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PERTURBATION TECHNIQUE IN THE BUCKLING OF AN ELASTIC SPHERICAL SHELL STRUCK BY A PERIODIC LOAD WITH SLOWLY VARYING FREQUENCY

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Abstract

This work discussed analytical determination of the dynamic buckling loads of an elastic spherical shell, subjected to a periodic load having slowly varying circular frequencies. The governing equation contains two small but independent parameters which are used in asymptotic expansions of the relevant variables. Besides, a two-timing regular perturbation procedure is adopted to analyze the governing equation, which has various degrees of nonlinearities in its formulation. The results obtained show that: (a) The dynamic buckling load decreases with increased imperfections.(b) The dynamic buckling load depends on the first degree first order derivative of the slowly varying function evaluated at the initial time.(c) As the small parameter depicting the slow time variation tends to zero, the result of the periodic loading tends to that of a step load.(d) Thus, a step load might justifiably be seen as the limiting process of a periodic load that has a slowly varying circular frequency.(e) The least dependence of the dynamic buckling load λ_D on the slow time parameters δ is quadratic in nature, that is λ_D is of the order δ^2.

1.0 INTRODUCTION

Buckling (and dynamic buckling in particular) presents formidable instabilities associated with engineering elastic structures under compressive loading. While sufficiently large investigations have been done on static loading (under static compressive loads), the same cannot be said of dynamic loading. Such materials studied include columns on nonlinear elastic foundations [1], columns on quadratic – cubic foundations [2], imperfect thin cylindrical shells [3] and imperfect spherical cap [4] among others. The method adopted for solution may partly depend on the loading history prescribed or on the geometrical configuration of the structure investigated or partly on the novelty and mathematical insight available to the investigator.

In this study, we aim at investigating the dynamic buckling analysis of a spherical shell trapped by a periodic load with slowly varying frequency**.** To our knowledge, this type of loading is infrequently discussed, at least, in the landscape of buckling. It however suffices to mention that the concept of periodic forcing with slowly varying frequency in the time variable was discussed [5]. In that study, the frequency was assumed to have a slow cubic variation with time and was discussed in the context of a weakly damped and weakly nonlinear excitation involving Duffing's equation.
This work is an extension of the work [6], to the case where the structure is a spherical shell. In this study, we first obtain the displacement and thereafter, obtain the maximum displacement,ξ_a.The approach adopted in this work (method of asymptotics and perturbation analysis) was first developed by Birman in [7], who investigated the problem of dynamic buckling of anti – symmetric rectangular laminates. Eventually, the maximization, $\frac{d\lambda}{d\xi_a} = 0$is lastly evaluated to obtain the dynamic buckling load,λ_D.

2.0 FORMULATION OF THE PROBLEM

The relevant differential equations of an elastic spherical shell, for an arbitrary time dependent load $\bar{F}$(T), was derived by [8], who extended the initial derivations obtained by Budiansky and Hutchinson in [9]. In his derivation, Danielson assumed the normal displacement $W(x, y, T)$ of the imperfect shell to be given as;

$$W(x,y,T) = \xi_0(T)W_0(x,y) + \xi_1(T)W_1(x,y) + \xi_2(T)W_2(x,y) \qquad (2.1)$$

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where, W_0 is the pre-buckling symmetric mode with its time dependent amplitude $\xi_0(T)$. Similarly $W_1(x,y)$ is the axisymmetric mode with its time dependent amplitude as $\xi_1(T)$, while $W_2(x,y)$ is the non-axisymmetric mode with its time dependent amplitude as $\xi_2(T)$. We note that $W_i(x,y)$ are dependent on the spatial coordinates (x,y), while the dependence on time is factored out into $\xi_i(T)$, $i = 0,1,2$. Danielson assumed the imperfection $\bar{W}$ of the shell to be of the shapes of $W_1(x,y)$ and $W_2(x,y)$ such that;

$$\bar{W}(x,y) = \bar{\xi}_1 W_1(x,y) + \bar{\xi}_2 W_2(x,y) \quad (2.2)$$

where, $0 < \bar{\xi}_1 < 1, \quad 0 < \bar{\xi}_2 < 1$

By substituting (2.1) and (2.2) into the relevant equations of motion and compatibility characterizing a spherical shell as in [8], Danielson obtained the following equations.

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2} + \xi_0 = \lambda \bar{F}(T) \quad (2.3a)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi^2 = \bar{\xi}_1\xi_0 \quad (2.3b)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{dT^2} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2\xi_0 \quad (2.3c)$$

$$\xi_i(0) = \frac{d\xi_i(0)}{dT} = 0;\ i = 0,1,2$$

where, λ is a non-dimensional load parameter satisfying the inequality $0 < \lambda < 1$, while ω_i, $i = 0,1,2.$ are the circular frequencies of the respective buckling modes $\xi = 0,1,2$, and k_1 and k_2 are positive constants, considered fractional. The equations (2.3a, b, c) have been severally studied for several loading conditions. For example, the work in [10] analyzed the case of an impulsive load, while the work in [11] studied the same system for step load without neglecting the terms which Danielson found necessary to neglect. Earlier on, the work in [4] had studied the same equations for the case where the structure was subjected to an impulse. In all of these, the method of solution depends on the type of loading. In this work, we are considering the case where the loading is explicitly time dependent and periodic but with slowly varying angular frequency.

3.0 PERTURBATION SOLUTION OF THE PROBLEM

As in [6], our procedure will be to first and foremost, asymptotically determine the modes ξ_i, $i = 0,1,2$ using multi-timing perturbation procedures. For our case, we shall let

$$\bar{F}(T) = cos(\varphi_0(\delta T)), 0 < \delta \ll 1 \quad (3.1a)$$

where, $\varphi_0(\delta T)$ is a slowly varying continuous function of T, such that

$$\varphi_0(0) = 0, |\varphi_0(\delta T)| << 1 \quad (3.1b)$$

Let,

$$\bar{t} = \omega_0 T \quad (3.1c)$$

$$\therefore \frac{d(\dots)}{dT} = \frac{d}{d\bar{t}}\frac{d\bar{t}}{dT} = \omega_0\frac{d(\dots)}{d\bar{t}};\ \frac{d^2(\dots)}{dT^2} = \omega_0^2\frac{d^2(\dots)}{d\bar{t}^2} \quad (3.1d)$$

If we substitute (3.1a, b, c, d) into (2.3a, b, c), we get

$$\frac{d^2\xi_0}{d\bar{t}^2} + \xi_0 = \lambda cos\left(\varphi_0\left(\frac{\bar{t}\delta}{\omega_0}\right)\right) \quad (3.2a)$$

$$\left(\frac{\omega_0}{\omega_1}\right)^2\frac{d^2\xi_1}{d\bar{t}^2} + \xi_1(1-\xi_0) - k_1\xi_0^2 + k_2\xi_2^2 = 0 \quad (3.2b)$$

Here in (3.2b) we have assumed $\bar{\xi}_1 = 0$ (as in the case [8]), and the third equation is

$$\left(\frac{\omega_0}{\omega_2}\right)^2\frac{d^2\xi_2}{d\bar{t}^2} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2\xi_0 \quad (3.2c)$$

Further simplification of (3.2a,b,c), yields

$$\frac{d^2\xi_0}{d\bar{t}^2} + \xi_0 = \lambda cos\left(\varphi_0\left(\frac{\bar{t}\delta}{\omega_0}\right)\right) \quad (3.3a)$$

$$\frac{d^2\xi_1}{d\bar{t}^2} + \left(\frac{\omega_1}{\omega_0}\right)^2\xi_1(1-\xi_0) + \left(\frac{\omega_1}{\omega_0}\right)^2[-k_1\xi^2 + k_2\xi_2^2] = 0 \quad (3.3b)$$

$$\frac{d^2\xi_2}{d\bar{t}^2} + \left(\frac{\omega_2}{\omega_0}\right)^2\xi_2(1-\xi_0) + \left(\frac{\omega_2}{\omega_0}\right)^2\xi_1\xi_2 = \left(\frac{\omega_2}{\omega_0}\right)^2\xi_2\xi_0 \quad (3.3c)$$

As in Danielson's study, we shall ignore the pre-buckling inertia, that is
$\frac{d^2\xi_0}{d\bar{t}^2} \equiv 0$(3.4)
Hence, we get from (3.2a)

$$\xi_0 = \lambda\cos\left(\varphi_0\left(\frac{\bar{t}\delta}{\omega_0}\right)\right) \tag{3.5}$$

If we substitute (3.5) into (3.3b, c), we get

$$\frac{d^2\xi_1}{d\bar{t}^2} + \left(\frac{\omega_1}{\omega_0}\right)^2 \xi_1\left\{1-\lambda\cos\left(\varphi_0\left(\frac{\delta\bar{t}}{\omega_0}\right)\right)\right\} + \left(\frac{\omega_1}{\omega_0}\right)^2[-k_1\xi_1^2 + k_2\xi_2^2] = 0 \tag{3.6}$$

$$\frac{d^2\xi_2}{d\bar{t}^2} + \left(\frac{\omega_2}{\omega_0}\right)^2 \xi_2\left\{1-\lambda\cos\left(\varphi_0\left(\frac{\delta\bar{t}}{\omega_0}\right)\right)\right\} + \left(\frac{\omega_2}{\omega_0}\right)^2 \bar{\xi}_1\xi_2 = \lambda\left(\frac{\omega_2}{\omega_0}\right)\bar{\xi}_2\cos\left(\varphi_0\left(\frac{\delta\bar{t}}{\omega_0}\right)\right) \tag{3.7}$$

We shall assume $0 < \left(\frac{\omega_2}{\omega_0}\right) < 1$.Let,

$$Q = \left(\frac{\omega_1}{\omega_0}\right), R = \left(\frac{\omega_2}{\omega_0}\right), \varphi(\delta\bar{t}) = \varphi_0\left(\frac{\delta\bar{t}}{\omega_0}\right) \tag{3.8}$$

Thus (3.6) and (3.7) give

$$\frac{d^2\xi_1}{d\bar{t}^2} + Q^2\xi_1\left[1-\lambda\cos\left(\varphi(\delta\bar{t})\right)\right] + Q^2[-k_1\xi_1^2 + k_2\xi_2^2] = 0 \tag{3.9}$$

and

$$\frac{d^2\xi_2}{d\bar{t}^2} + R^2\xi_2\left[1-\lambda\cos\left(\varphi(\delta\bar{t})\right)\right] + R^2\xi_1\xi_2 = \lambda R^2\bar{\xi}_2\cos\left(\varphi(\delta\bar{t})\right) \tag{3.10}$$

and

$$= \xi_i(0) = 0, \frac{d\xi_i(0)}{d\bar{t}} = 0, i = 1, 2$$

Let,

$$\frac{d\tilde{t}}{d\bar{t}} = \left[1-\lambda\cos\left(\varphi(\delta\bar{t})\right)\right]^{\frac{1}{2}} \tag{3.11a}$$

$$\tau = \delta\bar{t} \tag{3.11b}$$

and

$$t = \tilde{t} + \frac{1}{\delta}\left(\bar{\xi}_2 h_1(\tau) + \bar{\xi}_2 h_2(\tau) + \cdots\right) \tag{3.11c}$$

$h_i(0) = 0, i = 1, 2 \ldots$
Then,

$$\frac{d\xi_i}{d\bar{t}} = \frac{\partial\xi_i}{\partial t}\frac{\partial t}{\partial\tilde{t}}\frac{d\tilde{t}}{d\bar{t}} + \frac{\partial\xi_i}{\partial t}\frac{\partial t}{\partial\tau}\frac{d\tau}{d\bar{t}} + \frac{\partial\xi_i}{\partial\tau}\frac{d\tau}{d\bar{t}} \tag{3.12a}$$

$$\frac{d\xi_i}{d\bar{t}} = (1-\lambda\cos(\varphi))^{\frac{1}{2}}\xi_{i,t} + (\bar{\xi}_2 h_1' + \bar{\xi}_2^2 h_2' + \cdots)\xi_{i,t} + \delta\xi_{i,\tau} \tag{3.12b}$$

where,

$$(\ldots)_{,t} \equiv \frac{\partial}{\partial t}, (\ldots)' \equiv \frac{d(\ldots)}{d\tau}$$

Thus we get,

$$\begin{aligned}\frac{d^2\xi_i}{d\bar{t}^2} &= (1-\lambda\cos(\varphi))\xi_{i,tt} + (\bar{\xi}_2 h_1' + \bar{\xi}_2^2 h_2' + \cdots)^2\xi_{i,tt} + \delta^2\xi_{i,\tau\tau} + 2(1-\cos(\varphi))^{\frac{1}{2}}\left(\bar{\xi}_2 h_1' + \bar{\xi}^2 h_2' + \cdots\right)\xi_{i,tt} \\ &+ 2\delta\left(\bar{\xi}_2 h_1' + \bar{\xi}_2^2 h_2' + \cdots\right)\xi_{i,\tau\tau} + 2\delta(1-\lambda\cos(\varphi))^{1/2}\xi_{i,t\tau} + \frac{\delta}{2}\lambda\,\acute{\phi}\sin(\varphi)\,(1-\lambda\cos(\varphi))^{-1/2}\xi_{i,t} \\ &+ \delta\left(\bar{\xi}_2 h_1'' + \bar{\xi}_2^2 h_2'' + \ldots\right)\xi_{i,t}\end{aligned} \tag{3.13}$$

We note here that $\cos(\varphi) \equiv \cos(\varphi(\tau))$
If we substitute (3.13) in (3.2a, b) and simplify, we get

$$\begin{aligned}\xi_{1,tt} &+ \frac{1}{F}(\bar{\xi}_2 h_1' + \bar{\xi}_2^2 h_2' + \cdots)^2\xi_{1,tt} + \frac{\delta^2}{F}\xi_{1,\tau\tau} + \frac{2}{F^{\frac{1}{2}}}\left(\bar{\xi}_2 h_1' + \bar{\xi}_2^2 h_2' + \cdots\right)\xi_{1,tt} + \frac{2\delta}{F}\left(\bar{\xi}_2 h_1' + \bar{\xi}_2^2 h_2' + \cdots\right)\xi_{1,t\tau} + \frac{2\delta}{F^{\frac{1}{2}}}\xi_{1,t\tau} \\ &+ \frac{\delta\lambda\acute{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}\xi_{1,t} + \frac{\delta}{F}\left(\bar{\xi}_2 h_1'' + \bar{\xi}_2^2 h_2'' + \cdots\right)\xi_{1,t} + Q^2\xi_1 + \frac{Q^2}{F}(-k_1\xi_1^2 + k_2\xi_2^2 + \cdots) \\ &= 0\end{aligned} \tag{3.14}$$

and

$$\xi_{2,tt}+\frac{1}{F}(\bar{\xi}_2h_1'+\bar{\xi}_2^2h_2'+\cdots)^2\xi_{2,tt}+\frac{\delta^2}{F}\xi_{2,\tau\tau}+\frac{2}{F^{\frac{1}{2}}}(\bar{\xi}_2h_1'+\bar{\xi}_2^2h_1'+\cdots)\xi_{2,t\tau}+\frac{2\delta}{F^{\frac{1}{2}}}\xi_{2,t\tau}+\frac{\delta\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}\xi_{2,t}+\frac{\delta}{F}(\bar{\xi}_2h_1''+\bar{\xi}_2^2h_2''+\cdots)\xi_{2,t}$$
$$+R^2\xi_2+\frac{R^2}{F}\xi_1\xi_2=\bar{\xi}_2R^2B \qquad (3.15)$$

where,

$$F=(1-\lambda\cos(\varphi))\equiv F(\tau);B=\frac{\lambda\cos(\varphi)}{1-\lambda\cos(\varphi)}\equiv B(\tau)$$

