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DIRECT TENTH ORDER IMPLICIT MULTI-HYBRID BLOCK METHODS FOR SPECIAL SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract

This paper considered a two-step implicit multi-hybrid block methods of uniform order $3k+4$ for solving special second order delay differential equations. The methods were generated by interpolation and collocation techniques using a combination of power series and exponential function. The approximate basis function is interpolated at the first two grid points and collocated at both grid and off-grid points. The developed schemes and its derivatives were combined to form block methods to simultaneously solve special second order delay differential equations. The basic properties of the methods such as order, error constants, consistency and convergence were also examined. The developed methods were applied to solve two special second order delay differential equations and the numerical results perform better in terms of accuracy when compared with the methods in the literature.

Keywords**:** Block Methods, direct tenth order, second order Delay D.E, Consistency, Mathieu's equation
Subject Classification: 65L05;65L06;65L20

1. Introduction

The importance of mathematics in solving real life problems cannot be over emphasized. Mathematics has been in use for centuries to perform some complex and computational intensive tasks. Also in science and Engineering, usually mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called differential equations. Differential equations also arises in some other field such as physical sciences, Economics, Medicine, Psychology operation Research study of thin film flow of a liquid in fluid dynamics and anthropology. There are different types of differential equations. They are ordinary differential equations, partial differential equations, stochastic differential equations and delay differential equations. The methods can be adopted for the solution of initial value problems (IVPs) of ordinary differential equations (ODE) of the form

$$y^{(n)} = f(x, y, y' ... y^{(n-1)}),\ y(a) = \eta_0,\ y'(a) = \eta_1, ... y^{(n-1)}(a) = \eta_{n-1} \qquad (1)$$

on the interval $[a,b]$

The conventional way of solving higher order differential equation of type (1) is by reducing it to an equivalent system of lower order initial value problem of ODES. The success of this approach was faulted as a result of some short comings.

In recent years, there has been a growing interest in the numerical treatment of delay differential equations. It occupies a place of central importance due to their versatility in the mathematical modelling of processes in various application fields. It is also of central importance in biological sciences (e.g population dynamics and epidemiology in Adegboyega [1]. For example when the birth rate of predators is affected by prior levels of predators or prey rather than by only the current levels in a predator prey model. The manner in which the properties of system of delay differential equation differ from those of systems of ODE has remain an active area of research (Martin et al. [2]). Hoo et al. [3] uses spline collocation and Adomain decomposition method for solving delay differential equations. Also Ogunfiditimi [4] employed Adomain decomposition method (ADM) to solve both linear and Non Linear DDE. In recent times, Anakira et al. [5] employed the Optimal Homotopy Asymptotic Method (OHAM) in solving linear and nonlinear.

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DDE. Sumudu and Wu"s [6] used a new modified variational iteration method for the solution of Linear and Non linear DDE. DDEs are differing from ODEs in that the derivatives at any time depend on the solution at prior times. In this paper, special second order delay differential equation of the form.

$$y''(t) = f(t, y(t-\tau)),\ y(t_0) = \alpha,\ y'(t_0) = \beta,\ t \ge t_0,\ \tau > 0 \tag{2}$$

is considered where α is the initial function and τ is the delay term. Most of the methods for solving special second order odes can be adopted for solving special second order delay differential equation. In addition to the above, equation (2) with multiple delays can be written in the following form:

$$y''(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2)...y(t-\tau_n)),\ t > 0 \tag{3}$$

with initial conditions

$$y(t) = \phi(t),\ y'(t_0) = \phi'(t),\ t \le t_0$$

There are two different ways to calculate the delay term in the developed methods.

For $x_0 - \tau \le 0$, the delay term is calculated using the initial function given $\phi(x)$. For $x_0 - \tau \ge 0$, the delay terms are rely on the location of $x_0 - \tau$. from this location, we are able to recall the values which we had been stored earlier.

