Journal of the Nigerian Association of Mathematical Physics
Volume 52 (*July & Sept., 2019 Issue*), pp67 – 72
© *J. of NAMP*

DERIVATION OF OPTIMUM REPLENISHMENT POLICIES FOR INVENTORY MODELS

[1]S.A. Ogumeyo and [2]C.C. Nwamara

[1]Department of Mathematics, Delta State College of Education, Mosogar. Delta State, Nigeria.
[2]Department of General Studies, Petroleum Training Institute, Effurun. Delta State, Nigeria.

Abstract

The management and control of inventory is a problem common to all business organizations. It involves striking a balance between two extremes: holding of either too much or too little inventory. While too little inventory leads to decrease in profit, too much inventory tie down capital and increase overall cost. Hence many organizations are forced to search for proper inventory control techniques that will minimize ordering and holding costs. In this paper, we present inventory models which maintain appropriate inventory levels that enhance profitability and reduce inventory cost, including the derivation of an Optimum Replenishment Policy for inventory models.

Keywords: **Inventory, holding cost, order quantity, optimal policy, replenishment**

1.1 Introduction

Inventory as stated in [1], is an idle stock of items for future use. The two key issues in inventory models as remarked by Hillier and Lieberman [2] are the quantity and the time of the orders, with the objective of minimizing the total inventory cost consisting of carrying (holding) cost and ordering cost. While very small orders would result in frequent reordering and thereby incurring a considerable expense associated with processing and receiving the order (reorder cost), keeping large stock of idle goods tie up capital (holding cost) as stated in [3]. Optimal inventory policy is one that strikes the proper balance among the reorder, holding, and penalty costs [4]. Three main types of inventories as discussed in [5] are raw materials, work-in-progress and finished goods. A study focused on inventory management which involves price variable in inventory management is discussed in Lal [6].

Aniche and Agu [7] elaborate the two types of inventory calculations that determine the inventory level required for profitability. The two calculations are "cost to order" and "cost to keep". An inventory model for calculating the optimal order quantity that used the Economic Order Quantity method is remarked in Panigrahi [8] while Gaur et al [9] in their study examined firm-level inventory behaviour among retailing companies and observed that inventory turnover for retailing firms was positively related to capital intensity. Singh [10] opined that an increase in components of inventory lead to an increase in the proportion of inventory in current assets.

Capkun et al [11] statistically analyzed the relationship between inventory performance and financial performance in manufacturing companies and inferred that a significant relationship existed between inventory performance along with the performance of its components and profitability.

An inventory model which studies the link- age between the performance of the components of inventory such as raw material, work in progress and finished goods and financial performance is discussed in [12]. Eneje et al [13] investigated the effects of raw materials inventory management on the profitability of brewery firms in Nigeria and concluded that efficient management of raw material inventory is a major factor to be contented with by Nigerian brewers in enhancing or boosting their profitability. Regression analysis was adopted in [14] to determine the impact of inventory conversion period over gross operating profit.

Sitienei and Memba [15] remarked that firm's inventory systems must maintain an appropriate inventory level to enhance profitability and reduce the inventory costs associated with holding excessive stock in warehouses.

In this article, we present an inventory model which maintains an appropriate inventory level that will enhance profitability by reducing inventory cost: Optimum Replenishment Model.

2.1 Definitions of Mathematical Terms and Symbols

(i) **g(*y*/*i*)** ≡ expected cost when y is the inventory available to meet demand after ordering, given that initial inventory is i.

Corresponding Author: Ogumeyo S.A., Email: Simonogumeyo64@gmail.com, Tel: +2348052762209

(ii) **Holding cost** (sometimes called the storage cost) represents all the costs associated with the storage of the inventory unit sold or used.

(iii). **Shortage cost** (sometimes called the unsatisfied demand cost) is incurred when the amount of the commodity required (demanded) exceeds the available stock.