We assume the following asymptotic series expansions

$$\xi_1(t,\tau)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}L^{ij}(t,\tau)\bar{\xi}_2^i\delta^j \qquad (3.16a)$$

$$\xi_2(t,\tau)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}G^{ij}(t,\tau)\bar{\xi}_2^i\delta^j \qquad (3.16b)$$

where ij in L^{ij} and G^{ij} are superscripts and not powers. By substituting (3.16a, b) into (3.14) and (3.15) and equating powers of $\bar{\xi}_2\delta$, we get the following:

$$L_{,tt}^{10}+Q^2L^{10}=0 \qquad (3.174a)$$

$$L_{,tt}^{11}+Q^2L^{11}=\frac{-2L_{,t\tau}^{10}}{F^{\frac{1}{2}}}-\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}L_{,t}^{10} \qquad (3.17b)$$

$$L_{,tt}^{12}+Q^2L^{12}=\frac{-2L_{,t\tau}^{11}}{F^{\frac{1}{2}}}-\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}L_{,t}^{11}-\frac{L_{,\tau\tau}^{10}}{F} \qquad (3.17c)$$

$$=L_{,tt}^{20}+Q^2L^{20}=\frac{Q^2}{F^{\frac{1}{2}}}(k_1(L^{10})^2-k_2(G^{10})^2)-\frac{2h_1'}{F^{\frac{1}{2}}}L_{,tt}^{10} \qquad (3.17d)$$

$$L_{,tt}^{21}+Q^2L^{21}=\frac{2Q^2}{F^{\frac{1}{2}}}(k_1L^{10}L^{11}-k_2G^{10}G^{11})\frac{2h_1'L_{,tt}^{11}}{F^{\frac{1}{2}}}-\frac{2h_1'L_{,tt}^{10}}{F}-\frac{2h_1'L_{,tt}^{20}}{F^{\frac{1}{2}}}-\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}L_{,t}^{20}$$
$$-\frac{h_1'L_{,t}^{10}}{F} \qquad (3.17e)$$

$$L_{,tt}^{22}+Q^2L^{22}=\frac{Q^2}{F}\left[k_1\left(L^{11^2}+2L^{10}L^{12}\right)-k_2\left(G^{11^2}+2G^{10}G^{12}\right)\right]-\frac{2h_1'L_{,tt}^{12}}{F^{\frac{1}{2}}}-2h_1'L_{,t\tau}^{11}-\frac{2L_{,t\tau}^{21}}{F^{\frac{1}{2}}}-\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}L_{,t}^{21}-\frac{h_1''L_{,t}^{11}}{F}$$
$$-\frac{L_{,t\tau}^{20}}{F} \qquad (3.17f)$$

We also get the following

$$G_{,tt}^{10}+R^2G^{10}=R^2B \qquad (3.18a)$$

$$G_{,tt}^{11}+R^2G^{11}=-\frac{2G_{,t\tau}^{10}}{F^{\frac{1}{2}}}-\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}G_{,t}^{10} \qquad (3.18b)$$

$$=G_{,tt}^{12}+R^2G^{12}=-\frac{2G_{,t\tau}^{11}}{F^{\frac{1}{2}}}-\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}G_{,t}^{11}-\frac{G_{,\tau\tau}^{10}}{F} \qquad (3.18c)$$

$$G_{,tt}^{20}+R^2G^{20}=-\frac{2h_1'G_{,tt}^{10}}{F^{\frac{1}{2}}}-\frac{R^2}{F}(L^{10}G^{10}) \qquad (3.18d)$$

$$G_{,tt}^{21}+R^2G^{21}$$
$$=-\frac{2h_1'G_{,tt}^{11}}{F^{\frac{1}{2}}}-\frac{R^2}{F}(L^{10}G^{11}+L^{11}G^{10})-\frac{2G_{,tt}^{20}}{F^{\frac{1}{2}}}-\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}G_{,t}^{20}-\frac{2h_1'G_{,t\tau}^{10}}{F}$$
$$-\frac{h_1'''G_{,t}^{10}}{F} \qquad (3.18e)$$

$$G_{,tt}^{22}+R^2G^{22}=-2h_1'G_{,tt}^{12}-\frac{R^2}{F}(L^{10}G^{12}+L^{11}G^{11}+L^{12}G^{10})-\frac{2G_{,t\tau}^{21}}{F^{\frac{1}{2}}}-\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}G_{,t}^{21}-\frac{2h_1'G_{,t\tau}^{11}}{F}-\frac{2h_1''G_{,t}^{11}}{F}$$
$$-\frac{G_{,\tau\tau}^{20}}{F} \qquad (3.18f)$$

The initial conditions on L^{ij} and G^{ij} are as follows:
$L^{ij}(0,0) = G^{ij}(0,0) = 0, i = 1,2 \ldots, j = 0,1,2, \ldots$
$O(\bar{\xi}_2)$: $L_{,t}^{10}(0,0) = 0$
$O(\bar{\xi}_2\delta)$: $L_{,t}^{11}(0,0) + (1-\lambda)^{-\frac{1}{2}} L_{,\tau}^{10}(0,0) = 0$
$O(\bar{\xi}_2\delta^2)$: $L_{,t}^{12}(0,0) + (1-\lambda)^{-\frac{1}{2}} L_{,\tau}^{11}(0,0) = 0$
$O(\bar{\xi}_2^2)$: $L_{,t}^{20}(0,0) + (1-\lambda)^{-\frac{1}{2}} L_{,t}^{10}(0,0) = 0$ (v)
$O(\bar{\xi}_2^2\delta)$: $L_{,t}^{21}(0,0) + (1-\lambda)^{-\frac{1}{2}} h_1'(0)[L_{,t}^{11}(0,0) + L_{,\tau}^{20}(0,0)] = 0$
$O(\bar{\xi}_2^2\delta)$: $L_{,t}^{22}(0,0) + (1-\lambda)^{-\frac{1}{2}} h_1'(0)[L_{,t}^{12}(0,0) + L_{,\tau}^{21}(0,0)] = 0$
$O(\bar{\xi}_2)$: $G_{,t}^{10}(0,0) = 0$
$O(\bar{\xi}_2\delta)$: $G_{,t}^{11}(0,0) + (1-\lambda)^{-\frac{1}{2}} G_{,\tau}^{10}(0,0) = 0$
$O(\bar{\xi}_2\delta^2)$: $G_{,t}^{12}(0,0) + (1-\lambda)^{-\frac{1}{2}} G_{,\tau}^{11}(0,0) = 0$
$O(\bar{\xi}_2^2\delta)$: $G_{,t}^{21}(0,0) + (1-\lambda)^{-\frac{1}{2}} h_1'(0)[G_{,t}^{11}(0,0) + G_{,\tau}^{20}(0,0)] = 0$
$O(\bar{\xi}_2^2\delta)$: $G_{,t}^{22}(0,0) + (1-\lambda)^{-\frac{1}{2}} h_1'(0)[G_{,t}^{12}(0,0) + G_{,\tau}^{21}(0,0)] = 0$
From (3.18a), that is,

$$L_{,tt}^{10} + QL^{10} = 0 \qquad (3.19)$$

and the initial conditions,
$L^{10}(0,0) = 0 = L_{,t}^{10}(0,0)$
We now solve (3.19), and get

$$L^{10}(t,\tau) = \alpha_1(\tau)\, cosQt + \beta_1(\tau) sinQ \qquad (3.20)$$

Using the initial conditions of (3.18a) in (3.20), we get
$L^{10}(0,0) = \alpha_1(0) = 0 \ \therefore \alpha_1(0) = 0$
$L_{,t}^{10}(0,0) = \beta_1(0) = 0 \therefore \beta_1(0) = 0$
Solving (3.18b), we first substitute for L^{10}, and get

$$L_{,tt}^{11} + Q^2 L^{11} - \frac{2Q}{F^{\frac{1}{2}}}(-\alpha_1' sinQt + \beta_1' cosQt) - \frac{\lambda Q\acute{\varphi}}{2F^{\frac{3}{2}}}\sin(\varphi)(-\alpha_1 sinQt + \beta_1 cosQt) \qquad (3.21a)$$

with the initial conditions
$L^{11}(0,0) = 0$, and $L_{,tt}^{11}(0,0) + (1-\lambda)^{\frac{1}{2}} L_{,\tau}^{10}(0,0) = 0$
Simplifying(3.21a),we have

$$L_{,tt}^{11} + Q^2 L^{11} = \left(\frac{2Q\alpha_1'}{F^{\frac{1}{2}}} + \frac{Q\lambda\acute{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}\right) sinQt - \left(\frac{2Q\beta_1'}{F^{\frac{1}{2}}} + \frac{Q\lambda\acute{\varphi}\beta_1\sin(\varphi)}{2F^{\frac{3}{2}}}\right) cosQt \qquad (3.21b)$$

To ensure a uniformly valid solution in t, we equate to zero in (3.21b), the coefficients of $cosQt$ and $sinQt$
From the coefficient of $cosQt$, we have

$$\frac{2Q\beta_1'}{F^{\frac{1}{2}}} + \frac{Q\lambda\beta_1\acute{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \qquad (3.22a)$$

From the coefficient of $sinQt$, we have

$$\frac{2Q\alpha_1'}{F^{\frac{1}{2}}} + \frac{Q\lambda\alpha_1\acute{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \qquad (3.22b)$$

Solving (3.22a), we get

$$\frac{\beta_1}{\beta_1} + \frac{\lambda\acute{\varphi}\sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0$$

After integrating the above equation, we get

$$In\beta_1 + \frac{1}{4} In(1-\lambda\cos(\varphi)) = InC_1$$

Thus, we get

$$\beta_1(\tau) = C_1(1-\lambda\cos\left(\varphi)\right)^{-\frac{1}{4}}\Big) \qquad (3.23a)$$

$$\beta_1(0) = C_1(1-\lambda)^{-\frac{1}{4}} = 0 \therefore C_1 = 0$$

and
$\beta_1(\tau) = 0$
Similarly from (3.22b), we get
$$\frac{\alpha_1'}{\alpha_1} + \frac{\lambda\acute{\varphi}\sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0$$
$$\therefore In\alpha_1 + \frac{1}{4}In(1-\lambda\cos(\varphi)) = InC_2$$
$$\therefore \alpha_1(\tau) = C_2(1-\lambda\cos(\varphi))^{-\frac{1}{4}} \qquad (3.23b)$$
$\alpha_1(0) = C_2(1-\lambda)^{-\frac{1}{4}} = 0 \ \therefore C_2 = 0$
Substituting C_2 in (3.23b), we get, $\alpha_1(\tau) = 0$. Substituting $\alpha_1(\tau)$ and $\beta_1(\tau)$ in (3.20), we obtain
$$L^{10}(t,\tau) = 0 \qquad (3.24)$$
Solving the remaining equation in (3.21b), we get
$$L^{11}(t,\tau) = \alpha_2(\tau)cosQt + \beta_2(\tau)sinQt \qquad (3.25)$$
Using the initial conditions, we get, $L^{11}(0{,}0) = \alpha_2(0) = 0 \ \therefore \alpha_2(0) = 0$
$L_{,t}^{11}(0{,}0) = \beta_2(0) = 0, since\ L^{10} = 0, \therefore \beta_2(0) = 0$
Next we solve (3.17c), first substituting for $L^{10}\ and\ L^{11}$, and getting
$$L_{,tt}^{12} + Q^2L^{12} = -\frac{2Q}{F^{\frac{1}{2}}}(-\alpha_2'sinQt + \beta_2'\cos Qt) - \frac{\lambda Q\acute{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}(-\alpha_2 sinQt + \beta_2 cosQt) \qquad (3.26a)$$
The initial conditions are
$L^{12}(0{,}0) = 0$, and $L_{,t}^{12}(0{,}0) + (1-\lambda)^{-\frac{1}{2}}L_{,\tau}^{11}(0{,}0) = 0$
From (3.26a), we get
$$L_{,tt}^{12} + Q^2L^{12} = \left(\frac{2Q\alpha_2'}{F^{\frac{1}{2}}} + \frac{\lambda Q\acute{\varphi}\alpha_2\sin(\varphi)}{2F^{\frac{3}{2}}}\right)sinQt - \left(\frac{2Q\beta_2'}{F^{\frac{1}{2}}} + \frac{\lambda Q\acute{\varphi}\beta_2\sin(\varphi)}{2F^{\frac{3}{2}}}\right)cosQt \qquad (3.26b)$$
To ensure uniformly valid solution in t, we equate to zero in (3.26b), the coefficients of $cosQt$ and $sinQt$. From the coefficient of $sinQt$, we get
$$\frac{2Q\beta_2'}{F^{\frac{1}{2}}} + \frac{\lambda\acute{\varphi}\beta_2\sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \qquad (3.27a)$$
And from thecoefficient of $cosQt$, we have
$$\frac{2Q\alpha_2'}{F^{\frac{1}{2}}} + \frac{\lambda Q\acute{\varphi}\alpha_2\sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \qquad (3.27b)$$
Solving (3.27b), we get
$$\frac{\alpha_2'}{\alpha_2} + \frac{\lambda\acute{\varphi}\sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0 \qquad (3.27c)$$
Integrating both sides of (3.27c), we get
$$In\alpha_1 + \frac{1}{4}In(1-\lambda\cos(\varphi)) = InC_3$$
i.e., $\alpha_2(\tau) = C_3(1-\lambda\cos(\varphi))^{-\frac{1}{4}}$ (3.27d)
i.e., $\alpha_2(0) = C_3(1-\lambda)^{-\frac{1}{4}} = 0 \ \therefore C_3 = 0, \ \therefore \alpha_2(\tau) = 0$
Similarly from (3.27a), we get
$$\beta_2(\tau) = C_4(1-\lambda\cos(\varphi))^{-\frac{1}{4}} \qquad (3.28)$$
$\beta_2(0) = C_4(1-\lambda)^{-\frac{1}{4}} = 0 \ \therefore C_4 = 0, \quad \therefore \beta_2(\tau) = 0$
Substituting $\alpha_2(\tau)$ and $\beta_2(\tau)$ in (3.25), we get
$$L^{11}(t,\tau) = 0 \qquad (3.29)$$
Solving the remaining equation in (3.26a), we have
$$L^{12}(t,\tau) = \alpha_3(\tau)cosQt + \beta_3(\tau)sinQt \qquad (3.30)$$
Using the initial conditions of (3.17c) in (3.30), we get
$L^{12}(0{,}0) = \alpha_3(0) = 0 \ \therefore \ \alpha_3(0) = 0$
and
$L_{,t}^{12}(0{,}0) = \beta_2(0) = 0, since\ L^{11} = 0 \ \therefore \ \beta_2(0) = 0$

We next solve (3.18a), ie
$G,_{tt}^{10} + R^2G^{10} = R^2B$
with the initial conditions
$G^{10}(0,0) = 0 = G,_t^{10}(0,0)$
Solving (3.18a), we get
$G^{10}(t,\tau) = \Upsilon_1(\tau)cosRt + \gamma_2(\tau)sinRt + B$ (3.31a)
Using the initial conditions of (3.18a) in (3.31a), we get
$G^{10}(0,0) = \gamma_1(0) + B(0) = 0$
$\therefore \gamma_1(0) = -B(0) = \frac{-\lambda}{1-\lambda}$ (3.31b)
Using the second initial conditions of (3.18a) in (3.31a), we get
$\gamma_2(0) = 0$
We next solve (3.18b),and first substitute for$G^{10}$and get
$$G,_{tt}^{11} + R^2G^{11} = \left(\frac{2R\gamma_1'}{F^{\frac{1}{2}}} + \frac{R\lambda\acute{\varphi}\gamma_2\sin(\varphi)}{2F^{\frac{3}{2}}}\right)sinRt - \left(\frac{2R\gamma_2'}{F^{\frac{1}{2}}} + \frac{R\lambda\acute{\varphi}\gamma_2\sin(\varphi)}{2F^{\frac{3}{2}}}\right)cosRt \quad (3.32)$$
with the initial conditions
$G^{11}(0,0) = 0$
and
$G,_t^{11}(0,0) + G,_\tau^{10}(1-\lambda)^{-\frac{1}{2}} = 0$
For uniformly valid solution in t, we equate to zero in (3.32), the coefficients of $cosRt$ and $sinRt$.
From the coefficient of $sinRt$,we get
$$\frac{2R\gamma_1'}{F^{\frac{1}{2}}} + \frac{R\lambda\gamma_1\acute{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.33a)$$
And from the coefficient of $cosRt$, we get
$$\frac{2R\gamma_2'}{F^{\frac{1}{2}}} + \frac{R\lambda\gamma_2\acute{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.33b)$$
Solving (3.33a), we get
$$\frac{\gamma_1'}{\gamma_1} + \frac{\lambda\acute{\varphi}\sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0 \quad (3.33c)$$
Integrating both sides of (3.33c), we get
$$In\gamma_1 + \frac{1}{4}In(1-\lambda\cos(\varphi)) = InC_4$$
$$\therefore \gamma_1(\tau) = C_4(1-\lambda\cos(\varphi))^{-\frac{1}{4}} \quad (3.33d)$$
i.e., $\gamma_1(0) = C_4(1-\lambda)^{-\frac{1}{4}} = \frac{-\lambda}{1-\lambda} = -B(0)$
i.e., $C_4 = -B(0)(1-\lambda)^{\frac{1}{4}}$
$$\therefore \gamma_1(\tau) = -B(0)\left(\frac{1-\lambda}{1-\lambda\cos(\varphi)}\right)^{\frac{1}{4}} \quad (3.33e)$$
Solving (3.33b), we get
$$\frac{\gamma_2'}{\gamma_2} + \frac{\lambda\acute{\varphi}\sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0$$
Integrating both sides, we get
$$In\gamma_2 + \frac{1}{4}In(1-\lambda\cos(\varphi)) = InC_5$$
i.e, $\gamma_2(\tau) = C_5(1-\lambda\cos(\varphi))^{-\frac{1}{4}}$ (3.34a)
$\gamma_2(0) = C_5(1-\lambda) = 0 \;\therefore\; C_5 = 0, \therefore \gamma_2(\tau) = 0$
Solving the remaining equationof (3.18b),we get
$G^{11}(t,\tau) = \gamma_3(\tau)cosRt + \gamma_4(\tau)sinRt$ (3.34b)
Using the initial conditions in (3.34b), we get
$G^{11}(0,0) = \gamma_3(0) = 0 \;\therefore\; \gamma_3(0) = 0$

