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# A BLOCK-HYBRID THIRD DERIVATIVE METHOD OF UNIFORM ORDER K+11 FOR SOLVING INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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***Abstract***

***This paper proposes a new group of Block-Hybrid Third Derivative Method of uniform order k+11 for solving Initial Problems in Ordinary Differential Equations. The method was augmented by the introduction of offstep points in order to circumvent Dahlquist zero stability barrier and upgrade the order and accuracy of the method. Power series approximate solution as an interpolation polynomial and its first, second and third derivative as a collocation equation is considered in deriving the method. Properties of the methods were also investigated. The new method is then applied to solve first order-order ordinary differential equations of initial value problem and the reliability of the new methods and the results obtained show that the method computationally reliable, accurate and competes favourably with other existing ones.***

***Keywords*:** Block-Hybrid, Third Derivatives Methods, IVPs, ODEs, Order k+11, first order
**AMS (2010) subject Classification:** 65L05, 65L06, and 65L10

## 1.0 INTRODUCTION

The mathematical modeling of some physical phenomenon in celestial mechanics, scattering theory, theoretical physics, chemistry, series circuits, mechanical systems with several springs attached in series lead to a system of differential equation electronics [1-5]. These may result in the nth- order initial value problem of ordinary differential equations. Also problems in diverse fields like economics, medicine, psychology, operation research and even in anthropology are modeled mathematically [6]. Any equation which connect the derivatives of a differentiable function of one independent variable with respect to itself are called ordinary differential equations (ODEs) [3]. The general form of ordinary differential equations of first order ODEs is given in this form.

$$y'(x) = f(x,y), y(x_0) = y_0, a \le x \le b \qquad \textbf{(1)}$$

Numerical Method is very important because some differential equations arising from the modeling of physical phenomena, often do not have analytic solutions hence the development of numerical method to obtain approximate solutions becomes necessary [7]. To that extend, several numerical methods such as finite difference methods, finite element methods and finite volume methods among others, have been developed based on the nature and type of the differential equation to be solved.

In this research work, we are interested in developing the linear multistep method of the form:

$$y(x) = \alpha_n(t)y_n + h\left[\sum_{j=0}^{k}\beta_j(t)f_{n+j} + \beta_1(t)f_{n+1}\right] + h^2\left[\sum_{j=0}^{k}\gamma_j(t)g_{n+j} + \gamma_1(t)g_{n+1}\right] + h^3\left[\sum_{j=0}^{k}\tau_j(t)p_{n+j} + \tau_1(t)p_{n+1}\right] \qquad \textbf{(2)}$$

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on a given mesh $a = x_0 < x_1 < ... < x_n < x_{n+1} < ...x_{n+1} < .... < x_{n+k} < .... < x_N = b,$

where h=$x_{n+1}$-$x_n$ is a constant step size and k denotes the step number of the method (2), $\alpha_j, \beta_j, \gamma_j$ *and* $\tau_j$ are the continuous coefficients. So many authors have proposed solution to higher order initial value ordinary differential equations especially the first order differential equations by different approaches (see for examples early researchers like[8-12] . In particular, [9] have developed a symmetric linear multistep methods for second order differential equations with periodic solution. [13] studied p-stable linear multistep method for general third order initial value problems of ordinary differential equations which is to be used in form of predictor-corrector forms and like most linear multistep methods, they require starting values from Runge-kutta one–step methods or other one-step methods. Also, the predictors are developed in the same way we develop correctors.

The theoretical solution of (1) is usually highly oscillatory and severely restricts the mesh-size of the conventional linear multistep methods as noted by [14]. Several numerical integrators have been developed to solve the first order ordinary differential equations. Prominent among these are [14-23] with more recent works by [24] who did a survey & comparison study on the implementation of the method is in predictor-corrector mode.