2. Formulation of the Methods

This work considers an approximate solution that combines power series and exponential function of the form;

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j + a_{r+s} \sum_{j=0}^{r+s} \frac{\alpha^j x^j}{j!} \tag{4}$$

Interpolation and collocation procedures are used by choosing interpolation point s at a grid points and collocation points r at all points given rise to ζ = s+ r system of equations whose coefficients are determined by using appropriate procedures such as Gaussian elimination method. The first derivative and second derivative of (4) is given by (5) and (6) respectively.

$$y'(x) = \sum_{j=1}^{r+s-1} j a_j x^{j-1} + a_{r+s} \sum_{j=1}^{r+s} \frac{\alpha^j x^{j-1}}{(j-1)!} \tag{5}$$

$$y''(x) = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2} + a_{r+s} \sum_{j=2}^{r+s} \frac{\alpha^j x^{j-2}}{(j-2)!} \tag{6}$$

where $a_j, \alpha^j \varepsilon \Re$ for $j = 0(1)7$ and $y(x)$ is continuous differentiable. Let the solution of (4) be sought on the partition $\pi_N : a = x_0 < x_1 < x_2 <, ... x_n < x_{n+1}, ... < x_N = b$ of the integration interval [a,b] with a constant step-size h, given by $h = x_{n+1} - x_n, n = 0,1,......, N$

Then, substituting (6) into (4) gives;

$$f(x, y, y') = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2} + a_{r+s} \sum_{j=2}^{r+s} \frac{\alpha^j x^{j-2}}{(j-2)!} \tag{7}$$

Collocating (6) at $x_{n+j}, j = 0(\tfrac{1}{4})k$ and interpolating (5) at $x_{n+j}, j = 0(1)k-1$ to yield a system of equations which is solved for unknown a_j's, $j = 0(\tfrac{1}{4})k$ using Gaussian elimination method, and solving for the parameters a_j's and substituting their values into (4) leads to a class of continuous implicit hybrid linear multistep method of the form:

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \left(\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \sum_{vi=0} \beta_{vi}(x) f_{n+vi} \right) \tag{8}$$

where $y_{n+j} = y(x_{n+j})$, $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$ and $f_{n+vi} = f(x_{n+vi}, y_{n+vi}, y'_{n+vi})$ and τ is the delay term

For simplicity,

$$f_n = (t_n, y_n, y(t_n - \tau)),\ f_{n+1} = (t_{n+1}, y_{n+1}, y(t_{n+1} - \tau)),\ f_{n+2} = (t_{n+2}, y_{n+2}, y(t_{n+2} - \tau)),\ f_{n+\frac{1}{2}} = (t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}, y(t_{n+\frac{1}{2}} - \tau)),$$

$$f_{n+\frac{1}{4}} = (t_{n+\frac{1}{4}}, y_{n+\frac{1}{4}}, y(t_{n+\frac{1}{4}} - \tau)),\ f_{n+\frac{3}{4}} = (t_{n+\frac{3}{4}}, y_{n+\frac{3}{4}}, y(t_{n+\frac{3}{4}} - \tau)),\ f_{n+\frac{5}{4}} = (t_{n+\frac{5}{4}}, y_{n+\frac{5}{4}}, y(t_{n+\frac{5}{4}} - \tau)),$$

$$f_{n+\frac{3}{2}} = (t_{n+\frac{3}{2}}, y_{n+\frac{3}{2}}, y(t_{n+\frac{3}{2}} - \tau)),\ f_{n+\frac{7}{4}} = (t_{n+\frac{7}{4}}, y_{n+\frac{7}{4}}, y(t_{n+\frac{7}{4}} - \tau)),$$

The $\alpha_j(x),\ \beta_j(x), \beta_{vi}(x),$ in (8) is expressed as continuous function of t such that:

$t=\dfrac{x-x_{n+k-1}}{h}$, also $\dfrac{dt}{dx}=\dfrac{1}{h}$

Taking the derivative of (8) yields;

$$y'(x)=\frac{1}{h}\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+\frac{1}{h}\left(\sum_{j=0}^{k}\beta_j(x)f'_{n+j}+\sum_{vi=0}\beta_{vi}(x)f'_{n+vi}\right) \qquad (9)$$

Now, applying the block method as established in Shampine and Thompson [7], the block formula of the following form:

$$h^q\sum_{j=0}^{q}a_{ij}y^{\lambda}y_{n+j}=h^q\sum_{vi=0}^{\lambda}e_{ij}y^{\lambda}{}_n+\left(\sum_{j=1}^{q}d_{ij}f_n+\sum_{j=1}^{q}b_{ij}f_{n+ij}\right),\ i=0,1,...,q \qquad (10)$$

The λ represent the power of the derivative of the continuous method, p is the order of the problem to be solved, now, using vector notation, equation (9) becomes

$$h^{\lambda}AY_m=h^{\lambda}Ey_m+h^{p-\lambda}\left[Df(y_m)+BF(Y_m)\right] \qquad (11)$$