Also using the second condition, we get

$$R\gamma_4(0)+\gamma_1'(0)\,(1-\lambda)^{-\frac{1}{2}}+B'(0)=0 \qquad (3.35a)$$

We note that

$$B'(\tau)=-\frac{\lambda\dot{\varphi}\sin(\varphi)\{1-\lambda\cos(\varphi)+\lambda\cos(\varphi)\}}{(1-\lambda\cos(\varphi))^2} \qquad (3.35b)$$

$\therefore B'(0)=0;\varphi(0)=0\,,\sin(0)=0\,,\gamma_1'(0)=0$

From (3.35a), we have

$\gamma_4(0)=0$

Substituting $\gamma_2(\tau)$ in (3.31a), we get

$$G^{10}(t,\tau)=\gamma_1 cosR+B(\tau) \qquad (3.36)$$

Solving (3.18c), we get

$$G,_{tt}^{12}+\mathrm{R}^2\mathrm{G}^{12}=-\frac{2G,_{t\tau}^{11}}{F^{\frac{1}{2}}}+\frac{\lambda\dot{\varphi}\sin(\varphi)}{2F^{\frac{3}{2}}}G,_t^{11}-\frac{G,_{\tau\tau}^{10}}{F}$$

with the initial conditions

$G^{12}(0,0)=0$

and

$G,_t(0,0)+(1-\lambda)^{-\frac{1}{2}}G,_\tau(0,0)=0$

Substituting for G^{11} *and* G^{10} in (3.18c), we get

$$G,_{tt}^{12}+\mathrm{RG}^{12}=\left(\frac{2R\gamma_3'}{F^{\frac{1}{2}}}+\frac{R\gamma_3\lambda\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}}\right)sinRt$$

$$-\left(\frac{2R\gamma_4'}{F^{\frac{1}{2}}}+\frac{\gamma_4R\lambda\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}}+\frac{\gamma_1''}{F}\right)\cos\mathrm{Rt}+B'' \qquad (3.37)$$

To ensure uniformly valid solution in t, we equate to zero in (3.37), the coefficients of sinRt and cosRt

From the of coefficients of sinRt, we get

$$\frac{2R\gamma_3'}{F^{\frac{1}{2}}}+\frac{R\gamma_3\lambda\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}}=0 \qquad (3.38a)$$

And from the coefficients of cosRt, we get

$$\frac{2R\gamma_4'}{F^{\frac{1}{2}}}+\frac{R\gamma_4\lambda\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}}+\frac{\gamma_1''}{F}=0 \qquad (3.38b)$$

Solving (3.38a), we get

$$\frac{\gamma_3'}{\gamma_3}+\frac{\lambda\varphi'\sin(\varphi)}{4(1-\lambda\cos(\varphi)}=0 \qquad (3.38c)$$

Integrating both sides of (4.214c), we get

$$In\gamma_3+\frac{1}{4}\mathrm{In}In\big(1-\lambda\cos(\varphi)\big)=InC_6$$

i.e. $\gamma_3(\tau)=C_6(1-\lambda\cos(\varphi))^{-\frac{1}{4}}$ (3.38d)

i.e. $\gamma_3(0)=\mathrm{C}_6(1-\lambda)^{-\frac{1}{4}}=0\therefore\mathrm{C}_6=0$

$\therefore\gamma_3(\tau)=0$

Solving (3.38b), we get

$$\gamma_4'+\frac{\lambda\,\varphi'\gamma_4\sin(\varphi)}{4(1-\lambda\cos(\varphi))}=\frac{-\gamma_1''}{R(1-\lambda\cos(\varphi)^{\frac{1}{2}}} \qquad (3.39a)$$

The integrating factor, IF, is

$$\mathrm{I.F}=e^{\frac{1}{4}\int\left(\frac{\lambda\varphi'\sin(\varphi)}{1-\lambda\cos(\varphi)}\right)d\tau}=(1-\lambda\cos(\varphi))^{\frac{1}{4}}$$

Then, we have

$$\int\frac{d}{d\tau}\left(\gamma_4\big(1-\lambda\cos(\varphi)\big)^{\frac{1}{4}}\right)d\tau=-\int_0^\tau\frac{\gamma_1''}{R}\big(1-\lambda\cos(\varphi)\big)^{-\frac{1}{4}}\mathrm{ds}+C_7 \qquad (3.39b)$$

$$\gamma_4(\tau)=-(1-\lambda\cos(\varphi))^{-\frac{1}{4}}\left[\int_0^\tau\frac{\gamma_1''}{R}\big(1-\lambda\cos(\varphi)\big)^{-\frac{1}{4}}ds+C_7\right] \qquad (3.39c)$$

$\gamma_4(0) = -(1-\lambda)^{-\frac{1}{4}}C_7 = 0 \therefore C_7 = 0$
and we get

$$\gamma_4(\tau) = \left(1-\lambda\cos(\varphi)\right)^{-\frac{1}{4}}\int_0^{\tau}\frac{\gamma_1''}{R}\left(1-\lambda\cos(\varphi)\right)^{-\frac{1}{4}}ds \qquad (3.39d)$$

From (3.35a), we note that since $\gamma_3(\tau) = 0$, then

$$G^{11}(t,\tau) = \gamma_4 sinRt \qquad (3.40)$$

The remaining equation in (3.37) is

$$G^{12}_{,tt} + R^2G^{12} = \frac{B''}{F} \qquad (3.41)$$

Solving (3.41), we get

$$G^{12}(t,\tau) = \gamma_5(\tau)\cos Rt + \gamma_6(\tau)\sin Rt + \frac{B''}{R^2F} \qquad (3.42)$$

Using the initial conditions in (3.41), we get

$$G^{12}(0,0) = \gamma_5(0) + \frac{B''(0)}{R^2F(0)} = 0 \therefore \gamma_5(0) = \frac{B''(0)}{R^2F(0)} = \frac{B'^2(0)B\ (0)}{R^2(1-\lambda)^2}$$

We note that

$$B''(\tau) = -\frac{\lambda\varphi'^2(1-\lambda\cos(\varphi))^2\cos(\varphi))}{(1-\lambda\cos(\varphi))^3}, \quad B''(0) = -\frac{\lambda\varphi'^2(0)}{1-\lambda} \qquad (3.43)$$

Also using the second initial condition, we obtain, $\gamma_6(0) = 0$. On solving (3.18d), we get,

$$G^{20}_{,tt} + R^2G^{20} = -\frac{2h_1'G^{10}_{tt}}{F^{\frac{1}{2}}} - \frac{F^2}{F}L^{10}G^{10}$$

with the initial conditions

$$G^{20}(0,0) = O$$

and

$$G_{,t}^{20}(0,0) + (1-\lambda)^{-\frac{1}{2}}h_1'(0)G_{,t}^{10}(0,0) = 0$$

Substituting for $L^{10}$and G^{10} in (3.18d), we get

$$G^{20}_{,tt} + R^2G^{20} = \frac{2h'\gamma_1}{F^{\frac{1}{2}}}RsinRt \qquad (3.44)$$

To ensure uniformly valid solution in t, we equate to zero in (3.44), the coefficient of sinRt
For the coefficients of$sinRt$, we get

$$\frac{2h_1'\gamma_1}{F^{\frac{1}{2}}} = 0 \qquad (3.45a)$$

Solving (3.45a), we have

$$h_1'(\tau) = 0 \therefore h(\tau) = \text{constant} \qquad (3.45b)$$

Solving the remaining terms in (3.44), we get

$$G^{20}(t,\tau) = \gamma_7(\tau)\cos Rt + \gamma_8(\tau)\sin Rt \qquad (3.46)$$

Using the initial conditions, we get

$$G^{20}(0,0) = \gamma_7(0) = 0 \therefore \gamma_7(0) = 0$$

Also from the second condition
$\gamma_8(0) = 0$, since $h_1(0) = 0$
Solving (3.17d), we get

$$L_{,tt}^{20} + Q^2L^{20} = \frac{Q^2}{F}k_1{L^{10}}^2 - k_2{G^{10}}^2 - \frac{2h_1'}{F^{\frac{1}{2}}}L_{,tt}^{10}$$

with the initial conditions

$$L^{20}(0,0) = 0$$

and

$$L_{,tt}^{20}(0,0) + (1-\lambda)^{-\frac{1}{2}}h_1'(0)L_{,t}^{10} = 0$$

First substituting for L^{10}, $G^{10}$and$L_{,tt}^{10}$, we get

$$L_{,tt}^{20} + Q^2L^{20} = -\frac{Q^2k_2}{F}\left[\left(B^2 + \frac{\gamma_1}{2}\right) + 2\gamma_1\cos R + \frac{\gamma_1^2}{2}\cos 2Rt\right] \qquad (3.47)$$

Solving (3.47), we get

$$L^{20}(t,\tau) = \alpha_4 \cos Qt + \beta_4 \sin Qt - \frac{K_2 Q^2}{Q^2 F}\left(\frac{\gamma_1^2}{2} + B^2\right) - \frac{\gamma_1^2 k_2 Q^2}{2F(Q^2 - 4R^2)}\cos 2Rt - \frac{2k_2\gamma_1 Q^2}{F(Q^2 - R^2)} \quad (3.48)$$

where $Q \neq 2R$, and $Q \neq R$

$$L^{20}(t,\tau) = \gamma_4 \cos Qt + \beta_4 \sin Qt + r_0 + r_1 \cos 2Rt + r_2 \cos Rt \quad (3.49)$$

where,

$$r_0(\tau) = \frac{-k_2}{F}\left(\frac{\gamma_1^2}{2} + B^2\right); \quad r_1(\tau) = \frac{\gamma_2 Q^2 \gamma_1}{2F(Q^2-4R^2)}; \quad r_2(\tau) = \frac{-2k_2\gamma_1 Q^2}{F(Q^2-R^2)} \quad (3.50)$$

$$r_0(0) = \frac{-3k_2B^2}{1-\lambda}; r_1(0) = \frac{k_2 Q^2 B^2(0)}{2(1-\lambda)(Q^2-4R^2)}; \quad r_2(0) = \frac{2k_2 Q^2 B(0)}{(1-\lambda)(Q^2-R^2)} \quad (3.51)$$

Solving (3.18e), first wesubstitute for L^{10}, L^{10}, h_1', $G^{10}$and G^{20}, we have

$$G_{,tt}^{21} + R^2 G^{21} = \left(\frac{2\gamma_7' R}{F^{1/2}} + \frac{R\gamma_7 \lambda\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}}\right)\sin Rt - \left(\frac{2\gamma_8' R}{F^{\frac{1}{2}}} + \frac{R\gamma_8 \lambda\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}}\right)\cos Rt \quad (3.52)$$

with the initial conditions,

$G^{20}(0{,}0) = 0$

and

$G_{,t}^{21}(0{,}0) + (1-\lambda)^{-1} h_1'(0) G_{,t}^{11}(0{,}0) + (1-\lambda)^{-\frac{1}{2}} G_{,\tau}^{20}(0{,}0) = 0$

To solve (3.52), we equate to zero, the coefficient of $sinRt$ and $cosRt$ in (3.52),to ensure uniformly valid solution in t.From the coefficient of $sinRt$, we get

$$\frac{2\gamma_1' R}{F^{\frac{1}{2}}} + \frac{R\gamma_7 \lambda' \varphi \sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.53a)$$

And fromthe coefficient of $cosRt$, we get

$$\frac{2\gamma_8' R}{F^{\frac{1}{2}}} + \frac{R\gamma_8 \lambda\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.53b)$$

Solving (3.53a), we get

$$\frac{\gamma_7'}{\gamma_7} + \frac{\lambda\varphi' \sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0 \quad (3.54a)$$

i.e. $\text{In}\gamma_7 + \frac{1}{4} In\left(1 - \lambda cos(\varphi)\right) = \text{In}C_8$

i.e., $\gamma_7(\tau) = C_8(1 - \lambda cos(\varphi))^{-\frac{1}{4}}$ (3.54b)

i.e., $\gamma_7(0) = C_8(1-\lambda)^{-\frac{1}{4}} = 0 \therefore \gamma_7(\tau) = 0$

Also solving (3.53b), we get

$$\frac{\gamma_8'}{\gamma_8} + \frac{\lambda\varphi' \sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0 \quad (3.54c)$$

Integrating both sides, we get, $\text{In}\gamma_8 + \frac{1}{4} In\left(1 - \lambda\cos(\varphi)\right) = InC_9$

$\therefore \gamma_8(\tau) = C_9(1 - \lambda\cos(\varphi))^{-\frac{1}{4}}$ (3.54d)

$\gamma_8(0) = C_9(1 - \lambda(\varphi))^{-\frac{1}{4}} = 0 \therefore C_9 = 0 \therefore \gamma_8(\tau) = 0$

Substituting $\gamma_7(\tau)$ and$\gamma_8(\tau)$in (3.46), we get

$G^{20}(t,\tau) = 0$ (3.55)

Solving the remaining equation in (3.52), we get

$G^{21}(t,\tau) = \gamma_9 \cos Rt + \gamma_{10} \sin Rt$ (3.56)

To solve (3.17e), we first substitute for the values of L^{10}, L^{11}, h_1', and L^{20}, ie

$$L_{,tt}^{21} + Q^2 L^{21} = \frac{2Q^2 k_2}{F} G^{10} G^{11} - \frac{2L_{,t\tau}^{20}}{F^{\frac{1}{2}}} - \frac{\lambda\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}} L_{,t}^{20} \quad (3.57)$$

with the initial conditions

$L^{21}(0{,}0) = 0$

And

$L_{,tt}^{21}(0,0) + (1-\lambda)^{-\frac{1}{2}} h_1'(0) L_{,t}^{11}(0,0) + (1-\lambda)^{-\frac{1}{2}} L_{,\tau}^{20}(0,0) = 0$
Simplifying (3.57),we get

$$L_{,tt}^{21} + Q^2L^2 = \left\{\frac{2Q^2k_2\gamma_1\gamma_4}{2F} + \frac{4\gamma_1' R}{F^{\frac{1}{2}}} + \frac{\gamma_1 R\varphi' \sin(\varphi)}{F^{\frac{3}{2}}}\right\}\sin 2Rt + \left\{\frac{-2BQ^2k_2\gamma_4}{F} + \frac{2\gamma_2' R}{F^{\frac{1}{2}}} + \frac{\gamma_1 R\,\varphi' \sin(\varphi)}{F^{\frac{3}{2}}}\right\} sinRt$$
$$+ \left\{\frac{2\alpha_4' Q}{F^{\frac{1}{2}}} + \frac{Q\alpha_4\lambda\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}}\right\}\sin Qt + \left\{\frac{2\beta_4' Q}{F^{\frac{1}{2}}} + \frac{\beta_4 Q\,\lambda\,\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}}\right\}\cos Qt \quad (3.58)$$

To ensure uniformly valid solution in t, we equate to zeroin (3.58), the coefficients of sint and cost.
From the coefficient of sin$Q$t, we get

$$\frac{2\alpha_4' Q}{F^{\frac{1}{2}}} + \frac{\alpha_4 Q\,\lambda\,\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.59a)$$

And fromthe coefficient of cosQt, we get

$$\frac{2\beta_4'}{F^{\frac{1}{2}}} + \frac{\beta_4 Q\,\lambda\,\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.59b)$$

Solving (3.59a), we have

$$\frac{\alpha_4'}{\alpha_4} + \frac{\lambda\varphi\sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0 \quad (3.60a)$$