The block nature of the proposed methods produce simultaneously approximations to the IVP (1) at the block points $x_{n+1}$, $x_{n+2}$,….$x_{n+N}$. Although each integrator in the block is a linear multistep method, it is observed that, as a block method, it preserves the traditional Runge-Kutta advantages of being self –starting and of permitting easy change of step-length. Their advantage over conventional Runge-Kutta methods lies in the fact that they are less expensive in terms of function evaluation for a given order.

Multi-block r-point methods for second order ODEs are proposed in [22]Yusuph (2004) in the same line of thought as [14 and 17]; [23] also adopted the matrix inversion approach to multistep collocation methods which was started in [25-26] for initial value problems, for the first order system and extended to the special second order ODEs .

In the work of [27], a class of initial value methods for the direct solution of second order initial value problems, linear multistep methods with continuous coefficients were obtained and applied as simultaneous numerical integrators to $y'' = f(x, y, y')$.The methods are implemented efficiently by combining the IVMs as simultaneous integrators for IVPs without looking for any other methods to provide the starting values and then proceed by explicitly obtaining initial conditions at $x_{n+k}$, n = 0, k….N-K. using the computed values y($x_{n+k}$) = $y_{n+k}$ and $\sigma$ ($x_{n+k}$) = $\sigma_{n+k}$ over the subinterval [$x_0$, $x_k$], [$x_{N-k}$, $x_N$] which do not overlap. The implementation strategy is more efficient than those given in [28] which are applied over overlapping intervals in predictor-corrector mode. [29] constructed a variable step-size implementation of multistep Methods for, $y'' = f(x, y, y')$. [30] derived new multiple Finite Difference Methods (FDMs) through multistep collocation for $y'' = f(x, y)$. [31] also derived an accurate and efficient direct method for the initial value problem for general first order ordinary differential equations. In his paper, second and fourth order two-step discrete finite difference methods are derived by collocation for the first approximation and are combined with the Numerov method for a direct application to general second order initial value problem of ODEs. [32] in his note, reported a hybrid formula of order four for starting the Numerov method applied to special second order initial value problems that is $y'' = f(x, y)$ which is more accurate than the existing ones. [33] derived a new Butcher type two-step block hybrid multistep method for accurate and efficient parallel solution of ODEs.

However, little or no attention has been paid to the use of block multi-derivatives with single step and multi-hybrid points for the direct solution of first order initial value problem in ordinary differential equations. Hence we are attracted to work in this direction.

## 2.0 METHODOLOGY

In this section, we intend to construct the proposed One Step Block-Hybrid Third Derivative Method which will be used to generate the method. We consider an approximation of the form

$$y(x) = \sum_{j=0}^{2s+r-1} a_j x^j; \qquad \textbf{(3)}$$

Where r and s are the number of interpolation and collocation points respectively. Differentiate (3) once, twice and thrice yields

$$y'(x) = \sum_{j=1}^{2s+r-1} j a_j x^{j-1} = f(x,y); \quad (4)$$

$$y''(x) = \sum_{j=2}^{2s+r-1} j(j-1) a_j x^{j-2} = g(x,y); \quad (5)$$

$$y'''(x) = \sum_{j=3}^{2s+r-1} j(j-1)(j-2) a_j x^{j-3} = p(x,y); \quad (6)$$

Imposing the following conditions on (3), (4), (5) and (6) gives

$$y(x_n) = y_n \quad (7)$$

$$y'(x_{n+j}) = f_{n+j},\ j = 0, \frac{1}{3}, \frac{2}{3}, 1 \quad (8)$$

$$y''(x_{n+j}) = g_{n+j},\ j = 0, \frac{1}{3}, \frac{2}{3}, 1 \quad (9)$$

$$y'''(x_{n+j}) = p_{n+j},\ j = 0, \frac{1}{3}, \frac{2}{3}, 1 \quad (10)$$

Combining equations (7) – (10) and solve using Gaussian elimination method with the aid of maple software gives the values of $a_0 - a_{12}$ which are substituted into (3) gives to give a continuous hybrid third derivate method in the form;

$$y(x) = \alpha_0(t) y_n + h\left[\beta_i(t) f_{n+i} + \beta_1(t) f_{n+1}\right] + h^2\left[\gamma_i(t) g_{n+i} + \gamma_1(t) g_{n+1}\right] + h^3\left[\tau_i(t) p_{n+i} + \tau_1(t) p_{n+1}\right] \quad (11)$$