(iv) The **salvage value** of an item is the value of a leftover item when no further inventory is desired. The salvage value represents the disposal value of the item through discount.

(v) **Salvage cost** is the amount of losses due to discount.

(vi) **Discount rate** takes into account the time -value of money.

(vii) **Lead time** is the amount of time between the placement of an order to replenishment inventory.

To begin the discussion, we define the symbols

i = level of inventory *prior* to the ordering decision

x = amount ordered ($x \geq 0$)

$y = i + x$ = total inventory available to meet future demand

q = actual demand quantity (a random variable, where $q \geq 0$)

$p(q)$ = probability that demand equals q.

h = net holding cost

L(y) = values of holding cost h

π = penalty cost

For expository convenience, we assume $i, x, y,$ and q are integer-valued.

Typically, optimal values for x. and y will depend on the level of starting inventory i, and so the symbols $x(i)$ and $y(i)$ will be used to denote an optimal policy.

3.1 Model Formulation

Commonly, managers use the following simple type of replenishment rule:

$x(i) = 0$ and $y(i) = i$ for $i \geq S$ (do not order)

$x(i) = S - i$ and $y(i) = S$ for $i < S$ (place an order)…………………….(1)

According to (1), no order is placed if initial inventory $i \geq s$; but if i is small enough ($i < s$), then an order is placed so that the *total* inventory available for future demand is S. The rule in (1) is referred to as an (s, **S) policy, where *s*** denotes the **reorder point and *S* the reorder level.**

Linear holding and penalty cost. Suppose that the expected cost for the entire period is the sum of the purchase cost and the expected holding and penalty cost. Specifically, let the cost of ordering the amount x be

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ K + cx & \text{for } x > 0, \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Where $K \geq 0$ represents the setup cost and $c \geq$ o the unit purchasing cost.

Assume that the expected holding and penalty cost during the period depends only on the *total* amount of inventory available to meet demand, namely, the level y, which equals initial inventory i plus the order amount x. Let $L(y)$ denote this expected cost function, and in particular, assume that $L(y)$ is calculated from the formula

$$L(i) = \sum_{q=0}^{y} h(y-q)p(q) + \sum_{q>y} \pi.(q-y)p(q) \quad \text{for } y \geq 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where $h \geq 0$ is the unit holding cost per item left over at the end of the period, and $\pi \geq 0$ is the unit penalty cost per item short at the end of the period. Thus the first summation is the *expected* holding cost and the second summation is the *expected* penalty cost. We also make the reasonable assumptions that $c + h > 0$.

Note that the value of y that minimizes $L(y)$ increases as the penalty cost π increases.

When there is a salvage value v (where $0 < v \leq c$) for each item remaining at the end of the horizon, this value can be subtracted from the unit holding cost. Consequently, h in (3) really represents the *net* holding cost ($h = h' - v$), and can be negative. By the same token, if the item is being stocked for sale, and demand in excess of inventory on hand is lost, then π includes the sales price r ($\pi = \pi' + r$).

It can be shown that an (s, S) rules is the optimal form of policy special case using the data contained in Wagner [1]. To find optimal values for s and S, let

$$R \equiv \frac{\pi - c}{\pi + h}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Penalty cost = π and denote the value of $\boldsymbol{R}$ as the **critical ratio**. Observe that $0 < R < 1$ because we assumed that $\pi > c$ and $\pi + h > 0$. Then it can be shown that an optimal reorder level $S \geq 0$ is the *smallest* integer such that

$$P(S) = \sum_{q=0}^{S} p(q) \geq R \quad \text{(determination of reorder level } S\text{)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Since *P(S)* is simply the *cumulative* distribution function at $q = S$, an optimal reorder level *S* is set such that there is at least a probability *R* of satisfying all demand.
Observed from (4) that
(i) *R* increases as π increases, so that the reorder level *S* is a non-decreasing function of the penalty cost π.
(ii) *R* decreases either as the holding cost *h* or purchase cost *c* increases, so that the reorder level *S* is a non-increasing function of *h* and *c*.
(iii) If $\pi \geq 2c + h$, then $R \geq \frac{1}{2}$ and *S* is at least as large as the median of the demand distribution.