Integrating both sides of (3.60a), we have, $\ln\alpha_4 + \frac{1}{4}\ln\big(1-\lambda\cos(\varphi)\big) = \ln C_{10}$

$$\therefore\ \alpha_4(\tau) = C10(1-\lambda\cos(\varphi))^{-\frac{1}{4}} \quad (3.60b)$$

But from (3.49), we have that

$$\alpha_4(0) = -(r_0(0) - r_1(0) + r_2(0) = -P(0)\frac{K_2Q^2B^2(0)}{1-\lambda}$$

where,

$$P(0) = \left\{\frac{3}{Q^2} - \frac{1}{2(Q^2-4R^{2)}} + \frac{2}{Q^2-R^2}\right\} \quad (3.60c)$$

From (3.60b), we obtain

$$\alpha_4(0) = C_{10}(1-\lambda)^{-\frac{1}{4}} = \frac{-P(0)k_2Q^2B^2(0)}{1-\lambda}$$

$$\therefore\ C_{10} = \frac{-k_2Q^2B^2(0)(1-\lambda)^{\frac{1}{4}}P(0)}{(1-\lambda)} \quad (3.60d)$$

Solving (3.59b), we get

$$\frac{\beta_4'}{\beta 4} + \frac{\lambda\varphi' \sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0$$

By integration we have

$$\ln\beta_4 + \frac{1}{4}\ln\,(1-\lambda\cos(\varphi)) = InC_{11}$$

i.e, $\beta_4(\tau) = C_{11}(1-\lambda\cos(\varphi))^{-\frac{1}{4}}$, $\quad \beta_4(0) = C_{11}(1-\lambda)^{-\frac{1}{4}} = 0\ \therefore C_{11} = 0 \therefore \beta_4(\tau) = 0$
Substituting $\beta_4(\tau)$ in (3.49), we have

$$L^{20}(t,\tau) = \alpha_4\cos Qt + r_0(\tau) + r_1(\tau)\cos 2Rt + r_2(\tau)\cos Rt \quad (3.61)$$

Solving the remaining equation in (3.58), we have

$$L^{21} = \alpha_5(\tau)\cos Qt + \beta_5\sin Qt + r_3(\tau)\sin 2Rt + r_4(\tau)\sin Rt \quad (3.62)$$

where,

$$r_3(\tau) = \frac{1}{Q^2-4R^2}\left\{\frac{4\gamma_1' R}{F^{\frac{1}{2}}} + \frac{\gamma_1 R\lambda\varphi' \sin(\varphi)}{F^{\frac{3}{2}}} - \frac{Q^2k_2\gamma_1\gamma_4}{F}\right\} \quad (3.63a)$$

$$r_4(\tau) = \frac{1}{Q^2-R^2}\left\{\frac{Rr_2' R}{F^{\frac{1}{2}}} + \frac{Rr_2\lambda\varphi' \sin(\varphi)}{2F^{\frac{3}{2}}} - \frac{2BQ^2k_2\gamma_4}{F}\right\} \quad (3.63b)$$

$r_3(0) = r_4(0) = 0$
Using the initial conditions
$L^{21}(0,0) = \alpha_5(0) = 0 \therefore \alpha_5(0) = 0$

Also the second condition gives, $\beta_5(0) = 0$.
Solving (3.17f), but first substituting the values of $L^{10}, L^{11}, G^{10}G^{12}, h_1', L^{20}L^{21}$, we get

$$L_{,tt}^{22} + Q^2L^{22} = \frac{k_2Q^2}{F}\left(G^{11^2} + 2G^{10}G^{12}\right) - \frac{2L_{,t\tau}^{21}}{F^{\frac{1}{2}}} - \frac{\lambda\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}}L_{,t}^{21} - \frac{L_{,\tau\tau}^{20}}{F} \quad (3.64)$$

with the initial conditions
$L^{22}(0,0) = 0$
and
$L_{,t}^{22}(0,0) + (1-\lambda)^{-\frac{1}{2}}h_1'(0)\, L_{,t}^{12}(0,0) + (1-\lambda)^{-\frac{1}{2}}L_{,\tau}^{21}(0,0)$
Substituting G^{11}, G^{10} and G^{12} and simplifying, gives

$$\begin{aligned} L_{,tt}^{22} + Q^2L^{22} = &\left\{\frac{-k_2Q^2}{F}\left(\gamma_1\gamma_5 + \frac{2BB''}{R^2F} + \frac{\gamma_4^2}{2}\right)r_0''\right\} - \left(\frac{k_2Q^2\gamma_1\gamma_6}{F}\right)\sin 2Rt \\ &+ \left\{\frac{-k_2Q^2}{F}\left(\gamma_1\gamma_5 - \frac{\gamma_4}{2}\right) + \frac{4R\, r_3'}{F^{\frac{1}{2}}} - \frac{4R\, r_3\lambda\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}} - \frac{r_1''}{F}\right\} \\ &+ \cos 2Rt\left\{-\frac{k_2Q^2}{F}\left(\frac{2B''\gamma_1}{2F} + 2_B\gamma_5\right) + \frac{2R\, r_4'}{F^{\frac{1}{2}}} - \frac{R\, r_4\lambda\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}} - \frac{r_2''}{F}\right\}\cos Rt - \left(\frac{2k_2Q^2B\,\gamma_4}{F}\right)\sin Rt \\ &+ \left\{\frac{2\,\alpha_5'Q}{F^{\frac{1}{2}}} + \frac{\alpha_5Q\lambda\,\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}}\right\}\sin Qt \\ &- \left\{\frac{2\beta_{5Q}'}{F^{\frac{1}{2}}} + \frac{\beta_5Q\lambda\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}}\right\}\cos Qt \end{aligned} \quad (3.65)$$

To ensure uniformly valid solution in t, we equate to zero in (3.65), the coefficients of $sinQt$ and $cosQt$.
From the coefficients of $sinQt$, we get

$$\frac{2\alpha_5'Q}{F^{\frac{1}{2}}} + \frac{\alpha_5Q\,\lambda\,\varphi'\mathrm{Sin}(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.66a)$$

And for the coefficients of $cosQt$,we have

$$\frac{2\beta_5'Q}{F^{\frac{1}{2}}} + \frac{\beta_5Q\,\lambda\,\varphi'\mathrm{Sin}(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.66b)$$

Solving (3.66a), we get

$$\frac{\alpha_5'}{\alpha_5} + \frac{\lambda\,\varphi'\sin(\varphi)}{4(1-\lambda\cos(\varphi))} = 0 \quad (3.67a)$$

i. e, $\mathrm{In}\alpha_5 + \frac{1}{4}\mathrm{In}\big(1-\lambda\cos(\varphi)\big) = C_{12},$

$$\alpha_5(\tau) = C_{12}(1-\lambda\cos(\varphi))^{-\frac{1}{4}} \quad (3.67b)$$

$\alpha_5(0) = C_{12}(1-\lambda)^{-\frac{1}{4}} = 0 \therefore C_{12} = 0 \therefore \alpha_5(\tau) = 0$
Also solving (3.66b), we get

$$\frac{\beta_5'}{\beta_5} + \frac{\lambda 4'\mathrm{Sin}(\varphi)}{4(1-\lambda\mathrm{Cos}(\varphi))} = 0 \quad (3.68)$$

Integrating (3.68), we have
$\mathrm{In}\beta_5 + \frac{1}{4}\mathrm{In}\big(1-\lambda\cos(\varphi)\big) = \mathrm{In}C_{13},$ i. e $\beta_5(\tau) = C_{13}(1-\lambda\cos(\varphi))^{-\frac{1}{4}}$
i. e, $\beta_5(0) = 0 = C_{13}(1-\lambda)^{-\frac{1}{4}}, C_{13} = 0, \therefore \beta_5(\tau) = 0$
Substituting $\beta_5(\tau)$ and $\alpha_5(\tau)$ in (3.62), we have

$$L^{21}(t,\tau) = r_3(\tau)\,\sin 2Rt + r_4(\tau)\sin Rt \quad (3.69)$$

Solving the remaining equation in (3.65), we get

$$\begin{aligned} L^{22}(t,\tau) = &\ \gamma_6(\tau)\cos Qt + \beta_6(\tau)\sin Qt + r_5(\tau) + r_6(\tau)\cos 2Rt + r_7(\tau)\cos Rt + r_8(\tau)\sin Rt \\ &+ r_9(\tau)\sin Rt \end{aligned} \quad (3.70)$$

where,

$$r_5(\tau) = \frac{-k_2}{F}\left(\gamma_1\gamma_5 + \frac{2BB''}{R^2F} + \frac{\gamma_4^2}{2}\right) + \frac{\gamma_0''}{Q^2} \quad (3.71a)$$

$$r_5(\tau) = \frac{1}{Q^2 - 4R^2}\left\{\frac{k_2Q^2}{F}\left(\gamma_1\gamma_5 - \frac{\gamma_4}{2}\right) + \frac{4R\gamma_3'}{F^{\frac{1}{2}}} - \frac{4\lambda R\gamma_3\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}} - \frac{\gamma_1''}{F}\right\} \quad (3.71b)$$

$$r_7(\tau) = \frac{1}{Q^2R^2}\left\{\frac{k_2Q^2}{F}\left(2B''\gamma_1 + 2B\,\gamma_5\right) + \frac{2R\,\gamma_4'}{F^{\frac{1}{2}}} - \frac{R\gamma_4\lambda\,\varphi'\sin\,(\varphi)}{2F^{\frac{3}{2}}}\right\} (3.71c)$$

$$r_8(\tau) = \frac{-2}{Q^2-R^2}\left\{\frac{k_2Q^2B\,\gamma_6}{F}\right\}; r_9(\tau) = \frac{k_2Q^2\gamma_1\gamma_6}{F(Q^2-4R^2} \quad (3.71d)$$

$$r_5(0) = \frac{k_2\varphi^{1^2}(0)B^2(0)}{(1-\lambda)^3}\left\{\frac{4Q^2 - R^2(\lambda+7)}{4R^2Q^2}\right\} \quad (3.72a)$$

$$r_6(0) = \frac{Q^2B^2(0)\varphi'^2(0)}{(1-\lambda)^2(Q^2-4R^2)}\left\{\frac{-k_2}{R^2} + \frac{4R}{(1-\lambda)^{\frac{1}{2}}(Q^2-4R^2)}\left\{1-k_2Q^2\left(\frac{1}{2(1-\lambda)(Q^2-4R^2)} + \frac{B(0)}{R}\right)\right\} - \frac{3k_2}{8\,(Q^2-4R^2)}\right\} \quad (3.72b)$$

$$r_7(0) = \frac{2k_2Q^2\varphi'^2(0)B^2(0)}{(1-\lambda)^2(\varphi^2-R^2)}\left\{\frac{1-R^2}{R^2} + \frac{1}{Q^2-R^2}\left(\frac{-3R}{Q^2-4R^2} + \frac{B(0)}{2(1-\lambda)^{\frac{1}{2}}}\right)\right\} \quad (3.72c)$$

$$r_8(0) = 0; \qquad r_9(0) = 0 \quad (3.72d)$$

Applying the initial conditions in (3.70), we obtain

$L^{22}(0,0) = 0 = \alpha_6(0) + r_5(0) + r_6(0) + r_7(0) \ \therefore \alpha_6(0) = -\{r_5(0) + r_6(0) + r_7(0)\}$

Using the second initial condition gives, $\beta_6(0) = 0$.

Solving (3.18f), but first substituting the value of h_1', $L^{10}, G^{10}, L^{11}G^{20}, G^{21}$, we get

$$G_{,tt}^{22} + R^2G^2 = \frac{R^2}{F}L^{12}G^{10} - \frac{2G_{,t\tau}^{21}}{F^{\frac{1}{2}}} - \frac{\lambda\,\varphi'\sin(\varphi)}{2F^{\frac{3}{2}}}G_{,t}^{21} \quad (3.73)$$

with the initial conditions

$G^{22}(0,0) = 0$

and

$G_{,tt}^{22}(0,0) + (1-\lambda)^{-\frac{1}{2}}[h_1'(0)G_t^{12}(0,0) + G_{,\tau}^{21}(0,0)] = 0$

From (3.73), we have

$$G_{,tt}^{22} + R^2G^2 = \frac{R^2\alpha_3}{2F}\cos(Q+R)t - \frac{R^2\gamma_1\alpha_3}{2F}\cos(R-Q)t - \frac{R^2\gamma_1\beta_3}{2F}\sin(Q+R)t + \frac{R^2\gamma_1\beta_3}{2F}\sin(R-Q)t - \frac{R^2\alpha_3}{F}\cos Qt - \frac{R^2B\beta_3}{F}\sin Qt + \left(\frac{2R\gamma_9'}{F^{\frac{1}{2}}} + \frac{R\gamma_{9\gamma}\varphi'\mathrm{Sin}(\varphi)}{2F^{\frac{3}{2}}}\right)\sin Rt - \left(\frac{2R\,\gamma_{10}'}{F^{\frac{1}{2}}} + \frac{R\,\gamma_{10}\lambda\,\varphi'\mathrm{Sin}\,(\varphi)}{2F^{\frac{3}{2}}}\right)\cos Rt \quad (3.74)$$

To ensure uniformly valid solution in t, we equate to zero in (3.74) the coefficients of sinRt and cosRt, and get,

First for the coefficients of *sinRt*

$$\frac{2R\,\gamma_9'}{F^{\frac{1}{2}}} + \frac{2Y_9\,\lambda\varphi'\sin\,(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.75a)$$

And for the coefficients of *cosRt*,

$$\frac{2R\,\gamma_{10}'}{F^{\frac{1}{2}}} + \frac{2\,Y_{10}\,\lambda\varphi'\sin\,(\varphi)}{2F^{\frac{3}{2}}} = 0 \quad (3.75b)$$

Solving (3.75a), we get

$$\frac{\gamma_9'}{\gamma_9} + \frac{\lambda\,\varphi'\sin\varphi(\varphi)}{4(1-\lambda\mathrm{Cos}(\varphi))} = 0 \quad (3.76a)$$

Integrating both sides of (3.76a), we have

$$\mathrm{In}\gamma_9 + \frac{1}{4}\mathrm{In}\big(1-\lambda\cos(\varphi)\big) = \mathrm{In}C_{13}\text{ ,i. e }\gamma_9(\tau) = C_{13}\big(1-\lambda\cos(\varphi)\big)^{-\frac{1}{4}} \quad (3.76b)$$

$\gamma_9(0) = C_{13}(1-\lambda)^{-\frac{1}{4}} = 0 \therefore C_{13} = 0 \ \therefore \gamma_9(\tau) = 0$

Also solving (3.75b), we get

$$\frac{\Gamma_{10}'}{\gamma_{10}} + \frac{\lambda\,\varphi'\mathrm{Sin}(\varphi)}{4\big(1-\lambda\mathrm{Coc}(\varphi)\big)} = 0 \quad (3.77a)$$

$\therefore\ \mathrm{In}\gamma_{10} + \frac{1}{4}\mathrm{In}\big(1-\lambda\cos(\varphi)\big) = \mathrm{InC}_{14}$

$\gamma_{10}(\tau) = C_{14}(1-\lambda\cos(\varphi))^{-\frac{1}{4}}$ (3.77b)

$\gamma_{10}(0) = C_{14}(1-\lambda)^{-\frac{1}{4}} = 0 \therefore C_{14} = 0 \therefore \gamma_{10}(\tau) = 0$

From (3.56), if we substitute $\gamma_9(\tau)$, we have

$G^{21}(t,\tau) = 0$ (3.78)

Solving the remaining terms in (3.74), we get

$$G^{22}(t,\tau) = \gamma_{11}\cos Rt + \gamma_{12}\sin Rt + r_{10}\cos(Q+R)t + r_{11}\cos(R-Q)t + r_{12}\sin(Q+R) + r_{13}\sin(Q-R)t + r_{14}\cos Qt + r_{15}\sin Qt \quad (3.79)$$

where,

$$r_{10}(\tau) = \frac{-R^2\alpha_3}{2F(R^2-(Q+R)^2)},\quad R \neq (Q+R) \quad (3.80a)$$

$$r_{11}(\tau) = \frac{-R^2\gamma_1\alpha_3}{2F(R^2-(R-Q)^2)},\quad R \neq (Q-R) \quad (3.80b)$$

$$r_{12}(\tau) = \frac{R^2\gamma_1\beta_3}{2F(R^2-(Q+R)^2)},\quad r_{13}(\tau) = \frac{R^2\gamma_1\beta_3}{2F(R^2-(R-Q)^2)} \quad (3.80c)$$