The Matrices $A=(a_{ij}),\ B=(b_{ij}),\ E=(e_{ij}),\ D=(d_{ij})$, are square matrices are constant coefficient matrix and $Y_m=(y_{n+vi},y_{n+1},y'_{n+vi})^T,\ y_m=(y_n,y_{n-vi},y_{n-1},y'_{n-vi})^T,\ F(Y_m)=(f_{n+vi},f_{n+1})^T,F(y_m)=(f_{n-1},f_n)^T,$ $i=1(1)q.$ The normalized form of (10) is then given as:

$$h^{\lambda}AY_m=h^{\lambda}Ey_m+h^{\mu-\lambda}\left[Df(y_m)+BF(Y_m)\right] \qquad (12)$$

This equation is solved for $y_{n+vi},\ \ y_{n+1}$. After some simplification, we obtain discrete schemes which is used to implement the hybrid schemes (8) without any need for predictors or Taylor series as starting values.

2.1 Derivation of Two -step method with six-off step points

Collocating (6) at $x_{n+j},\ j=0(\tfrac{1}{4})k$ and interpolating (5) at $x_{n+j},\ j=0(1)k-1$ to yield a system of equations which is solved for unknown a_j's, $j=0(\tfrac{1}{4})k$ using Gaussian elimination method, to obtain values for the parameters: $a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,..,a_{10}$ and substituting the values of the parameters into equation (4) and simplifying the result, to obtain a continuous scheme of the general form:

$$y(t)=\sum_{j=0}^{1}\alpha_j(t)y_{n+j}(t)+\sum_{j=0}^{k}\beta_j f_{n+j}(t) \qquad (13)$$

where $t=\dfrac{x-x_{n+k-1}}{h}, k=2$ and Setting $x_n=0,\ \ x_{n+1}=h,\ \ x_{n+2}=2h$

The coefficients of $\alpha_j(t)$ and $\beta_j(t)$ are:

$$\alpha_0(t)=-t+1,\quad \alpha_1(t)=t$$

$$\beta_0(t)=-\frac{761}{420}h^2t^3+\frac{104}{105}t^8h^2+\frac{1}{2}h^2t^2-\frac{96}{35}h^2t^7-\frac{64}{315}h^2t^9-\frac{7703}{113400}h^2t$$
$$-\frac{267}{50}h^2t^5+\frac{256}{14175}h^2t^{10}+\frac{29531}{7560}h^2t^4+\frac{1069}{225}h^2t^6$$

$$\beta_{\frac{1}{4}}(t)=\left(-\frac{2336}{315}t^8h^2-\frac{1924}{105}h^2t^4+\frac{1396}{45}h^2t^5+\frac{16}{3}h^2t^3-\frac{21056}{675}h^2t^6\right.$$
$$\left.-\frac{2048}{14175}h^2t^{10}-\frac{1552}{4725}h^2t+\frac{3680}{189}h^2t^7+\frac{128}{81}h^2t^9\right)$$

$$\beta_{\frac{1}{2}}(t)=\left(-\frac{18353}{225}h^2t^5+\frac{58}{567}h^2t+\frac{7648}{315}t^8h^2+\frac{61156}{675}h^2t^6-\frac{2176}{405}h^2t^9-\frac{28}{3}h^2t^2\right.$$
$$\left.-\frac{11456}{189}h^2t^7+\frac{207}{5}h^2t^4+\frac{1024}{2025}h^2t^{10}\right)$$

$$\beta_{\frac{3}{4}}(t) = \Big(-\frac{4768}{105} t^8 h^2 - \frac{8012}{135} h^2 t^4 - \frac{2048}{2025} h^2 t^{10} + \frac{2272}{21} h^2 t^7 + \frac{3188}{25} h^2 t^5 + \frac{112}{9} h^2 t^3 - \frac{34304}{225} h^2 t^6 + \frac{1408}{135} h^2 t^9 - \frac{5008}{14175} h^2 t \Big) \quad (4.6)$$

$$\beta_1(t) = \Big(-\frac{5828}{45} h^2 t^5 - \frac{1024}{81} h^2 t^9 - \frac{35}{3} h^2 t^3 + \frac{3344}{63} t^8 h^2 + \frac{21986}{135} h^2 t^6 + \frac{691}{12} h^2 t^4 + \frac{512}{405} h^2 t^{10} + \frac{47}{180} h^2 t - \frac{22912}{189} h^2 t^7 \Big) \quad (14)$$