Where $i = \frac{1}{3}$ and $\frac{2}{3}$

$$\alpha_0(t) = 1$$

$$\beta_0(t) = -\frac{1}{9856} ht \begin{pmatrix} 11563398t^{11} - 74340504t^{10} + 208590228t^9 - 334552680t^8 + 3369160503t^7 \\ -219649320t^6 + 91544068t^5 - 22732248t^4 + 2669128t^3 - 9856 \end{pmatrix}$$

$$\beta_{\frac{1}{3}}(t) = \frac{729}{9856} ht^4 \begin{pmatrix} 112266t^8 - 680400t^7 + 1779624t^6 - 2624160t^5 + 2388771t^4 \\ -1378960t^3 + 496496t^2 - 103488t + -9856 \end{pmatrix}$$

$$\beta_{\frac{2}{3}}(t) = -\frac{729}{9856} ht^4 \begin{pmatrix} 112266t^8 - 666792t^7 + 1704780t^6 - 2448600t^5 + 2160081t^4 \\ -1199000t^3 + 409948t^2 - 79464t + 6776 \end{pmatrix}$$

$$\beta_1(t) = \frac{1}{9856} ht^4 \begin{pmatrix} 11563398t^8 - 64420272t^7 + 154028952t^6 - 206569440t^5 + 170201493t^4 \\ -88458480t^3 + 28450576t^2 - 5218752t + 423808 \end{pmatrix} \quad (12)$$

$$\gamma_0(t) = -\frac{1}{49280} h^2t^2 \begin{pmatrix} 6174630t^{10} - 40007520t^9 + 113388660t^8 - 18428260t^7 + 188949915t^6 \\ -126355680t^5 + 54702340t^4 - 14448896t^3 + 1903440t^2 - 24640 \end{pmatrix}$$

$$\gamma_{\frac{1}{3}}(t) = \frac{243}{49280} h^2t^4 \begin{pmatrix} 187110t^8 - 1111320t^7 + 2827440t^6 - 4010160t^5 + 3449985t^4 \\ -1829080t^3 + 575960t^2 - 96096t + 6160 \end{pmatrix}$$

$$\gamma_{\frac{2}{3}}(t) = \frac{243}{49280} h^2t^4 \begin{pmatrix} 187110t^8 - 1134000t^7 + 2952180t^6 - 4305840t^5 + 38449955t^4 \\ -2153360t^3 + 740740t^2 - 144144t + 12320 \end{pmatrix}$$

$$\gamma_1(t) = -\frac{1}{49280} h^2t^4 \begin{pmatrix} 6174630t^8 - 34088040t^7 + 80831520t^6 - 107609040t^5 + 88097625t^4 \\ -45536040t^3 + 14577640t^2 - 2663584t + 215600 \end{pmatrix}$$

$$\tau_0(t) = -\frac{1}{147840} h^3t^3 \begin{pmatrix} 561330t^9 - 3674160t^8 + 10553004t^7 - 17463600t^6 + 18367965t^5 \\ -12759120t^4 + 5875100t^3 - 1740816t^2 + 304920T - 24640 \end{pmatrix}$$

$$\tau_{\frac{1}{3}}(t) = -\frac{81}{49280}h^3t^4\begin{pmatrix} 62370t^8 - 385560t^7 + 10311847t^6 - 1558480t^5 + 1456455t^4 \\ -862840t^3 + 317240t^2 - 66528t + 6160 \end{pmatrix}$$