Also,

$r_{10}(0) = r_{11}(0) = r_{12}(0) = r_{14}(0) = r_{15}(0) = 0$

Applying the initial conditions in (3.79), we get

$G^{22}(0,0) = \gamma_{11} = 0 \therefore \gamma_{11} = 0$

The second condition, gives $\gamma_{12}(0) = 0$. From (3.16a), we obtain

$$\xi_1 = \bar{\xi}_2(L^{10} + \delta L^{11} + \delta^2 + L^{12} + \cdots) + \bar{\xi}_2^2(L^{20} + \delta L^{21} + \delta L^{22} + \cdots) + \cdots \quad (3.81)$$

From (3.16b), we get

$$\xi_2 = \bar{\xi}_2(G^{10} + \delta G^{11} + \delta^2 G^{12} + \cdots) + \bar{\xi}_2^2(G^{20} + \delta G^{21} + \delta^2 G^{22} + \cdots) + \cdots \quad (3.82)$$

At maximum displacement for ξ_1, we need

$$\frac{d\xi_1}{d\bar{t}} = 0 \quad (3.83)$$

This implies (from (3.12b)) and for ξ_1

$$\therefore \xi_{1,t} + \delta(1-\lambda\mathrm{Cos}(\varphi))^{-\frac{1}{2}}\xi_{,\tau} = 0 \quad (3.84)$$

Here, we have used the fact that, $h_i(\tau) \equiv 0$

Let the values of t and τ for which ξ_1 is a maximum be t_{1a} and τ_{1a} respectively. We assume the following a asymptotic expansions of t_{1a} and τ_{1a}

$$t_{1a} = t_0^{(1)} + \delta t_{01}^{(1)} + \delta^2 t_{02}^{(1)} + \cdots + \bar{\xi}_2\big(t_{10}^{(1)} + \delta t_{11}^{(1)} + \delta^2 t_{12}^{(1)} + \cdots\big) + \bar{\xi}_2^2\big(t_{20}^{(1)} + \delta t_{21}^{(1)} + \delta t_{22}^{(1)} + \cdots\big) + \cdots \quad (3.85)$$

Let the value of $\bar{t}$ at maximum of ξ_1 be $\bar{t}_{1a}$, where ,

$$\bar{t}_{1a} = \Big[\bar{t}_0^{(1)} + \delta\bar{t}_{01}^{(1)} + \delta^2\bar{t}_{02}^{(1)} + \cdots + \bar{\xi}_2\big(\bar{t}_{10}^{(1)} + \delta\bar{t}_{11}^{(1)} + \delta^2 t_{12}^{(1)} + \cdots\big) + \bar{\xi}_2^2\big(\bar{t}_{20}^{(1)} + \delta\bar{t}_{21}^{(1)} + \delta^2\bar{t}_{22}^{(1)} + \cdots\big)\Big] + \cdots \quad (3.86)$$

Therefore, we get from (3.11b) that,

$$\tau_{1a} = \delta\bar{t}_a^{(1)} = \delta\Big[\bar{t}_0^{(1)} + \delta\bar{t}_{01}^{(1)} + \delta^2\bar{t}_{02}^{(1)} + \cdots + \bar{\xi}_2\big(\bar{t}_{10}^{(1)} + \delta\bar{t}_{11}^{(1)} + \delta^2\bar{t}_{12}^{(1)} + \cdots\big) + \bar{\xi}_2^2\big(\bar{t}_{20}^{(1)} + \delta\bar{t}_{21}^{(1)} + \delta^2\bar{t}_{22}^{(1)} + \cdots\big) + \cdots\Big] \quad (3.87)$$

Expanding asymptotically each term of (3.82) and noting that $L^{10} = L^{11} = L^{12} = 0$ we obtain the following:

$$\bar{\xi}_2^2 L_{,t}^{20} = \bar{\xi}_2^2\Bigg[L_{,t}^{20} + \big\{\delta\bar{t}_{01}^{(1)} + \cdots\big\}L_{,tt}^{20} + \delta\big\{\bar{t}_0^{(1)} + \delta\bar{t}_{01}^{(1)} + \cdots\big\}L_{,tt}^{20} + \delta\big\{\bar{t}_0^{(1)} + \delta\bar{t}_{01}^{(1)} + \cdots\big\}L_{,t\tau}^{20} + \frac{1}{2}\Big\{\Big\{\big\{\delta\bar{t}_{01}^{(1)} + \delta^2 t_{02}^{(1)} + \cdots\big\}^2 L_{,ttt}^{20} + 2\delta\big\{\bar{t}_0^{(1)} + \delta\bar{t}_{01}^{(1)} + \cdots\big\}\big\{\{\delta t_{01}^{(1)} + \delta^2 t_{02}^{(1)} + \cdots\big\}L_{,tt\tau}^{20} + \delta^2\{\bar{t}_0^{(1)} + \cdots\}^2 L_{,t\tau\tau}^{20}\Big\}\Big\}\Bigg] = 0 \quad (3.88a)$$

$$\bar{\xi}_2^2\delta L_t^{21} = \bar{\xi}_2^2\boldsymbol{\delta}\Big[L_{,tt}^{21} + \big\{\delta t_0^{(1)} + \cdots\big\}L_{,tt}^{21} + \cdots\delta\big(\bar{t}_0^{(1)} + \cdots\big)L_{,\tau\tau}^{21}\Big] = 0,\ \bar{\xi}_2^2\delta^2 L_{,t}^{22} = \bar{\xi}_2^2\delta^2 L_{,t}^{22} = 0 \quad (3.88b)$$

Equating the orders of $(\delta^j\bar{\xi}^i{}_2)$, we get

$$O\left(\bar{\xi}_2^2\right):\quad L_{,t\tau}^{20}\left(t_0^{(1)},0\right) = 0 \quad (3.89a)$$

$$0\left(\bar{\xi}_2^2\delta\right): L_{,t}^{20} + \bar{t}_0^{(1)}L_{,t\tau}^{20} + L_{,t}^{21} + (1-\lambda)^{-\frac{1}{2}}L_{,\tau}^{20} = 0 \quad (3.89b)$$

$0\left(\bar{\xi}_2^2\delta^2\right): L_{,t}^{22} = 0$ (3.89c)

From (3.89a), we get $L_{,t}^{20}(t_{0,}^{(1)}0) = 0$.

$L_{,t}^{20}\left(t_0^{(1)}, 0\right) = -\alpha_4(0)Q\sin Qt_0^{(1)} - 2Rr_1(0)\sin 2Rt_0^{(1)} - Rr_2(0)\sin 2Rt_0^{(1)} = 0, \quad \alpha_4(0) = 0$

$\therefore -R[2Rr_1(0)\sin 2Rt_0^{(1)} + r_2(0)\sin Rt_0^{(1)}] = 0$, since $R \neq 0$ (3.90)

$\therefore\ 2r_1(0)\sin 2Rt_0^{(1)} + r_2(0)\sin Rt_0^{(1)} = 0$

Also $\sin t_0^{(1)} \cong t_0^{(1)}$

$$\therefore 2r_1(0)\left\{2Rt_0^{(1)} - \frac{\left(2Rt_0^{(1)}\right)^3}{3!} + \cdots\right\} + r_2(0)\left\{Rt_0^{(1)} - \frac{\left(Rt_0^{(1)}\right)^3}{3!} + \cdots\right\} = 0 \qquad (3.91)$$

$$r_1(0)\left\{4R - \frac{8R^3 t_0^{(1)^2}}{3} + \cdots\right\} + r_2(0)\left\{R - \frac{R^3 t_0^{(1)^2}}{6} + \cdots\right\} = 0$$

$$i.e, \qquad \{4Rr_1(0) + Rr_2(0)\} = \left\{\frac{8r_1(0)}{3} + \frac{r_3(0)R^3}{6}\right\} t_0^{(1)^2}$$

$$\therefore t_0^{(1)} = \pm\sqrt{\frac{4Rr_1(0) + Rr_2(0)}{\frac{8r_1(0)R^3}{3} + \frac{r_2(0)R^3}{6}}} \qquad (3.92)$$

We shall however take the positive value of $t_0^{(1)}$. From (3.89b), we have

$t_{01}^{(1)}L_{,tt}^{20} + \bar{t}_0 L_{,t\tau}^{20} + L_{,t}^{21} + (1-\lambda)^{-\frac{1}{2}} = 0$

$\therefore t_{01}^{(1)} = \frac{-1}{L_{,tt}^{20}}\{\bar{t}_0 L_{,t\tau}^{20} + L_{,t}^{21} + (1-\lambda)^{-\frac{1}{2}}L_{,\tau}^{20}\}|_{(t_0^{(1)},0)}$

We note that, $\xi_{1a} = \xi(t_{1a}^{(1)}, \tau_{1a})$, where, ξ_{1a} is the maximum of ξ_a. However, we equally note that

$\xi_1 = \xi_2(L^{10} + \delta L^{11} + \delta^2 L^{12} + \cdots) + \bar{\xi}_2^2(L^{20} + \delta L^{21} + \delta^2 L^{22} + \cdots) + \cdots$ (3.93a)

and that,

$L^{10} = L^{11} = L^{12} = 0$

From (3.93a), we therefore get

$\xi_1 = \bar{\xi}_2^2(L^{20} + \delta L^{21} + \delta^2 L^{22} + \cdots)$ (3.93b)

Expanding asymptotically each term of (3.93a), we obtain the following:

$$\bar{\xi}_2^2 L^{20} = \bar{\xi}_2^2\Bigg[L^{20} + \{\delta t_{01}^{(1)} + \delta^2 t_{02}^{(1)} + \cdots\}L_{,t}^{20} + \delta(\bar{t}_0^{(1)} + \bar{t}_{01}^{(1)} + \delta^2\bar{t}_{02}^{(1)} + \cdots)L_{,\tau}^{20}$$
$$+ \frac{1}{2}\left\{\begin{Bmatrix}\left\{\delta\bar{t}_{01}^{(1)} + \delta^2\bar{t}_{02}^{(1)} + \cdots\right\}^2 L_{,tt}^{20} + +2\delta\left\{\bar{t}_0^{(1)} + \bar{t}_{01}^{(1)} + \cdots\right\} \\ \left\{\delta\bar{t}_{01}^{(1)} + \cdots\right\}L_{,t\tau}^{20} + \delta^2\bar{\xi}\left(t_0^{(1)} + \cdots\right)^2 L_{,\tau\tau}^{20} + \cdots\end{Bmatrix}\right\}\Bigg] = 0$$

$\delta\bar{\xi}_2^2 L^{21} = \delta\bar{\xi}_2^2\left[L^{21} + \left\{\delta t_{01}^{(1)} + \cdots +\right\}L_{,t}^{21} + \delta(t_{01}^{(1)} + \cdots)L_{,t\tau}^{21}\right] = 0,\ \delta^2\bar{\xi}_2^2 L^{22} = 0.$

Then, we go on to obtain

$$\xi_{1a} = \bar{\xi}_2^2\Bigg[L^{20} + \delta\left\{t_{01}^{(1)}L_{,t}^{20^0} + \bar{t}_0^{(1)}L_{,\tau}^{20^0} + L^{21}\right\}$$
$$+ \delta^2\left\{\left\{\left\{t_{02}^{(1)}L_{,t}^{20^0} + \bar{t}_{01}^{(1)}L_{,\tau}^{20^0}\right\} + \frac{1}{2}\left\{t_{01}^{(1)^2}L_{,tt}^{20} + 2\bar{t}_0^{(1)}t_{01}^{(1)}L_{,t\tau}^{20^0} + \bar{t}_0^{(1)}L_{,\tau\tau}^{20}\right\} + t_{01}^{(1)}L_{,t}^{21^0} + \bar{t}_0^{(1)}L_{,\tau}^{21}\right.\right.$$
$$\left.\left. + L^{22}\right\}\right\}\Bigg]\Big|_{(t_{01}^{(1)})} + \cdots \qquad (3.94)$$

Equating non-zero terms of (3.94), evaluated at $(t_0^{(1)}, 0)$, we obtain the following:

$$L^{20}\left(t_0^{(1)}, 0\right) = \frac{\theta_1 B(0)}{1-\lambda} + \frac{\theta_2 B^2(0)}{1-\lambda} \qquad (3.95)$$

where,

$$\theta_1 = \frac{2k_2}{Q^2 - R^2}\left\{\cos Rt_{01}^{(1)} - \cos Qt_0^{(1)}\right\} \quad (3.96)$$

$$\theta_2 = -3k_2\left\{1 - \cos Qt_0^{(1)}\right\} - \frac{k_2Q^2}{2(Q^2 - 4R^2)}\left\{\cos 2Rt_0^{(1)} - \cos Qt_{01}^{(1)}\right\} \quad (3.97)$$

$$L_{,tt}^{20}\left(t_0^{(1)}, 0\right) = \theta_3 B(0) + \theta_4 B^2(0) \quad (3.98)$$

where

$$\theta_3 = \frac{3k_2Q^2}{1-\lambda}\, cost - \frac{k_2Q^2}{2(1-\lambda)(Q^2 - 4R^2)}\left\{Q^2 cosQt_0^{(1)} - 4R^2\, cos2Rt_0^{(1)}\right\} \quad (3.99)$$

$$\theta_4 = \frac{2k_2Q^2}{(1-\lambda)(Q^2 - R^2)}\left\{Q^2 cosQt_0^{(1)} - R^2 cosRt_0^{(1)}\right\} \quad (3.100)$$

$$L_{,\tau\tau}^{20}\left(t_0^{(1)}, 0\right) = \theta_5 B^2(0) + \theta_6 B^3(0) \quad (3.101)$$

where,

$$\theta_6 = -\left\{\frac{3k_2\varphi'^2_{(0)}}{4} + \frac{k_2Q^2\varphi'^2_{(0)}}{4(Q^2 - 4R^2)}\right\} cosQt_0^{(1)} \quad (3.102)$$

$$\theta_5 = \frac{k_2Q^2\varphi'^2_{(0)}}{2(Q^2 - R^2)}\, cosQt_0^{(1)} - \frac{3k_2Q^2\varphi'^2_{(0)}}{8(1-\lambda)(Q^2 - 4R^2)} cos2Rt_0^{(1)} - \frac{3k_2\varphi'^2_{(0)}}{2(1-\lambda)(Q^2 - R^2)} cosRt_0^{(1)}$$
$$- \frac{k_2\varphi'^2_{(0)}}{(1-\lambda)}\left\{\frac{4Q^2 - R(\lambda + 7)}{4R^2Q^2}\right\} \quad (3.103)$$

$$L_{,\tau}^{21}\left(t_0^{(1)}, 0\right) = \theta_7 B^2(0) + \theta_8 B^3(0) \quad (3.104)$$

where,

$$\theta_7 = -\left\{\frac{2k_2Q^2\varphi'^2_{(0)}R}{(1-\lambda)^{\frac{3}{2}}(Q^2 - R^2)}\right\} sinRt_0^{(1)} \quad (3.105)$$

$$\theta_8 = \frac{1}{Q^2 - 4R^2}\left\{\frac{-2k_2Q^2\varphi'^2_{(0)}R}{(1-\lambda)^{\frac{3}{2}}(Q^2 - 4R^2)} + \frac{k_2Q^2\varphi'^2_{(0)}}{4R(1-\lambda)^{\frac{3}{2}}}\right\}\sin 2Rt_0^{(1)}$$
$$- \frac{1}{Q^2 - R^2}\left\{\frac{k_2Q^2\varphi'^2_{(0)}}{4R(1-\lambda)^{\frac{3}{2}}}\right\} sinRt_0^{(1)} \quad (3.106)$$

$$L^{22}\left(t_0^{(1)}, 0\right) = \theta_9 B^2(0) \quad (3.107)$$

where,

$$\theta_9 = \frac{k_2\varphi'^2_{(0)}}{(1-\lambda)^2}\left\{\frac{1}{1-\lambda} - 2\right\} \quad (3.108)$$

Thus (3.94) becomes

$$\xi_{1a} = \xi_2^2\Bigg[\theta_1 B(0) + \theta_2 B^2(0)$$
$$+ \delta^2\left\{\frac{t_{01}^{(1)}}{2}\left(\theta_3 B(0) + \theta_4 B^2(0)\right) + \left(\theta_7 B^2(0)\right) + (\theta_5 B^2(0) + \theta_6 B^3(0))\, t_0^{-(1)^2}\right.$$
$$\left. + \theta_9 B^2(0)\right\}\Bigg] \quad (3.109)$$