$$\beta_{\frac{5}{4}}(t) = \Big(-\frac{12512}{315} t^8 h^2 - \frac{188}{5} h^2 t^4 - \frac{2048}{2025} h^2 t^{10} + \frac{82592}{945} h^2 t^7 + \frac{19564}{225} h^2 t^5 + \frac{112}{15} h^2 t^3 - \frac{76352}{675} h^2 t^6 + \frac{3968}{405} h^2 t^9 - \frac{2384}{14175} h^2 t \Big)$$

$$\beta_{\frac{3}{2}}(t) = \Big(\frac{1952}{105} t^8 h^2 + \frac{1024}{2025} h^2 t^{10} + \frac{2143}{135} h^2 t^4 - \frac{187}{5} h^2 t^5 - \frac{28}{9} h^2 t^3 + \frac{11212}{225} h^2 t^6 - \frac{128}{27} h^2 t^9 + \frac{986}{14175} h^2 t - \frac{832}{21} h^2 t^7 \Big)$$

$$\beta_{\frac{7}{4}}(t) = \Big(-\frac{224}{45} t^8 h^2 - \frac{412}{105} h^2 t^4 - \frac{2048}{14175} h^2 t^{10} + \frac{1952}{189} h^2 t^7 + \frac{2108}{225} h^2 t^5 + \frac{16}{21} h^2 t^3 - \frac{16}{945} h^2 t - \frac{8576}{675} h^2 t^6 + \frac{3712}{2835} h^2 t^9 \Big)$$

$$\beta_2(t) = \Big(-\frac{469}{450} h^2 t^5 + \frac{184}{315} t^8 h^2 + \frac{967}{675} h^2 t^6 - \frac{64}{405} h^2 t^9 - \frac{1}{12} h^2 t^3 - \frac{32}{27} h^2 t^7 + \frac{121}{280} h^2 t^4 + \frac{256}{14175} h^2 t^{10} + \frac{209}{113400} h^2 t \Big)$$

2.2 Derivation of Two-step Block method with six off-step points

The general block formula proposed by (6) in the normalized form is given by

$$A^{(0)} Y_m = e y_n + h^{\mu-\lambda} d f(y_n) + h^{\mu-\lambda} d F(y_m) \quad (15)$$

Evaluating (14) at $x = x_{n+j},\ j = 1, 2;$ the first derivative at $x = x_{n+j},\ j = 0\left(\frac{1}{4}\right)2,$ using Shampine and Thompson [7] gives the coefficient of (15) as

$$b = \begin{bmatrix}
\frac{8183}{230400} & \frac{-653203}{14515200} & \frac{50689}{907200} & \frac{-196277}{2903040} & \frac{-95167}{7257600} & \frac{92473}{2903040} & \frac{7703}{2419200} & \frac{-5741}{16588800} \\
\frac{3673}{28350} & \frac{-81}{800} & \frac{7729}{56700} & \frac{-22703}{181440} & \frac{373}{4725} & \frac{-14773}{453600} & \frac{449}{56700} & \frac{-521}{604800} \\
\frac{1467}{6400} & \frac{-4707}{44800} & \frac{225}{1024} & \frac{-28143}{143360} & \frac{11079}{89600} & \frac{-9141}{179200} & \frac{2223}{179200} & \frac{-387}{286720} \\
\frac{1552}{4725} & \frac{-58}{567} & \frac{5008}{14175} & \frac{-47}{180} & \frac{2384}{14175} & \frac{-986}{14175} & \frac{16}{945} & \frac{-209}{113400} \\
\frac{248375}{580608} & \frac{-19375}{193536} & \frac{143375}{290304} & \frac{-641875}{2322432} & \frac{225}{1024} & \frac{-12875}{145152} & \frac{3125}{145152} & \frac{-3625}{1548288} \\
\frac{369}{700} & \frac{-549}{5600} & \frac{111}{175} & \frac{-639}{2240} & \frac{9}{28} & \frac{-81}{800} & \frac{9}{350} & \frac{-9}{3200} \\
\frac{216433}{345600} & \frac{-98441}{1036800} & \frac{1601467}{2073600} & \frac{-160867}{552960} & \frac{55223}{129600} & \frac{-127253}{2073600} & \frac{8183}{230400} & \frac{-57281}{16588800} \\
\frac{1472}{2025} & \frac{-464}{4725} & \frac{2624}{2835} & \frac{-908}{2835} & \frac{2924}{4725} & \frac{-464}{14175} & \frac{1472}{14175} & 0
\end{bmatrix}, \quad d = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{324901}{23224320} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{58193}{1814400} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{71661}{1433600} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7703}{113400} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{56975}{663552} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{93}{896} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2019731}{16588800} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1978}{14175}
\end{bmatrix}$$

$$A^{(0)}=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix},\quad e=\begin{bmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{bmatrix}$$