To summarize, an optimal reorder level *S* can be found by first calculating the critical ratio *R* in (4) and then using a cumulative function *P(y)*.
Suppose the setup cost $K = 0$. Then, as we shall demonstrate later in this section, it is optimal to let $s = S$. This means that if initial inventory *i* is *any* amount less than *S*, you order the quantity $x\ x = S - i$. And if initial inventory is larger than *S*, you order nothing.
Consider the more challenging case where the setup cost $K > 0$. Now at optimal value for the reorder point s may be strictly less than *S*. The reason is that if the setup cost *K* is relatively large and initial inventory *i* (≤ S) is sufficiently close to *S*, then the additional expense of the letting $y = S$ (that is, amount ordered $x = S$-s) which is $K + c(S — s) + L(S)$. But since you do order when initial inventory is less than s, the preceding cost advantage must go the other way for $y = s – 1$.
We can summarize the reasoning so far by stating that you choose s to be the smallest number such that

$L(s) \leq K + c(S - s) + L(S)$ (determination of reorder point *s*)……………….(8)

Thus, when $K > 0$, you calculate the reorder point *s* by computing and comparing the expected holding and penalty cost function *L(y)* with the sum $[K + c(S - y) + L(S)]$, for successively smaller trial values of *y*.
* **Optimal (s, S) policy.** First, we need to review the notion of a convex function. A function *L*(*y*) defined for integer values of *y* is said to be convex if

$L(y + 1) - L(y) \geq L(y) - L(y - 1)$ for all *y* <u>convex</u>………………………(9)

Suppose that the expected cost function is the sum of an ordering cost and a term representing expected holding and penalty costs:

$g(y | i) \equiv c(y – i) + L(y)$………………………………………………...(10)

Let the ordering cost $c(y - i)$ consist of a setup cost $K \geq 0$ and a unit purchase Cost $c \geq 0$:

$$c(y - i) = \begin{cases} 0 & \text{for } y = i\ (x = 0) \\ K + c.\,(y – i) & \text{for } y > i \quad (x = y - i > 0) \end{cases}$$ <u>Ordering cost</u>. …………………………(11)

Assume that $cy + L(y)$ grows without bound as $|y| \to \infty$, and that the expected holding and penalty cost function *L(y)* is convex.
Suppose that holding cost is expressed as an increasing function $h(y - q)$ of inventory left over $(y - q)$ provided that actual demand $q \leq y$; in formal terms, assume

$$h(j) \begin{cases} \geq 0 \text{ for } j \geq 0 \\ = 0 \text{ for } j < 0 \end{cases}$$ ………………………………………..………………(12),

and *h(j)* is increasing for $j \geq 0$. Similarly, suppose that the penalty cost is an increasing function $\pi(q - y)$ of unfilled demand $(q - y)$ provided $q > y$; again, in formal terms, assume that

$$\pi(j) \begin{cases} \geq 0 \text{ for } j > 0 \\ = 0 \text{ for } j \leq 0, \end{cases}$$ ………………………………………………………..(13)

and $\pi(j)$ is increasing for $j \geq 0$. Then the expected holding and penalty cost function is

$$L(y) \equiv \begin{cases} \sum_{q=0}^{y} h(y-q)\, p(q) + \sum_{q>y} \pi(q-y) & \text{for } y \geq 0 \\ \sum_{q\geq 0} \pi(q-y)\, p(q) & \text{for } y < 0 \end{cases}$$ …………………………..(14)

It can be proved that if *the sum of the actual holding and penalty cost functions h(j) + π(j) is convex for every integer j, then the expected holding and penalty cost function L(y) in (14) is convex.* When holding cost is to be assessed on the level of inventory before demand occurs, the function *h*(*y*) replaces the first summation in (14).
The following result is easy to demonstrate.

