition of $[a, b]$. E.g., the *trivial partition* of $[a, b]$, $P^{\text{triv.}} = \{a = x_0, x_1 = b\} \in \wp[a, b]$.
Let $P, P' \in \wp[a, b]$. If $P \supseteq P'$, we say that P is *finer* than P' and that P *refines* P', and so call P a *refinement* of P'.

The function $f : [a, b] \to \mathbb{R}$ is said to be *bounded* and called a *bounded function*, if its range, $f([a, b])$, is a bounded set. We denote by $\mathcal{B}[a, b]$ the collection of all the bounded functions $f : [a, b] \to \mathbb{R}$, and so, by
$f \in \mathcal{B}[a, b]$
shall be meant that $f : [a, b] \to \mathbb{R}$ is a bounded function.

Suppose $f \in \mathcal{B}[a, b]$ and $P = \{a = x_0, x_1, \ldots, x_n = b\} \in \wp[a, b]$.

Since $f \in \mathcal{B}[a, b]$, then the sets $f([a, b])$, $f([x_{k-1}, x_k])$, $k = 1, 2, \ldots, n$, are bounded sets. Define
$M(f) = \sup f([a, b])$
$m(f) = \inf f([a, b])$
$M_k(f) = \sup f([x_{k-1}, x_k])$
$m_k(f) = \inf f([x_{k-1}, x_k])$.
$L(f, P) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1})$, called the *lower Riemann sum of f w.r.t P*,
and
$U(f, P) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1})$, called the *upper Riemann sum of f w.r.t P*.

FACT 1 Let $P \in \wp[a, b]$ and $f \in \mathcal{B}[a, b]$. Then,
$$m(f)(b-a) \le L(f, P) \le U(f, P) \le M(f)(b-a). \ ///$$

FACT 2 Let $f \in \mathcal{B}[a, b]$ and $P, P' \in \wp[a, b]$. If P refines P', then,
(i) $L(f, P') \le L(f, P)$,
and
(ii) $U(f, P) \le U(f, P')$. ///

FACT 3 Let $f \in \mathcal{B}[a, b]$ and $P, P' \in \wp[a, b]$. Then,
$$L(f, P) \le U(f, P'). \ ///$$

From all the preceding, we have that

(i) the $L(f, P)$ *s* increase with finer partition, and
(ii) the collection
$\{L(f, P) : P \in \wp[a, b]\}$
of all the lower Riemann sums of f, is bounded above by $M(f)(b-a)$. By the LUB Axiom, therefore,
$\sup\{L(f, P) : P \in \wp[a, b]\}$
exists, and called the *lower Riemann Integral of f* and denoted $\underline{\int_a^b} f$. Similarly,

$\inf\{U(f, P) : P \in \wp[a, b]\}$

exists and called the *upper Riemann Integral* of f and denoted $\overline{\int_a^b} f$.

FACT 4 For $f \in \mathcal{B}[a, b]$,
(i) $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ exist, and

(ii) $\underline{\int}_a^b f \leq \overline{\int}_a^b f$. ///

DEFINITION 5 Let $f \in \mathcal{B}[a, b]$.
(i) f is said to be *integrable*, and called a *Reimann integrable function*, provided
$\underline{\int}_a^b f = \overline{\int}_a^b f$, and

(ii) If f is integrable, its *integral*, denoted $\int_a^b f$, is defined as the common value of the lower, $\underline{\int}_a^b f$, and upper, $\overline{\int}_a^b f$, Riemann integrals;
that is, $\int_a^b f = \underline{\int}_a^b f = \overline{\int}_a^b f$.

Integrability Criterion

The Rieman integrability Criterion 6 Let $f \in \mathcal{B}[a, b]$. Then, f is integrable if and only if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon \in \wp[a, b]$ such that
$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$. ///

Let $f : [a, b] \to \mathbb{R}$. Define
$|f| : [a, b] \to \mathbb{R}, \ x \mapsto |f(x)|, \ x \in [a, b]$.

We now come to the advertised application. First, a

Property *Sidewise Addition of Inequalities* 7 Let $a_1, a_2, \ldots, a_n, \quad b_1, b_2, \ldots, b_n \in \mathbb{R}$. Then,

$$\left.\begin{array}{l} a_1 \leq b_1 \\ a_2 \leq b_2 \\ . \\ . \\ . \\ a_n \leq b_n \end{array}\right\} \Rightarrow a_1 + a_2 + \ldots + a_n \leq b_1 + b_2 + \ldots + b_n. ///$$

And finally,

THEOREM 8 *Integrability of* $|f|$ Let $f \in \mathcal{B}[a, b]$. If f is integrable, so is $|f|$

Proof *Hypothesis* $f \in \mathcal{B}[a, b]$ is integrable.

We want to show that $|f|$ is, consequently, integrable. We employ the Riemann Integrability Criterion twice. First, by the *Hypothesis*, if $\varepsilon > 0$ is given, there exists a partition $P_\varepsilon \in \wp[a, b]$ such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon \qquad \ldots(\alpha)$$

If $P_\varepsilon = \{a = x_o, x_1, \ldots, x_n = b\}$, then (α) can be written as

$$\sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) < \varepsilon.$$

That is, as

$$\sum_{k=1}^{n} (M_k(f) - m_k(f))(x_k - x_{k-1}) < \varepsilon \qquad \ldots(\beta)$$

From the Sup-Inf Property $[|\sup|S| - \inf|S|| \leq \sup S - \inf S|]$, we clearly have,

$$M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f) \qquad \ldots(\gamma)$$

And so, from (γ), we have

$$(M_k(|f|) - m_k(|f|))(x_k - x_{k-1})$$
$$\leq (M_k(f) - m_k(f))(x_k - x_{k-1}) \qquad \ldots(\gamma\gamma)$$

Hence, by the *Sidewise Addition Property*, (γγ) gives

$$\sum_{k=1}^{n}(M_k(|f|) - m_k(|f|))(x_k - x_{k-1}) \leq \sum_{k=1}^{n}(M_k(f) - m_k(f))(x_k - x_{k-1}).$$

That is,

$U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) \leq U(f, P_\varepsilon) - L(f, P_\varepsilon)$

which by (β),

$< \varepsilon$

And so,

$U(|f|, P_\varepsilon) - L(|f|, P_\varepsilon) < \varepsilon$.

And so, again, by the Riemann Integrability Criterion, $|f|$ is integrable. ///

REFERENCES

[1] Adegoke Olubummo, *Introduction to Real Analysis*, Heneman Books Publishers, Ibadan, 1979, 2008.