We recall that,

$$\frac{d\tilde{t}}{d\bar{t}} = [1 - \lambda cos\{\varphi(\delta\bar{t})\}]^{\frac{1}{2}}, \quad \tilde{t} = \int [1 - \lambda \cos\{\varphi(\delta\bar{t})\}]^{\frac{1}{2}}\, d\bar{t} \quad (3.110)$$

Now,

$$\cos\{\varphi(\delta\bar{t})\} = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \cdots \quad (3.111)$$

$$\therefore \lambda cos\{\varphi(\delta\bar{t})\} = \lambda\left[1-\frac{\varphi^2}{2!}+\frac{\varphi^4}{4!}-\frac{\varphi^6}{6!}+\cdots\right] \quad (3.112)$$

$$[1-\lambda cos\{\varphi(\delta\bar{t})\}] = 1-\lambda\left[1-\frac{\varphi^2}{2}+\frac{\varphi^4}{24}-\frac{\varphi^6}{6!}+\cdots\right] \quad (3.113)$$

$$=\left[(1-\lambda)+\ \lambda\left\{\frac{\varphi^2}{2}-\frac{\varphi^4}{24}+\frac{\varphi^6}{6!}+\cdots\right\}\right] \quad (3.114)$$

$$=(1-\lambda)\left[1+\frac{\lambda}{(1-\lambda)}\left\{\frac{\varphi^2}{2}-\frac{\varphi^4}{24}+\frac{\varphi^6}{6!}+\cdots\right\}\right] \quad (3.115)$$

$$\therefore [1-\lambda cos\{\varphi(\delta\bar{t})\}]^{\frac{1}{2}} = (1-\lambda)^{\frac{1}{2}}\left[1+\frac{\lambda}{(1-\lambda)}\left\{\frac{\varphi^2}{2}-\frac{\varphi^4}{24}+\frac{\varphi^6}{6!}-\cdots\right\}\right]^{\frac{1}{2}} (3.116)$$

$$=(1-\lambda)^{\frac{1}{2}}\left[1+\frac{\lambda}{2(1-\lambda)}\left\{\frac{\varphi^2}{2!}-\frac{\varphi^4}{4!}+\frac{\varphi^6}{6!}-\cdots\right\}-\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\left\{\frac{\varphi^2}{2}-\frac{\varphi^4}{24}+\frac{\varphi^6}{6!}-\cdots\right\}^2+\cdots\right] \quad (3.117)$$

Now,

$$\varphi(\delta\bar{t}) = \varphi(0)^0+\ \delta\bar{t}\acute{\varphi}(0)+\ \frac{(\delta\bar{t})^2}{2!}\varphi''(0)+\frac{(\delta\bar{t})^3}{3!}\varphi'''(0)+\cdots \quad (3.118)$$

$$=\delta\bar{t}\acute{\varphi}(0)+\ \frac{(\delta\bar{t})^2}{2}\varphi''(0)+\ \frac{(\delta\bar{t})^3}{3!}\varphi'''(0)+\cdots \quad (3.119)$$

Thus, we have

$$\varphi^2=\left[[(\delta\bar{t})\ \acute{\varphi}(0)+\ \frac{(\delta\bar{t})^2\varphi''(0)}{2}+\ \frac{(\delta\bar{t})^3\varphi'''(0)}{3!}\ +\ ...\right]^2 \quad (3.120)$$

$$=\left[(\delta\bar{t})^2\acute{\varphi}^2(0)+(\delta\bar{t})^3\acute{\varphi}(0)\varphi''(0)+\frac{(\delta\bar{t})^4}{4}\varphi''^2(0)+\ \frac{(\delta\bar{t})^4}{3}\acute{\varphi}(0)\varphi'''(0)+\cdots\right] \quad (3.121)$$

$$\varphi^2\cong[(\delta\bar{t})^2\acute{\varphi}^2(0)+\ (\delta\bar{t})^3\acute{\varphi}(0)\varphi''(0)+(\delta\bar{t})^4\left\{\frac{\varphi''^2(0)}{4}+\frac{\acute{\varphi}(0)\varphi'''(0)}{3}\right\}+\cdots] \quad (3.122)$$

Similarly, we get

$$\varphi^4=(\delta\bar{t})^4\acute{\varphi}^4(0)+\ O(\delta^5) \quad (3.123)$$

While,

$$[1-\lambda cos\{\varphi(\delta\bar{t})\}]^{\frac{1}{2}}=(1-\lambda)^{\frac{1}{2}}\left[1+\frac{\lambda\varphi^2}{4(1-\lambda)}+\left\{\frac{\lambda}{48(1-\lambda)}-\frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2\right\}\varphi^4+\cdots\right] \quad (3.124)$$

$$=(1-\lambda)^{\frac{1}{2}}\left[1-\frac{\lambda}{4(1-\lambda)}\left\{(\delta\bar{t})^2\acute{\varphi}^2(0)+\ (\delta\bar{t})^3\acute{\varphi}(0)+(\delta\bar{t})^4\left(\frac{\varphi''^2(0)}{4}+\acute{\varphi}(0)\varphi'''(0)\right)\right\}\right.$$
$$\left.-\frac{1}{48}\left(\left(\frac{\lambda}{1-\lambda}\right)-\frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2\right)\{(\delta\bar{t})^4\acute{\varphi}^4(0)+\ O(\delta^5)\}\right] (3.125)$$

$$=(1-\lambda)^{\frac{1}{2}}\left[1+\frac{\lambda}{4(1-\lambda)}\left\{(\delta\bar{t})^2\acute{\varphi}^2(0)+(\delta\bar{t})^3\acute{\varphi}(0)\varphi''(0)+(\delta\bar{t})^4\left(\frac{\varphi''^2(0)}{4}+\ \frac{\acute{\varphi}(0)\varphi'''(0)}{3}\right)\right\}-\frac{1}{48}\left(\frac{\lambda}{1-\lambda}\right)\right.$$
$$\left.+\frac{1}{32}\left(\frac{\lambda}{1-\lambda})^2\right)(\delta\bar{t})^4\acute{\varphi}^4(0)+0(\delta^5)\right] \quad (3.126)$$

$$=(1-\lambda)^{\frac{1}{2}}\left[1+\frac{\lambda}{4(1-\lambda)}\{(\delta\bar{t})^2\acute{\varphi}^2(0)+(\delta\bar{t})^3\acute{\varphi}(0)\varphi''(0)\}\right.$$
$$-\left\{\frac{\lambda}{48(1-\lambda)}+\ \frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2-\frac{\lambda}{4(1-\lambda)}\left(\frac{\varphi''^2}{4}+\frac{\acute{\varphi}(0)\varphi''(0)}{3}\right)\right\}(\delta\bar{t})^4$$
$$\left.+\ \mathrm{O}(\delta^5)\right] \quad (3.127)$$

$$\therefore \tilde{t}=\int[1-\lambda\cos\{\varphi(\delta\bar{t})\}]^{\frac{1}{2}}d\bar{t}$$

$$\tilde{t}=(1-\lambda)^{\frac{1}{2}}\int\left[1-\frac{\lambda}{4(1-\lambda)}\{(\delta\bar{t})^2\acute{\varphi}^2(0)+\ (\delta\bar{t})^3\acute{\varphi}(0)\varphi''(0)\}\left\{\frac{\lambda}{48(1-\lambda)}-\frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2-\frac{\lambda}{4(1-\lambda)}\left(\frac{\varphi''^2(0)}{4}+\right.\right.\right.$$

φ0φ′′′03(δt)4dt (3.128)

$$\tilde{t} = (1-\lambda)^{\frac{1}{2}}\left[\bar{t} - \frac{\lambda}{4(1-\lambda)}\left\{\frac{\delta^2 t^3}{3}\acute{\varphi}^2(0) + \frac{\delta^2\bar{t}^4}{4}\acute{\varphi}(0)\varphi''(0)\right\} - \left\{\frac{\lambda}{48(1-\lambda)} - \frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2 - \frac{\lambda}{4(1-\lambda)}\left(\frac{\varphi''^{\,2}(0)}{4} + \frac{\acute{\varphi}(0)\varphi'''(0)}{3}\right)\frac{\delta^4\bar{t}^5}{5}\right\}\right] \quad (3.129)$$

But, $t = \tilde{t}$

$\therefore t_{1a} = \tilde{t}_{1a} =$

$$(1-\lambda)^{\frac{1}{2}}\left[\bar{t}_{1a} + \frac{\lambda}{4(1-\lambda)}\left\{\frac{\delta^2\bar{t}_{1a}^3\acute{\varphi}^2(0)}{3} + \frac{\delta^2\bar{t}_{1a}}{4}\acute{\varphi}(0)\varphi''(0)\right\} - \left\{\frac{\lambda}{48(1-\lambda)} + \frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2 - \frac{\lambda}{4(1-\lambda)}\left(\frac{\varphi''^2(0)}{4} + \frac{\acute{\varphi}(0)\varphi'''(0)}{3}\right)\frac{\delta^4\bar{t}_{1a}^5}{5}\right\}\right]$$

(3.130)

Expanding both sides of (3.130) asymptotically, we have

$$t_0^{(1)} + \delta t_{01}^{(1)} + \delta^2 t_{02}^{(1)} + \cdots + \bar{\xi}_2(t_{10}^{(1)} + \delta t_{11}^{(1)} + \delta^2 t_{12}^{(1)} + \ \ldots + \bar{\xi}_2(t_{20}^{(1)} + \delta t_{21}^{(1)} + \delta^2 t_{22}^{(1)} + \ \ldots$$

$$= (1-\lambda)^{\frac{1}{2}}\Bigg[\bar{t}_0^{(1)} + \delta\bar{t}_{01}^{(1)} + \delta^2 t_{02}^{(1)} + \cdots + \bar{\xi}_2\left(\bar{t}_{10}^{(1)} + \delta\bar{t}_{11}^{(1)} + \delta^2\bar{t}_{12}^{(1)} + \cdots\right) + \bar{\xi}_2^2\left(\bar{t}_{20}^{(1)} + \delta\bar{t}_{21}^{(1)} + \delta^2\bar{t}_{22}^{(1)} + \cdots\right)$$

$$- \frac{\lambda}{4(1-\lambda)}\left\{\frac{\delta^2\acute{\varphi}^2(0)}{3}\left(\bar{t}_0^{(1)} + \delta\bar{t}_{21}^{(1)} + \cdots + \bar{\xi}_2\bar{t}_{10}^{(1)} + \cdots\right)^3 + \frac{\delta^2\acute{\varphi}^2(0)\varphi''(0)}{4}\left(\bar{t}_0^{(1)} + \delta\bar{t}_{01}^{(1)} + \cdots\right)^4\right\}$$

$$+ \left\{\frac{\lambda}{48(1-\lambda)} - \frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2 - \frac{\lambda}{48(1-\lambda)}\left(\frac{\varphi''^2(0)}{4} + \frac{\acute{\varphi}(0)\varphi'''(0)}{3}\right)\delta^5\left(\bar{t}_0^{(1)} + \delta\bar{t}_{01}^{(1)} + \cdots\right)^5\right\}\Bigg]$$

On equating coefficients of $(\xi_2^i\delta^j)$, we get

$0(1)$: $t_0^{(1)} = (1-\lambda)^{\frac{1}{2}}t_0^{(1)}$ (3.131a)

$0(\delta)$: $t_{01}^{(1)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{01}^{(1)}$ (3.131b)

$0(\delta^2)$: $t_{02}^{(1)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{02}^{(1)}$ (3.131c)

$0(\bar{\xi}_2)$: $t_{10}^{(1)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{10}^{(1)}$ (3.131d)

$0(\bar{\xi}_2\delta)$: $t_{11}^{(1)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{11}^{(1)}$ (3.131e)

$0(\bar{\xi}_2\delta^2)$; $t_{11}^{(1)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{11}^{(1)}$ (3.131f)

From (3.131a), we have

$\bar{t}_0^{(1)} = (1-\lambda)^{\frac{-1}{2}}t_0^{(1)}$ (3.132a)

From (3.131b), we have

$\bar{t}_{01}^{(1)} = (1-\lambda)^{\frac{-1}{2}}t_{01}^{(1)}$ (3.132b)

From (3.131c), we have

$\bar{t}_{02}^{(1)} = (1-\lambda)^{\frac{-1}{2}}t_{02}^{(1)}$ (3.132c)

Maximization of ξ_2

Let t_{2a} and τ_{2a} be the values of t and τ respectively at the maximum of ξ_2. Expanding t_{2a} and τ_{2a} asympotically, we get

$$t_{2a} = t_0^{(2)} + \delta t_{01}^{(2)} + \delta^2 t_{02}^{(2)} + \cdots + \bar{\xi}_2\left(t_{10}^{(2)} + \delta t_{11}^{(2)} + \delta^2 t_{12}^{(2)} + \cdots\right) + \xi_2^{(2)}\left(t_{20}^{(2)} + \delta t_{21}^{(2)} + \delta^2 t_{22}^{(2)} + \cdots\right) \quad (3.133)$$

Again, we let $\bar{t}_a^2$ be the value of $\bar{t}$ at the maximum of ξ_2 and let it be expanded as,

$$\bar{t}_a^{(2)} = \bar{t}_0^{(2)} + \delta^2\bar{t}_{01}^{(2)} + \delta^2\bar{t}_{02}^{(2)} + \cdots + \bar{\xi}_2\left(\bar{t}_{10}^{(2)} + \delta\bar{t}_{11}^{(2)} + \cdots\right) + \bar{\xi}_2^2\left(\bar{t}_{20}^{(2)} + \delta\bar{t}_{21}^{(2)} + \delta^2\bar{t}_{22}^{(2)} + \cdots\right) \quad (3.134)$$

Therefore, we get

$$\tau_{2a} = \delta\bar{t}_a^{(2)} = \delta\{\bar{t}_0^{(2)} + \delta\bar{t}_{01}^{(2)} + \delta^2\bar{t}_{20}^{(2)} + \cdots + \bar{\xi}_2\left(\bar{t}_{10}^{(2)} + \delta\bar{t}_{11}^{(2)} + \delta^2 t_{12}^{(2)} + \cdots\right) + \bar{\xi}_2^2\left(t_{20}^{(2)} + \delta\bar{t}_{21}^{(2)} + \cdots\right) \quad (3.135a)$$

From (3.82), we have

$$\xi_2 = \bar{\xi}_2(G^{10} + \delta G^{11} + \delta^2 G^{11} + \delta^2 G^{12} + \cdots) + \bar{\xi}_2^2(G^{20} + \delta G^{21} + \delta^2 G^{22} + \cdots) + \cdots \quad (3.135b)$$

For maximum displacement of ξ_2, we get, as in (3.84)

$$\xi_{2,t} + \delta(1 - \lambda\cos(\varphi))^{-\frac{1}{2}}\xi_{,\tau} = 0 \quad (3.136)$$

Then, expanding each term of (3.136) asymptotically, we have

$\bar{\xi}_2 G,_t^{10} = \bar{\xi}_2[G,_t^{10} + \{\delta t_{01}^{(2)} + \delta^2 t_{02}^{(2)} + \cdots + \bar{\xi}_2\big(t_{10}^{(20)} + \delta t_{11}^{(2)} + \delta^2 t_{12}^{(2)} + \cdots\big) + \bar{\xi}_2^2(t_{20}^{(2)} + \delta t_{21}^{(2)} + \delta^2 t_{22}^{(2)} + \cdots)\}G,_{tt}^{10} + \delta\{(\bar{t}_0^{(2)} + \delta\bar{t}_{20}^{(2)} + \cdots) + \bar{\xi}_2\big(\bar{t}_{10}^2 + \delta\bar{t}_{10}^{(2)} + \delta\bar{t}_{11}^{(2)} + \cdots\big) + \bar{\xi}_2^2\big(\bar{t}_{20}^{(2)} + \delta\bar{t}_{21}^{(2)} + \cdots\big)\}G,_{t\tau}^{10} + \frac{1}{2}\{\{\{\delta t_{01}^{(2)} + \delta t_{01}^{(2)} + \delta^2 t_{02}^{(2)} + \cdots + \bar{\xi}_2(\bar{t}_{10}^{(2)} + \delta t_{11}^{(2)} + \cdots) + \bar{\xi}_2^2(t_{20}^{(2)} + \delta t_{21}^{(2)} + \cdots)\}^2 G,_{ttt}^{10} + 2\delta\,\{\bar{t}_0^{(2)} + \delta\bar{t}_{21}^{(2)} + \cdots + \bar{\xi}_2(\bar{t}_2^{(2)}\delta\bar{t}_{11}^{(2)} + \cdots) + \bar{\xi}_2^2\Big\{\bar{t}_{20}^{(2)} + \delta\bar{t}_{21}^{(2)} + \cdots\Big\}\{\delta t_{01}^{(2)} + \delta^2 t_{02}^2 + \cdots + \bar{\xi}_2\big(\bar{t}_{10}^{(2)} + \delta\bar{t}_{11}^{(2)} + \cdots\big) + \bar{\xi}_2^2\big(\bar{t}_{20}^{(2)} + \delta t_{21}^{(2)} + \cdots\big)\}G,_{tt\tau}^{10} + \delta^2\{\bar{t}_0^{(2)} + \cdots + \bar{\xi}_2\big(\bar{t}_{10}^{(2)} + \cdots\big) + \bar{\xi}_2^2(\bar{t}_{20}^{(2)} + \cdots)\}^2 G,_{t\tau\tau}^{10}\}\}] + \cdots = 0$