$$\tau_{\frac{2}{3}}(t) = -\frac{81}{49280}h^3t^4\begin{pmatrix} 62370t^8 - 385560t^7 + 906444t^6 - 1268960t^5 + 1089165t^4 \\ -587840t^3 + 195580t^2 - 36960t + 3080 \end{pmatrix}$$

$$\tau_1(t) = -\frac{1}{147840}h^3t^4\begin{pmatrix} 561330t^8 - 3061800t^7 + 7185024t^6 - 9480240t^5 + 7702695t^4 \\ -3956040t^3 + 1259720t^2 - 229152t + 18480 \end{pmatrix}$$

Evaluating (12) at $t = \frac{1}{3}, \frac{2}{3}$ and 1 and solve simultaneously gives

$$y_{n+\frac{1}{3}} = \begin{pmatrix} \frac{1513}{107775360}p_{n+1} - \frac{7453}{11975040}p_{n+\frac{2}{3}} \\ +\frac{4423}{2395008}p_{n+\frac{1}{3}} + \frac{11369}{107775360}p_n \end{pmatrix}h^3 + \begin{pmatrix} -\frac{5941}{11975040}g_{n+1} + \frac{10657}{1330560}g_{n+\frac{2}{3}} - \\ \frac{10657}{1330560}g_{n+\frac{1}{3}} + \frac{21493}{35925120}g_n \end{pmatrix}h^2 \quad (13)$$
$$+\begin{pmatrix} -\frac{5851}{88704}f_{n+\frac{2}{3}} + \frac{23717}{88704}f_{n+\frac{1}{3}} + \\ \frac{35339}{7185024}f_{n+1} + \frac{912523}{7185024}f_n \end{pmatrix}h + y_n$$

$$y_{n+\frac{2}{3}} = \begin{pmatrix} \frac{8}{841995}p_{n+1} - \frac{19}{93555}p_{n+\frac{2}{3}} \\ +\frac{212}{93555}p_{n+\frac{1}{3}} + \frac{17}{168399}p_n \end{pmatrix}h^3 + \begin{pmatrix} -\frac{92}{280665}g_{n+1} - \frac{32}{10395}g_{n+\frac{2}{3}} \\ +\frac{32}{10395}g_{n+\frac{1}{3}} + \frac{544}{93555}g_n \end{pmatrix}h^2 +$$
$$\begin{pmatrix} \frac{71}{693}f_{n+\frac{2}{3}} + \frac{302}{693}f_{n+\frac{1}{3}} + \\ \frac{178}{56133}f_{n+1} + \frac{7031}{56133}f_n \end{pmatrix}h + y_n \quad \textbf{(14)}$$

$$y_{n+1} = \begin{pmatrix} \frac{17}{147840}p_{n+1} + \frac{81}{49280}p_{n+\frac{2}{3}} + \\ \frac{81}{49280}p_{n+\frac{1}{3}} + \frac{17}{147840}p_n \end{pmatrix}h^3 + \begin{pmatrix} -\frac{311}{49280}g_{n+1} + \frac{243}{49280}g_{n+\frac{2}{3}} \\ -\frac{243}{49280}g_{n+\frac{1}{3}} + \frac{311}{49280}g_n \end{pmatrix}h^2 \quad \textbf{(15)}$$
$$+\begin{pmatrix} \frac{3645}{9856}f_{n+\frac{2}{3}} + \frac{3645}{9856}f_{n+\frac{1}{3}} + \\ \frac{1283}{9856}f_{n+1} + \frac{1283}{9856}f_n \end{pmatrix}h + y_n$$

The equations (13) to (15) forms the One-Step Block Hybrid Third Derivatives Method

## 3.0 Convergence Analysis of Method

### 3.1 Order and error Constants of the Method

According to [8], the order of the new method in Equation (13)-(15) is obtained by using the Taylor series and it is found that the developed method has uniformly order twelve, with an error constants vector of:

$$C_{12} = \left[-\frac{3617}{655240243872614400}, -\frac{23}{51190644052548000}, -\frac{1}{998689595904000}\right]^T$$

### 3.2 Consistency

**Definition 3.1:** The hybrid block method (11) is said to be consistent if it has an order more than or equal to one i.e. $P \geq 1$. Therefore, the method is consistent [8].