3. Properties of the method

3.1 Order and Error Constant of the Block Method

Let the linear Operator defined on the method be $\varsigma[y(x);h],$ where

$$\Delta[y(x);h]=A^{(o)}Y_m^{(i)}-\sum_{i=0}^{k}\frac{jh}{i}y_n^{'}-h^{(2-i)}[d_i f(y_n)+b_i F(y_m)], \tag{16}$$

Expanding the form Y_m and $F(y_m)$ in Taylor Series and comparing coefficients of h,
we obtained

$$\Delta[y(x);h]=C_0 y(x)+C_1 hy'(x)+\ldots\ C_p h^p y^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\ldots \tag{17}$$

Theorem 1: The linear operator and the associated block method are said to be of order p if $C_0=C_1=\ldots C_p=C_{p+1}=0, C_{p+2}\neq 0$ C_{p+2} is called the error constant. It implies that the local truncation error is given by

$$T_{n+k}=C_{p+2}h^{p+2}y^{p+2}(x)+O(h^{p+3})$$

(18)

Expanding the block method (15) in Taylor Series expansion and comparing the coefficients of h, the order of the block is $p=10$ with error constant

$$C_{p+2}=\left(\frac{1129981}{100459163443200},\frac{22063}{7847962214400},\frac{3649}{826781204480},\frac{37}{6131220480},\frac{307625}{40181566537728},\frac{299}{32296140800},\frac{1570597}{1435055947776000},\frac{37}{3065610240}\right)^T$$

3.2 Consistency

Consistency of the Main Method

A Linear Multistep method is consistent if the following conditions are satisfied. Lambert [8] in Areo [9]

1) The order is $p\geq 1$
2) $\sum_{j=0}^{k}\alpha_j=0$
3) $\rho(r)=\rho'(r)=\rho''(r)=\rho^{(n-1)}(r)=0$
4) $\rho^n(r)=n!\,\sigma(r)$ and for the principal root $r=1$ and $n=2$

Hence, The method satisfies the necessary and sufficient conditions for consistency of a numerical method.

3.3 Zero stability of the method

$$\left\|\left[\lambda A^{(0)}-A^{(i)}\right]\right\|=\left\|\left[\begin{pmatrix}\lambda&0&0&0&0&0&0&0\\0&\lambda&0&0&0&0&0&0\\0&0&\lambda&0&0&0&0&0\\0&0&0&\lambda&0&0&0&0\\0&0&0&0&\lambda&0&0&0\\0&0&0&0&0&\lambda&0&0\\0&0&0&0&0&0&\lambda&0\\0&0&0&0&0&0&0&\lambda\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\right]\right\|=0$$

$$\lambda^8-\lambda^7=0,\ \lambda=0,0,0,0,0,0,0,1.$$

Since no root has modulus greater than one and $|\lambda|=1$ is simple, hence the block method is zero stable in the $h\to 0$

3.4 Convergence

Zero stability and consistency are sufficient conditions for a linear multistep method to be convergent. Areo [9]. Since the new method is zero-stable and consistent, it can be concluded that the method is convergent

3.5 Stability domain of the block method

Following the stability domain as discussed in Ibijola *et al.*[10]
The stability domain of the Two-step six-off step block method is shown below

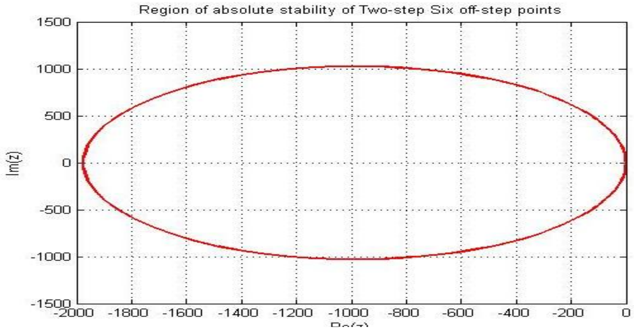

Figure 3: Showing Stability domain of Two-step six off-step points

4. Implementation of Numerical Examples

The methods was adopted on some delay differential equation of special second order to assess the accuracy and efficiency of the methods and the results was compared with that of other researchers that proposed existing methods Hoo *et al.* [3], San *et al.* [11].