$\bar{\xi}_2\delta G,_t^{11} = \bar{\xi}_2\delta[G,_t^{11} + \{\delta t_{01}^{(2)} + \cdots + \bar{\xi}_2(t_{10}^{(2)} + \cdots) + \bar{\xi}_2^2(\bar{t}_{20}^{(2)} + \cdots)\}\, G,_{tt}^{11} + \delta(\bar{t}_0^{(2)} + \cdots + \bar{\xi}_2\bar{t}_{10}^{(2)} + \cdots + \bar{\xi}_2^2\bar{t}_{20}^{(2)} + \cdots\}\, G,_{tt\tau}^{11} + \frac{1}{2}\{\{\delta\bar{t}_{01}^{(2)} + \cdots + \bar{\xi}_2 t_{10}^{(2)} + \cdots + \bar{\xi}_2\big(t_{20}^{(2)} + \cdots\big)\}^2 G,_{ttt}^{11} + 2\delta\{\bar{t}_{10}^{(2)} + \cdots \bar{\xi}_2 t_{10}^{(2)} + \cdots + \bar{\xi}_2^2 t_{20}^{(2)} + \cdots\}\,\{\delta t_{01}^{(2)} + \cdots + \bar{\xi}_2 t_{10}^2 + \ldots + \bar{\xi}_2^2 t_{20}^{(2)} + \cdots\}G,_{tt\tau}^{11} + \cdots] + \cdots = 0$

$\bar{\xi}_2\delta^2 G,_t^{12} = \bar{\xi}_2\delta^2[G,_t^{12} + \Big\{\delta t_{01}^{(2)} + \cdots + \bar{\xi}_2 t_{10}^{(2)} + \cdots + \cdots\Big\}G,_{tt}^{12} + \cdots = 0;\;\; \bar{\xi}_2^2\delta^2 G,_t^{22} = 0$

$\delta\bar{\xi}_2(1-\lambda Cos(\varphi))^{-\frac{1}{2}}G,_\tau^{10} = \delta\bar{\xi}_2[(1-\lambda)^{-\frac{1}{2}}G,_\tau^{10} + (1-\lambda)^{-\frac{1}{2}}\{\delta t_{01}^{(2)} + \delta^2 t_{02}^{(2)} + \cdots + \bar{\xi}_2(t_{10}^{(2)} + \delta t_{11}^{(2)} + \cdots)\,\bar{\xi}_2^2\big(t_{20}^{(2)} + \delta t_{21}^{(2)} +$
…G,tτ10+δ{t02+…+ξ2(t102+ …)} { 1−λcos(φ)12 G,τ10},τ+12 {{ (1−λ)12 {δ2t02+…+
$\bar{\xi}_2(t_{10}^{(2)} + \delta t_{11}^{(2)} + \cdots\}^2 G,_{\tau tt}^{10} + 2\delta\{\{\{\,\bar{t}_0^{(2)} + \cdots + \bar{\xi}_2 t_{10}^{(2)} + \cdots\}\Big\{\bar{\xi}_2 t_{10}^{(2)} + \cdots\Big\}\Big(1-\lambda Cos(\varphi)\big)^{-\frac{1}{2}}G_{,\tau t}^{10}\Big), \tau\}\}] = 0$

$\delta^2\bar{\xi}_2(1-\lambda cos(\varphi))^{-\frac{1}{2}}G,_\tau^{11} = \bar{\xi}_2\delta^2\,[(1-\lambda)^{-\frac{1}{2}}G,_\tau^{11} + (1-\lambda)^{-\frac{1}{2}}\Big\{\bar{\xi}_2 t_{10}^{(2)} + \cdots\Big\}G,_{t\tau}^{11}\,] = 0$

Substituting all these equations in (3.136) and equating the coefficients of $(\bar{\xi}^i{}_2\delta^j)$, we obtain

$0(\bar{\xi}_2)$: $G,_t^{10} = 0$ (3.137a)

$0(\delta\bar{\xi}_2)$: $t_{01}^{(2)}G,_{tt}^{10} + \bar{t}_0^{(2)}G,_{t\tau}^{10} + G,_t^{11} + (1-\lambda)^{-\frac{1}{2}}G,_\tau^{10} = 0$ (3.137b)

$0(\delta^2\bar{\xi}_2)$: $t_{02}^{(2)}G,_{tt}^{10} + \bar{t}_{01}G,_{t\tau}^{10} + \frac{{t_{01}^{(2)}}^2}{2}G,_{ttt}^{10} + t_{01}^{(2)}\bar{t}_0^{(2)}G,_{tt\tau}^{10} + \bar{t}_0^{(2)}G,_{tt\tau}^{10} + t_{01}^{(2)}G,_{t\tau}^{11} + \bar{t}_0^{(2)}G,_{t\tau}^{10} + G,_t^{12} + t_{01}^{(1)}(1-\lambda)^{-1/2}G,_{tt} + \bar{t}_0^{(2)}\{1-\lambda cos(\varphi))^{-1/2}G,_\tau^{10}\},\tau + (1-\lambda)^{-1/2}G,_\tau^{11} = 0$ (3.137c)

$0(\bar{\xi}_2^2)$: $t_{10}^{(2)}G,_{tt}^{10} + G,_t^{20} = 0$ (3.137d)

$0(\delta\xi_2^2)$: $t^2 G,_{tt}^{10} + \bar{t}_{10}^{(2)}G,_{t\tau}^{10} + t_{01}^{(2)}t_{10}^{(2)}G,_{ttt}^{10} + \bar{t}_0^{(2)}t_{10}^{(2)}\,G,_{tt\tau}^{10} + t_{10}^{(2)}G,_{tt}^{11} + (1-\lambda)^{-\frac{1}{2}}t_{10}^{(2)}G,_{t\tau}^{10} = 0$(3.137e)

From (3.137a), we have

$G,_t^{10}\,\big(t_0^{(2)},0\big) = 0;\quad \therefore -R\gamma_1 sinRt_0^{(2)} = 0;\quad \therefore sinRt_0^{(1)} = 0; since\ \gamma_1(0) \neq 0$

i.e, $Rt_0^{(2)} = n\pi; n = 1,2,3, \ldots$

We take the first non-trivial value of n, i.e, n=1 , $\therefore t_0^{(2)} = \frac{\pi}{R}$

From (3.137b), we get

$t_{01}^{(2)} = \frac{-1}{G,_{tt}^{10}}\,[\bar{t}_0^{(2)}G,_{t\tau}^{10} + G,_t^{11} + (1-\lambda)^{-\frac{1}{2}}G,_\tau^{10}\,]$

since,

$G,_\tau^{11}\,\big(t_o^{(2)},0\big) = G,_t^{11}\,\big(t_0^{(2)},0\big) = G,_\tau^{10}\,\big(t_0^{(2)},0\big) = 0,\quad \therefore t_{01}^{(2)} = 0$

From (3.137d), we have

$t_{10}^{(2)} = 0;\ since\ G,_t^{20} = 0$ and $G,_t^{20} \neq 0$

Similarly,

$t_{02}^{(2)} = t_{10}^{(2)} = t_{11}^{(2)} = t_{20}^{(2)} = t_{21}^{(2)} = 0$

We note that,

$\xi_{2a} = \xi_2(t_{2a}, \tau_{2a})$ (3.138)

where

$\xi_2 = \bar{\xi}_2(G^{10} + \delta G^{11} + \delta^2 G^{11} + \cdots) + \bar{\xi}_2^2(G^{20} + \delta G^{21} + \delta G^{22} + \cdots) + \cdots$ (3.139)

We now expand each term of (3.139), and get

$$\bar{\xi}_2 G^{10}(t_{2a},\tau_{2a}) = \bar{\xi}_2[G_{10}(t_0^{(2)},0) + \{\delta G,_t^{10} + \delta^2 t_{02}^{(2)} + \cdots + \bar{\xi}_2(t_{10}^{(2)} + \delta t_{11}^{(2)} + \delta^2 t_{12}^{(2)} + \cdots)\}G,_t^{10} + \delta\{\bar{t}_0^{(2)} + \delta\bar{t}_{01}^{(2)} + \cdots + \bar{\xi}_2\,(\bar{t}_{10}^{(2)} + \delta\bar{t}_{11}^{(2)} + \delta^2\bar{t}_{12}^{(2)} + \cdots)\}\,G,_\tau^{10} + \frac{1}{2}\{\{\{\,\delta t_{01}^{(2)} + \delta^2 t_{02}^{(1)} + \cdots + \bar{\xi}_2(t_{10}^{(2)} + \delta t_{11}^{(2)} + \cdots\}^2 G,_{tt}^{10} + 2\delta\{(\bar{t}_0^{(2)} + \delta\bar{t}_{01}^{(2)} + \cdots + \bar{\xi}_2(\bar{t}_{10} + \delta\bar{t}_{02}^{(2)} + \cdots) + \bar{\xi}_2^2(\bar{t}_{20}^{(2)} + \delta\bar{t}_{21}^{(2)} + \cdots)\}\,\{\delta t_{01}^{(2)} + \delta^2 t_{02}^{(2)} + \cdots\bar{\xi}_2(t_{10}^{(2)} + \delta t_{11}^{(2)} + \cdots)\}\,G,_{t\tau}^{10} + \delta^2\{\bar{t}_0^{(2)} + \cdots + \bar{\xi}_2 t_{10}^2 + \cdots\}^2 G,_{\tau\tau}^{10}\}\}]$$

$$\bar{\xi}_2\delta G^{11} = \bar{\xi}_2\delta[G^{11} + \{\delta t_{01}^{(2)} + \cdots + \bar{\xi}_2(t_{10}^{(2)} + + \delta t_{11}^{(2)} + \cdots)\}\,G,_t^{11} + \delta\{\bar{t}_0^{(2)} + \bar{\xi}_2\bar{t}_{10}^{(2)} + \cdots\}G,_\tau^{11} + \frac{1}{2}\,\{\{\delta t_{01}^{(2)} + \cdots + \bar{\xi}_2\,(t_{10}^{(2)} + \delta t_{11}^{(2)} + \cdots\}^2 G,_{tt}^{11} + 2\delta\left\{t_{10}^{(2)} + \cdots + \bar{\xi}_2\bar{t}_{10}^{(2)} + \cdots\bar{\xi}_2^2\bar{t}_{20}^{(2)} + \cdots\right\}\{\delta t_{01}^{12} + \cdots + \bar{\xi}_2(t_{10}^{(2)} + \cdots)\}G,_{t\tau}^{11}\,]$$

$$\bar{\xi}_2\delta^2 G^{12} = \bar{\xi}_2\delta^2[G^{12} + \left\{\ldots\bar{\xi}_2(t_{10}^{(2)} + \cdots)\right\}G,_t^{12}\,];\;\; \delta^2\bar{\xi}_2^2 G^{22} = \delta^2\xi_2^2 G^{22}$$

Grouping further, we obtain

$$\xi_{2a} = \bar{\xi}_2[G^{10} + \delta\{t_0^{(2)}G,_t^{10^0} + \bar{t}_0^{(2)}G,_\tau^{10^0} + G^{11^0}\} + \delta^2\{t_{02}^{(2)}G,_t^{10^0} + \bar{t}_{01}^{(2)}G,_\tau^{10^0} + \frac{1}{2}t_{01}^{(2)^2}G,_{tt}^{10} + \bar{t}_0^{(2)}t_{01}^{(2)^0}G,_t^{10} + \bar{t}_0^{(2)^2}G,_{\tau\tau}^{10} + t_{01}^{(2)^0}G,_t^{11} + \bar{t}_0^{(2)}G,_\tau^{11^0} + G^{12} + \cdots] + \bar{\xi}_2^2[t_{10}^{(2)}G,_t^{10} + \delta\{t_{11}^{(2)}G,_t^{10^0} + \bar{t}_{10}^{(2)}G,_\tau^{10} + \bar{t}_{01}^{(2)}t_{10}^{(2)^0}G,_{tt}^{10} + \bar{t}_0^{(2)}t_{10}^{(2)^0}G,_{tt}^{10} + \bar{t}_0^{(2)}t_{10}^{(2)^0}G,_{t\tau}^{10} + t_{10}^{(2)^0}G,_t^{11}\} + \delta\{\{\{t_{12}^{(2)^0}G,_t^{10} + \bar{t}_{11}^{(2)}G,_\tau^{10^0} + \frac{1}{2}\{(2t_{02}^{(2)^0}t_{10}^{(2)} + + t_{01}^{(2)^0}t_{11}^{(2)})G,_{tt}^{10} + 2(\bar{t}^{(2)}t_{11}^{(2)} + \bar{t}_{01}^{(2)}t_{10}^{(2)} + \bar{t}_{10}^{(2)}t_{01}^{(2)})G,_{t\tau}^{10^0} + \bar{t}_{10}^{(2)}G,_{\tau\tau}^{10}\} + t_{11}^{(2)^0}G,_t^{11} + \bar{t}_{10}^{(2)}G,_\tau^{11^0} + \frac{1}{2}\{\{t_{01}^{(2)}t_{10}^{(2)^0}G,_{tt}^{11} + 2\bar{t}_{10}^{(2)}t_{10}^{(2)^0}G,_{t\tau}^{11} + t_{10}^{(2)^0}G,_t^{12} + G^{22^0}\}\}\} + \cdots \quad (3.140)$$

i. e, $$\xi_{2a} = \bar{\xi}_2\left[G^{10} + \delta^2\left\{\frac{1}{2}t_{01}^{(2)^2}G,_{tt}^{10} + t_0^{(2)^2}G,_{\tau\tau}^{10} + G^{12}\right\}\right] + \delta^2\bar{\xi}_2^2\{\bar{t}_{10}^{(2)}G,_{\tau\tau}^{10}\} \quad (3.141)$$

where (3.141) is evaluated at $(t_0^{(2)},0)$, that is, $t_0^{(2)} = \frac{\pi}{R},\ \tau = 0$,

$$\therefore\ \xi_{2a} = \bar{\xi}_2\left[2B(0) + \delta^2\left\{\bar{t}_0^{(2)^2}\left(B(0) - \frac{\acute{\varphi}^2(0)B^2(0)}{4}\right) - \frac{2\acute{\varphi}^2 B(0)}{R^2(1-\lambda)^2}\right\}\right] + \delta^2\bar{\xi}_2^2\left\{B(0) - \frac{\acute{\varphi}^2(0)B^2(0)}{4}\right\}\bar{t}_{10}^{(2)} \quad (3.142)$$

To obtain $\bar{t}_0^{(2)}$ that occurs in (3.142), we recall that,

$$\frac{d\tilde{t}}{d\bar{t}} = [1 - \lambda cos\,\{\varphi(\delta\bar{t})\}]^{\frac{1}{2}},\quad \therefore \tilde{t} = \int[1 - \lambda cos\{\varphi(\delta\bar{t})\}]^{\frac{1}{2}}\,d\bar{t} \quad (3.143)$$