### 3.3 Zero Stability

**Definition 3.2:** The hybrid block method (11) said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1$ *and if* $|r_z| = 1$, then the multiplicity of $r_z$ must not greater than two [8]. In order to find the zero-stability of hybrid block method (11), we only consider the first characteristic polynomial of the method according to definition (3.2) as follows

$$\prod(r) = \left| r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right| = r^2(r-1)$$

Which implies $r = 0, 0, 0, 1$. Hence the method is zero-stable since $|r_z| \leq 1$.

### 3.4 Convergence

**Theorem (3.1):** Consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the method (11) is consistent and zero stable, it implies the method is convergent for all point [8].

### 3.5 Regions of Absolute Stability (RAS)

The absolute stability region of the new method is found according to [34] and [8] and is shown in figure 1 below;

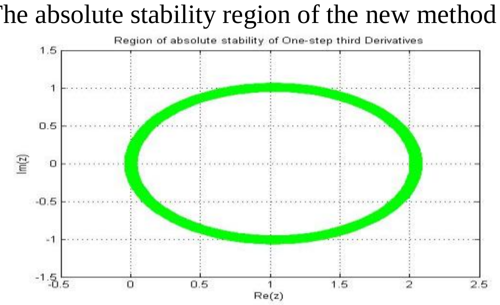


Figure 1: Absolute Stability Region of the Method

The region of the absolute stability of the method is A-stable since the regions consist of the complex plane outside the enclosed Figure

### 4. 0 Numerical Examples

In this section, practical performance of the new method is examined on some test examples. We present the results obtained from the test examples which include linear and nonlinear stiff and oscillatory problem of initial value problems found in the literature. The results are compared with the exact solutions. The results or absolute errors $|y(x) - yn(x)|$ are presented side by side in the Table of values. We used MATLAB codes for the computational purposes.

**Example I:**

$y' = -y, (0) = 0, h = 0.1$

$y(x) = 5 - 3e^{-4x}$

Source: [35]

**Table 1: Comparison of the newly developed method with [35]**

| X | Exact solution | Computed solution | Error in our Method | [35] |
|---|---|---|---|---|
| 0.100000 | 0.90483741803596 | 0.90483741803603 | 7.4940e-014 | 0.00 |
| 0.200000 | 0.81873075307798 | 0.81873075307812 | 1.3545e-013 | 1.0e-010 |
| 0.300000 | 0.74081822068172 | 0.74081822068190 | 1.8374e-013 | 0.00 |
| 0.400000 | 0.67032004603564 | 0.67032004603586 | 2.2116e-013 | 0.00 |
| 0.500000 | 0.60653065971263 | 0.60653065971288 | 2.5013e-013 | 0.00 |
| 0.600000 | 0.54881163609403 | 0.54881163609430 | 2.7245e-013 | 1.0e-010 |
| 0.700000 | 0.49658530379141 | 0.49658530379170 | 2.8866e-013 | 0.00 |
| 0.800000 | 0.44932896411722 | 0.44932896411752 | 2.9898e-013 | 0.00 |
| 0.900000 | 0.40656965974060 | 0.40656965974090 | 3.0476e-013 | 0.00 |
| 1.000000 | 0.36787944117144 | 0.36787944117175 | 3.0681e-013 | 1.0e-010 |

We solve this problem using the newly derived methods and the results obtained are presented in Table 1.