Problem1: Consider the linear delay equation

$$y''(x) = y(x-\pi),\ x\varepsilon[0,\pi],$$

$$y(x) = \sin(x),\ y'(x) = -e^{-x},\ -\pi \le x \le 0$$

Exact solution: $y(x) = \sin(x)$

Problem2: Consider the linear delay equation

$$y''(x) = -\frac{1}{2}y(x) + \frac{1}{2}y(x-\pi),\ x\varepsilon[0,\pi],$$

$$y(x) = 1-\sin(x),\ y'(x) = -e^{-x},\ -\pi \le x \le 0$$

Exact solution: $y(x) = 1-\sin(x)$

Problem 3: Application to Matheiu's Equation, in this section we apply our developed methods to solve a well-known equation in engineering, the Matheiu's equation, which defined as follows:

$$y''(t) + (\delta + a\cos t)y(t) + cy^3(t) = by(t-T) \quad (4.1)$$

which is a nonlinear delay differential equation. Where δ, a,b,c and T are parameters. δ is the frequency squared of the simple harmonic oscillator, and a is the amplitude of the parametric resonance, and b is the amplitude of delay which c is the amplitude of the cubic nonlinearity and T is the time delay.

Equation (4.1) is a model for high speed milling, a kind of parametrically interrupted cutting as opposed to the self-interrupted cutting arising in an unstable turning process.

According to Morisson and Rand [12], various special cases of (4.1) have been studied, depending on which parameters is zero.

when δ=a=b=1 and c=0 we obtained the following Linear Mathieu equation:

$$y''(t) = (1+\cos t)y(t) = y(t-T);\ tE[0,10],\ y(t) = \sin(t),\ y'(t) = \cos(t),\ t<0 \quad (4.2)$$

where T=τ=h/10 is the delay term, the exact solution does not exist.

when δ=a=b=c=1 we obtained the following Nonlinear Mathieu equation:

$$y''(t) = (1+\cos t)y(t) + y^3(t) = y(t-T);\ y(t) = \sin(t),\ y'(t) = \cos(t),\ t<0 \quad (4.3)$$

where T=τ=h/10 is the delay term, the exact solution does not exist.

Source: Mechee *et al* [13]

Both the linear and non linear Mathieu equations are solved using the developed methods and the results are presented in table 5 and 6.

Notations and their meaning

MTD= New Method Employed, Two-step with six off step points, Error=(y-Exact) – (y-Computed)

Problem3: Consider the linear delay equation

Table 1: Showing results for Problem 1 using MTD

S/N	X	y-exact	y-computed	Error in MTD	Time
1	**0.1**	0.900166583353171790	0.900166583353171790	0.00000000e+00	0.0747
2	**0.2**	0.801330669204938780	0.801330669204938780	0.00000000e+00	0.0901
3	**0.3**	0.704479793338660400	0.704479793338660400	0.00000000e+00	0.0917
4	**0.4**	0.610581657691349420	0.610581657691349530	1.11022302e-16	0.0928
5	**0.5**	0.520574461395796990	0.520574461395796990	0.00000000e+00	0.0939
6	**0.6**	0.435357526604964520	0.435357526604964630	1.11022302e-16	0.0949
7	**0.7**	0.414902727059537790	0.414902727059537850	5.55111512e-17	0.0952
8	**0.8**	0.355782312762308870	0.355782312762308930	5.55111512e-17	0.0960
9	**0.9**	0.282643909100477210	0.282643909100477210	0.00000000e+00	0.0970
10	**1.0**	0.158529015192103500	0.158529015192103440	5.55111512e-17	0.0984