Expanding $[1 - \lambda cos\{\varphi(\delta\bar{t})\}]^{\frac{1}{2}}$ and substitutingin (3.143), we have,

$$\tilde{t} = (1-\lambda)^{\frac{1}{2}}\int[1 + \frac{\lambda}{4(1-\lambda)}\{(\delta\bar{t})^2\acute{\varphi}^2(0) + \delta\bar{t})^3\acute{\varphi}(0)\varphi''(0)\} - \left\{\frac{\lambda}{48(1-\lambda)} + \frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2 - \frac{\lambda}{4(1-\lambda)}\left(\frac{\varphi''^2(0)}{4} + \frac{\acute{\varphi}(0)\varphi'''(0)}{3}\right)\right\}(\delta\bar{t}^4)]d\bar{t} \quad (3.144)$$

$$\tilde{t}_{2a} = (1-\lambda)^{\frac{1}{2}}\left[\bar{t}_{2a} - \frac{\lambda}{4(1-\lambda)}\left\{\frac{\delta^2\bar{t}_{2a}^3}{3}\acute{\varphi}^2(0) + \frac{\delta^3\bar{t}_{2a}^4}{4}\acute{\varphi}(0)\varphi''(0)\right\} + \left\{\frac{\lambda}{48(1-\lambda)} - \frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2 - \frac{\lambda}{4(1-\lambda)}\left(\frac{\varphi''^2(0)}{4} + \frac{\acute{\varphi}(0)\varphi'''(0)}{3}\right)\frac{\delta^4\bar{t}_{2a}^5}{5}\right\}\right] \quad (3.145)$$

But $t_{2a} = \tilde{t}_{2a}$, since $t = \tilde{t}$. Therefore, on expanding both sides of (3.145) asympotically, we get

$$t_0^{(2)} + \delta t_{01}^{(2)} + \delta^2 t_{02}^{(2)} + \cdots + \bar{\xi}_2(t_{10}^{(2)} + \delta t_{11}^{(2)} + \delta^2 t_{12}^{(2)} + \cdots) + \bar{\xi}_2^2(t_{20}^{(2)} + \delta t_{21}^{(2)} + \delta^2 t_{22}^{(2)} + \cdots)$$
$$= (1-\lambda)^{\frac{1}{2}}\Bigg[\bar{t}_0^{(2)} + \delta\bar{t}_{01}^{(2)} + \delta^2\bar{t}_{02}^{(2)} + \cdots + \bar{\xi}_2(\bar{t}_{10}^{(2)} + \delta\bar{t}_{11}^{(2)} + \delta^2\bar{t}_{12}^{(2)} + \cdots) + \bar{\xi}_2^2(\bar{t}_{20}^{(2)} + \delta\bar{t}_{21}^{(2)} + \delta^2\bar{t}_{22}^{(2)} + \cdots) - \frac{\lambda}{4(1-\lambda)}\left\{\frac{\delta^2\acute{\varphi}^2(0)}{3}(\bar{t}_0^{(2)} + \delta\bar{t}_{01}^{(2)} + \cdots) + \bar{\xi}_2(t_{10}^{(2)} + \cdots)^3 + \frac{\delta^2}{4}\acute{\varphi}(0)\varphi''(0)(\bar{t}_0^{(2)} + \delta\bar{t}_{01}^{(2)} + \cdots)^4\right\} + \left\{\frac{\lambda}{48(1-\lambda)} + \frac{1}{32}\left(\frac{\lambda}{1-\lambda}\right)^2 - \frac{\lambda}{4(1-\lambda)}\left\{\frac{\varphi''(0)}{4} + \acute{\varphi}(0)\varphi''(0)\frac{\delta^5}{5}\right\}(\bar{t}_0^{(2)} + \delta\bar{t}_{01}^{(2)} + \cdots)^5\right\}\Bigg]$$

On equating coefficients of $\xi_2^i\delta^j$, we get

$O(1): t_0^{(2)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_0^{(2)}$ (3.146a)

$O(\delta): t_{01}^{(2)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{01}^{(2)}$ (3.146b)

$O(\delta^2): t_{02}^{(2)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{02}^{(2)}$ (3.146c)

$O(\bar{\xi}_2): t_{10}^{(2)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{10}^{(2)}$ (3.146d)

$O(\bar{\xi}_2\delta): t_{11}^{(2)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{11}^{(2)}$ (3.146e)

$O(\bar{\xi}_2\delta^2): t_{12}^{(2)} = (1-\lambda)^{\frac{1}{2}}\bar{t}_{12}^{(2)}$ (3.146f)

From (3.146a), we have

$\bar{t}_0^{(2)} = (1-\lambda)^{\frac{-1}{2}}t_0^{(2)} = (1-\lambda)^{\frac{-1}{2}}\frac{\pi}{R}$

From (3.146b),we have

$\bar{t}_{10}^{(2)} = 0, since\ t_{10}^{(2)} = 0$

Thus, from (3.142), we have

$\xi_{2a} = \bar{\xi}_2[2B(0) + \delta^2\{(\frac{\pi}{R})^2\left(1-\lambda\right)^{-1}\left\{B(0) - \frac{\phi^2(0)B^2(0)}{4}\right\} - \frac{2\phi^2(0)B(0)}{R^2(1-\lambda)^2}\right]$ (3.147)

Net displacement of the spherical shell

Let the net displacement be ζ, where

$\zeta = \xi_{1a} + \xi_{2a}$ (3.148)

$\zeta = \bar{\xi}_2\left[2B(0) + \delta^2\left\{\left(\frac{\pi}{R}\right)^2(1-\lambda)^{-1}(B(0) - \frac{\phi^2(0)B^2(0)}{4} - \frac{2\phi^2(0)B}{R^2(1-\lambda)^2}\right\}\right] + \bar{\xi}_2^2\left[\frac{B^2(0)\theta_2}{1-\lambda} + \frac{\theta_1 B(0)}{1-\lambda} + \delta^2\left\{\frac{t_{01}^{(1)^2}}{2}\left(\theta_3 B(0) + \right.\right.\right.$

θ4B20+ t012θ5B20+θ6B30+ θ9B20+… (3.149)

We can further write ζ as,

$\zeta = \mathrm{e}_1\bar{\xi}_2 + \mathrm{e}_2\bar{\xi}_2^2 + \cdots$ (3.150)

where ,

$\mathrm{e}_1 = 2B(0)\left[1 + \frac{\delta^2}{2B}\left\{\left(\frac{\pi}{R}\right)^2(1-\lambda)^{-1}(B(0) - \frac{\phi^2(0)B^2(0)}{4} - \frac{2\phi^2(0)B(0)}{R^2(1-\lambda)^2}\right\}\right] + \cdots$ (3.151a)

$\mathrm{e}_2 = \frac{B^2\theta_2}{1-\lambda}\left[1 + \frac{\theta_1}{\theta_2 B(0)} + \frac{(1-\lambda)\delta^2}{B^2(0)\theta_2}\left\{\frac{t_{01}^{(1)^2}}{2}(\theta_3 B(0) + \bar{t}_0^{(1)^2}\{\theta_2 B^2(0) + \theta_6 B^3(0) + \theta_9 B^2(0)\}\right\}\right]$

$+ \cdots$ (3.151b)

Reversing the series (3.150), we have

$\bar{\xi}_2 = f_1\zeta + f_2\zeta^2 + \cdots$ (3.152)

Substituting for ζ, we have

$\bar{\xi}_2 = f_1(\bar{\xi}_2\mathrm{e}_1 + \bar{\xi}_2^2\mathrm{e}_2 + \cdots) + f_2(\bar{\xi}_2\mathrm{e}_1 + \bar{\xi}_2^2\mathrm{e}_2)^2 + \cdots$ (3.153)

Equating the coefficients of powers of $\bar{\xi}_2$, we get

$O(\bar{\xi}_2):\quad 1 = f_1\mathrm{e}_1$ (3.154a)

$O(\bar{\xi}_2^2): 0 = f_1\mathrm{e}_1 + f_2\mathrm{e}_1^2$ (3.154b)

From (3.154a), we get

$f_1 = \frac{1}{\mathrm{e}_1}$ (3.155a)

From (3.154b), we get

$f_2 = \frac{-f_1\mathrm{e}_2}{\mathrm{e}_1^2} = \frac{-\mathrm{e}_2}{\mathrm{e}_1^3}$ (3.155b)

We now evaluate (3.150) at dynamic buckling.Let ζ_c be the value of ζ at dynamic buckling. To determine the dynamic buckling load we use the maximization,

$$\frac{d\lambda}{d\zeta_c} = 0 \qquad (3.156a)$$

Thus, we get by differentiating both sides of (3.152) that

$$\frac{d(\xi_2)}{d\zeta_c} = \frac{d}{d\zeta_c}\{f_1\zeta + f_2\zeta^2 + \cdots\} \qquad (3.156b)$$

Following the procedure in [6],for the case $\zeta = \zeta_c$, we obtain

$$0 = \frac{d(f_1)}{d\lambda}\frac{d\lambda}{d\zeta_c}\zeta_c + f_1 + \frac{d(f_2)}{d\lambda}\frac{d\lambda}{d\zeta_c}\zeta_c + 2f_2\zeta_c \qquad (3.156c)$$

$$\therefore 0 = f_1 + 2f_2\zeta_c \ , \quad \therefore \quad \zeta_c = \frac{-f_1}{2f_2} = \frac{e_1^2}{2e_2} \qquad (3.157)$$

If we evaluate (3.152) at bluckling, we get

$$\bar{\xi}_2 = f_1\zeta_c + f_2\zeta_c^2 + \cdots \qquad (3.158)$$

Subtituting forf_1, ζ_c *and* f_2, we get

$$\bar{\xi}_2 = \frac{\zeta_c}{2e_1} = \frac{e_1}{4e_2} \qquad (3.159)$$

$$i.e, \ \ \bar{\xi}_2 = \frac{e_1}{4e_2} = \frac{2B\left\{1 + \frac{\delta^2 f_{12}}{2B(0)}\right\}}{\frac{4B^2(0)\theta_2}{1-\lambda}\left\{1 + \frac{\theta_1}{\theta_2 B(0)} + \frac{(1-\lambda)\delta^2 f_{22}}{\theta_2 B^2\theta_2}\right\}} \qquad (3.160)$$

Further simplication of (3.160) gives

$$(1-\lambda_D)^2 = 2\lambda_D\theta_2\bar{\xi}_2\left\{\frac{(1 + \frac{\theta_1}{\theta_2 B(0)} + \frac{(1-\lambda_D)\delta^2 f_{22}}{B^2(0)\theta_2}}{1 + \frac{\delta^2 f_{12}}{2B(0)}}\right\} \qquad (3.161)$$

where,

$$f_{12} = (\frac{\pi}{R})^2(1-\lambda)^{-1}(B(0) - \frac{\acute{\varphi}^2(0)B^2(0)}{4} - \frac{2\acute{\varphi}^2(0)B(0)}{R^2(1-\lambda)^2} \qquad (3.162)$$

$$f_{22} = \frac{{t_{01}^{(1)}}^2}{2}\{\theta_3 B(0) + \theta_4 B^2(0)\} + {\bar{t}_0^{(1)}}^2\{\theta_5 B^2(0) + \theta_6 B^3(0) + \theta_9 B^2(0)\} \qquad (3.163)$$

λ_D is the dynamic buckling load and it depends on δ^2 and not on δ.

4.0 RESULTS AND CONCLUSION

A simple computer program written with Qbasic, gives the values of λ_D at different values of $\bar{\xi}_2$ and δ. And $\varphi(\tau) = 1 - e^{-\alpha^2\tau}$, $\omega'(0) = \alpha^2, \alpha = 0.01, \delta = 0.01$, 0.02 and 0.03, we have the values in Table 1.

Table 1: The relationship between the dynamic buckling load, λ_D and the Imperfection parameter, $\bar{\xi}_2$for some fixed values of damping factors, δusing equation (3.161).

$\bar{\xi}_2$	$\lambda_D\ for\ \delta = 0.01$	$\lambda_D\ for\ \delta = 0.02$	$\lambda_D\ for\ \delta = 0.03$
0.01	0.900000	0.900000	0.900000
0.02	0.875839	0.875839	0.875839
0.03	0.863434	0.863434	0.863434
0.04	0.857440	0.857440	0.857440
0.05	0.846083	0.846083	0.846083
0.06	0.833333	0.833333	0.833333
0.07	0.833333	0.833333	0.833333
0.08	0.833333	0.833333	0.833333
0.09	0.827411	0.827411	0.827411
0.10	0.819002	0.819002	0.819002

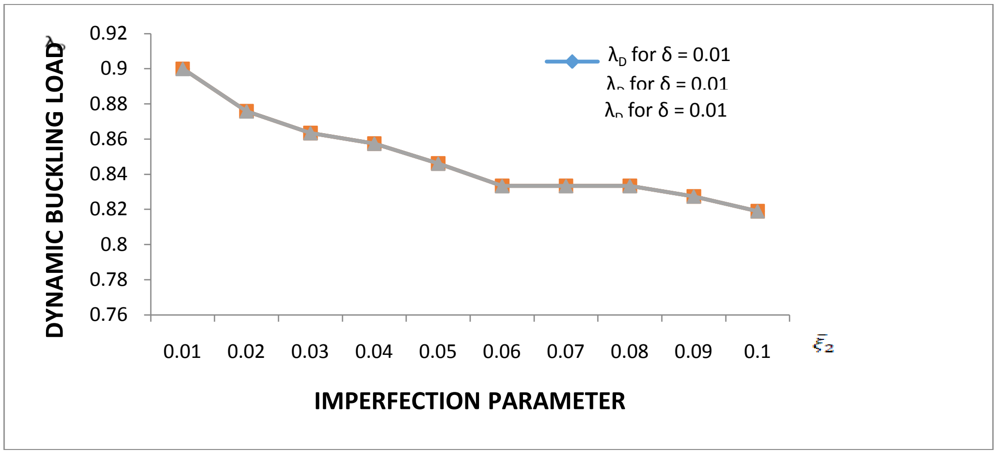

Figure 1: Graphical Plot showing the relationship between the dynamic buckling load, λ_D and the Imperfection parameter, $\bar{\xi}_2$for some fixed values of damping factors, δ using equation (3.161).

Analyses of the results of the elastic spherical shell are hereby presented. In general, the dynamic buckling load decreases with increased imperfection amplitude and vice-versa. This is equivalent to saying that, the nearer the structure is to a perfect nature, the more stable it is for a periodic load with slowly varying circular frequency. Besides, the least dependence of the dynamic buckling load on the circular frequency is of order two with respect to the slow time parameter$\boldsymbol{\delta}$, that is, λ_D is of order δ^2. Thus, if δ is very small relative to unity (as we have assumed in this work), then, δ^2 is even smaller so that in the limit as δ tends to zero, the results of the periodic load having a slowly varying circular frequency tend to those of a step load. Besides, we clearly observe that, within the limit of accuracy retained in this work, there is no marked difference in the values of λ_D for the different cases of $\delta = 0.01,\ 0.02\ and\ 0.03$. This IS because since λ_D is of the order δ^2, and δ happens to be small relative to unity, then $\delta^2$is even smaller so that its effect is very minimal and hence negligible. This is clearly observed, even in the table and graph of the results of the spherical shell structure with various values of $\bar{\xi}_2$ for some fixed values of δ. In such table and graph, we clearly observe that the difference in the value ofλ_D for different values of$\bar{\xi}_2$ is minimal and as δ becomes increasingly small, such results tend to those of a step load. Thus, we can justifiably say that the step load is a limiting process of the periodic load with slowly varying circular frequencies. Besides, it is observed that the dynamic buckling load λ_D depends on the first derivative of the slowly varying function $\varphi(\tau)$ (evaluated at the initial time $t = 0$).We thus demand that $\varphi(\tau)$ be continuous and differentiable and also have a right hand derivatives of all orders at $t = 0$. In this study, the results are given in a form that shows the dependence of the load degradation (i.e. the difference $(1 - \lambda_D)$) on the imperfection parameter. In the case of the model structure in [6], the load degradation is of order$\epsilon^{\frac{2}{3}}$while for the spherical shell, it is of order $\bar{\xi}_2^{\frac{1}{2}}$. Hence, the smaller the imperfection, the greater the value of λ_D and so, the smaller the load degradation.All results so far obtained are strictly asymptotic and so are valid as the small parameters δ and $\bar{\xi}_2$ become increasingly small relative to unity. We note, specifically that in this analysis, we have assumed that the two small parameters $\bar{\xi}_2$ and δ be independent and that such an assumption prompted the adoption of a two-timing regular perturbation procedure. The extreme nonlinearity of the this problem and the occurrence of these two small parameters $\bar{\xi}_2$ translate to our adopting asymptotic series expansions whereby higher order terms in the asymptotic expansions are neglected relative to lower order ones. The asymptotic results so far obtained in this problem turn out to be in line with expectations.

Though we have limited our analysis to a spherical shell with quadratic nonlinearity, we can, in principle, extend this analysis to any other elastic structure while taking care of whatever nonlinearities that are inherent in such problems. We expect this to be another phase of development in subsequent investigations.

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