**Example II:**

Consider the Prothero-Robinson oscillatory ODE

$y' = Ly + \cos(x) - L\sin(x), L = -1, y(0) = 0, h = 0.1$

$y(x) = \sin(x)$

Source: [36]

**Table 2:** Comparison of the newly developed method with [36]

| X | Exact solution | Computed solution | Error in our Method | [36] |
|---|---|---|---|---|
| 0.100000 | 0.09983341664683 | 0.09983341664683 | 0.0000e-000 | 1.342236e-011 |
| 0.200000 | 0.19866933079506 | 0.19866933079508 | 1.5488e-014 | 2.146397e-011 |
| 0.300000 | 0.29552020666134 | 0.29552020666137 | 3.3695e-014 | 3.235895e-011 |
| 0.400000 | 0.38941834230865 | 0.38941834230871 | 5.7787e-014 | 4.187661e-011 |
| 0.500000 | 0.47942553860420 | 0.47942553860429 | 8.6653e-014 | 4.637712e-011 |
| 0.600000 | 0.56464247339504 | 0.56464247339515 | 1.1835e-013 | 5.336764e-011 |
| 0.700000 | 0.64421768723769 | 0.64421768723785 | 1.5365e-013 | 5.893563e-011 |
| 0.800000 | 0.71735609089952 | 0.71735609089972 | 1.9151e-013 | 6.022138e-011 |
| 0.900000 | 0.78332690962749 | 0.78332690962772 | 2.3104e-013 | 6.334211e-011 |
| 1.000000 | 0.84147098480790 | 0.84147098480817 | 2.7223e-013 | 6. 505940e-011 |

We solve this problem using the newly derived methods and the results obtained are presented in Table 2.

**Example III:**

$y' = x - y, y(0) = 0, h = 0.1$

$y(x) = x + e^{-x} - 1$

Source: [37]

**Table 3:** Comparison of the newly developed method with [37]

| X | Exact solution | Computed solution | Error in our Method | [37] |
|---|---|---|---|---|
| 0.100000 | 0.00483741803596 | 0.00483741803596 | 0.0000e-000 | 1.9595e-11 |
| 0.200000 | 0.01873075307798 | 0.01873075307797 | 1.4194e-014 | 3.54623e-11 |
| 0.300000 | 0.04081822068172 | 0.04081822068169 | 3.0122e-014 | 4.81315e-11 |
| 0.400000 | 0.07032004603564 | 0.07032004603559 | 5.0251e-014 | 5.80680e-11 |
| 0.500000 | 0.10653065971263 | 0.10653065971256 | 7.3858e-014 | 6.56779e-11 |
| 0.600000 | 0.14881163609403 | 0.14881163609393 | 1.0064e-013 | 7.13132e-11 |
| 0.700000 | 0.19658530379141 | 0.19658530379128 | 1.2923e-013 | 7.52814e-11 |
| 0.800000 | 0.24932896411722 | 0.24932896411706 | 1.5912e-013 | 7.78485e-11 |
| 0.900000 | 0.30656965974060 | 0.30656965974041 | 1.8963e-013 | 7.92403e-11 |
| 1.000000 | 0.36787944117145 | 0.36787944117123 | 2.2049e-013 | 7.96712e-11 |

We solve this problem using the newly derived methods and the results obtained are presented in Table 3.

## 4.1 Discussion

Computer program is written for the implementation of the newly developed Hybrid Block Third derivatives Method. The method developed where tested respectively on three numerical examples for first order ordinary differential equation in the last section. The derived method converge Faster to the exact solution, when compared to [35-37] as we can see in TABLE 1, TABLE 2 and TABLE 3. Therefore our method is comparable with the existent methods.

## 5. 0 Conclusion

The approach adopted for the derivation of the block method involves interpolation and collocation at appropriate selected points via third derivative. The proposed order twelve hybrid block third derivative method for first order ODEs was found to be zero-stable, consistent and convergent. The stability nature of the method shows that it is A-stable. Numerical evidences shows that the method proposed here perform favorable when compared with existing scheme as it yielded better accuracy.

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