Table 2: Showing results for Problem 2 using MTD

S/N	X	y-exact	y-computed	Error in MTD	Time
1	**0.1**	0.099833416646828155	0.099833416646828155	0.00000000e+00	0.0223
2	**0.2**	0.198669330795061220	0.198669330795061220	0.00000000e+00	0.0286
3	**0.3**	0.295520206661339600	0.295520206661339600	0.00000000e+00	0.0294
4	**0.4**	0.389418342308650520	0.389418342308650470	5.55111512e-17	0.0298
5	**0.5**	0.479425538604203010	0.479425538604203010	0.00000000e+00	0.0302
6	**0.6**	0.564642473395035480	0.564642473395035260	2.22044605e-16	0.0306
7	**0.7**	0.644217687237691130	0.644217687237691020	1.11022302e-16	0.0310
8	**0.8**	0.717356090899522790	0.717356090899522680	1.11022302e-16	0.0314
9	**0.9**	0.783326909627483410	0.783326909627483300	1.11022302e-16	0.0318
10	**1.0**	0.841470984807896500	0.841470984807896390	1.11022302e-16	0.0322

Table 3: Showing the comparison of the numerical errors for problem 1 using MTD

$h=\pi/(10*2^i)$ i	**Cubic Spline (San [11])**	**Direct Method (Hoo*etal.* [3])**	**Error in MTD**
0	1.84E-02	1.84E-06	1.69864123e-14
1	4.62E-03	4.62E-07	0.00000000e+0
2	1.16E-03	1.16E-08	0.00000000e+00
3	2.89E-04	2.89E-09	1.11022302e-16
4	7.23E-05	7.23E-10	0.00000000e+00
5	1.81E-05	1.81E-11	0.00000000e+00
6	4.52E-06	4.52E-12	0.00000000e+00

Table 4: Showing the comparison of the numerical errors for problem 2 using MTD

H	y-exact	y-computed	**Error in MTD**	**Error in Hoo*et al* [3]**
$\pi/10$	0.309016994374947400	0.309016994374964380	1.69864123e-14	9.9016e-05
$\pi/20$	0.156434465040230870	0.156434465040230870	0.00000000e+00	7.7395e+06
$\pi/50$	0.062790519529313374	0.062790519529313374	0.00000000e+00	2.4408e-07
$\pi/100$	0.031410759078128292	0.031410759078128292	0.00000000e+00	1.6530e-08
$\pi/200$	.015707317311820675	0.015707317311820675	0.00000000e+00	1.0953e-09
$\pi/250$	0.012566039883352607	0.012566039883352607	0.00000000e+00	4.6437e-10
$\pi/500$	0.006283143965558951	0.006283143965558951	0.00000000e+00	4.8654e-11

Table 5: Showing the computed non linear solution for problem 3

X	Computed Non linear solution	Time of Execusion
0.1000000	0.100058792433976180	0.0017
0.2000000	0.200348223977677570	0.0021
0.3000000	0.301236737230030920	0.0028
0.4000000	0.403393465598161690	0.0041
0.5000000	0.507819663062459180	0.0056
0.6000000	0.615861136779212610	0.0060
0.7000000	0.729199011592285400	0.0065
0.8000000	0.849817854390083620	0.0068
0.9000000	0.979951996771661450	0.0715
1.0000000	1.122012658450579600	0.0719

Table 6: Showing the computed linear solution for problem 3

X	Computed Linear solution	Time of Execution
0.1000000	0.094717048839893861	0.0017
0.2000000	0.177546021015915540	0.0021
0.3000000	0.246547367488382100	0.0026
0.4000000	0.299877759653102370	0.0038
0.5000000	0.335834594397979140	0.0053
0.6000000	0.352895741742854600	0.0057
0.7000000	0.349753231513026450	0.0066
0.8000000	0.325339805798630420	0.0070
0.9000000	0.278847534345766200	0.0074
1.0000000	0.209737989871232070	0.0078

4.1 Discussion of Results

The exact solution, computed solution and error of sample problem 1-2 are shown in the Table 1–2. The table 3-4 shows the error comparison of the developed methods with the existing method in the literature. The developed methods are compared with the results of works of other researchers who proposed the existing methods, It can be seen that the developed methods performed better in terms of accuracy

5. Conclusion

This paper considers the derivation, analysis and implementation of direct solution of special second order delay differential equation by a two-step Implicit Multi-Hybrid block methods of uniform order. The results shown that the new method is more accurate and suitable for solving special second order delay differential equations. The numerical results generated when the new developed methods was applied on some second order delay differential equations have shown the high accuracy of the method. In addition, the method were applied to solve an engineering problem, namely Mathieus Equation, the results of the linear and non linear mathieus equations are displayed in table 5 and 6.

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