4.1 Derivation of Optimality of (*s, S)*

Given the cost functions in (10) and (11), and that *L*(*y*) is convex, then the form of an optimal policy is (*s*, *S*), defined in (1). Further, the value for the reorder level S does not depend on the setup cost *K* in (11), and if *K*= 0, then the reorder point $s = S$.

A proof of the optimal policy theorem as well as a general computational technique can be established by the following line of argument. Observe first that

$$f(i) = \text{minimum } g(y|\, i) = \text{minimum} \begin{cases} \text{Minimum } g(y/i) \\ \quad y>i \\ g(i|\, i) \end{cases} \quad \text{………………………….(15)}$$

$$= \text{minimum} \begin{cases} K - ci + \text{minimum } [cy + L(y)] \\ \quad\quad y>i \\ L(i). \end{cases} \quad \text{……………………………………….(16)}$$

Since both cy and $L(y)$ arc convex, their sum is convex and by assumption increases without bound as $|y|$ gets very large. Consequently, there exists an S such that

$$\underset{y}{\text{minimum}}\, [cy + L(y)] = cS + L(S) \quad \text{……………………………………….(17)}$$

Further, a locally optimal value S is also globally optimal.

Suppose $i \geq S$; then according to (17),

$$\text{minimum } [cy + L(y)] \geq ci + L(i) \quad (i > S) \text{……………………………….(18)}$$

So that

$$K - ci + \text{minimum } [cy + L(y)] \geq L(i) \quad (i \geq S) \text{…………………………….(19)}$$

Therefore from (16), we have for $i>$ S

$$f(i) = L(i) \quad y(i) = i \quad \text{and} \quad x(i) = 0 \quad (i \geq S) \text{…………………………(20)}$$

Now suppose $i \leq S$; then according to (16) and (17),

$$f(i) = \text{minimum} \begin{cases} (K - ci + cS + L(S) & \text{for } y = S \\ & (i < S) \text{………………….(21)} \\ (L(i) & \text{for } y = i \end{cases}$$

Consider the case $K = 0$, (17) implies that

$$L(i) \geq -ci + cS + L(S) \text{………………………………….(22)}$$

So that the minimum in (21) is given by

$$\left.\begin{aligned} f(i) &= c(S - i) + L(S) \\ y(i) &= S \\ x(i) &= S - i \end{aligned}\right\} \quad \text{for } i \leq S \text{ and } K = 0 \text{………………………….(23)}$$

Next consider the case $K > 0$; now $L(i)$ may be the smaller term in (21) for i near S. Let s be the smallest number such that

$$L(s) \leq K - cs + cS + L(S) \text{……………………………………………….(24)}$$

or, equivalently,

$$L(s) + cs \leq K + cS + L(S) \text{……………………………………………………….(25)}$$

Then (20) also holds for $i \geq s$. But for $i < s$,

$$f(i) = K + c(S - i) + L(S) \quad y(s) = S \quad \text{and } x(i) = S - i \text{……(26)}$$

In summary, the values for the reorder level S and the reorder point s in (8) are found according to (17) and (25), and

$$f(i) = \begin{cases} K + c(S - i) + L(S) & \text{for } i < s \\ L(i) & \text{for } i \geq s \text{………………………….(27)} \end{cases}$$

Using the formulas (4) and (5) we can find an optimal value of S in the case of linear holding and penalty cost model. The result can be derived as follows,

A consequence of (17) is that S must satisfy

$$[c(S + 1) + L(S + 1)] \geq [cS + L(S)], \text{………..(28)}$$

or

$$L(S + 1) - L(S) \geq -c \text{……………………………………………………….(29)}$$

The holding cost component on the left of (29) is

$$h\left[\sum_{q=0}^{S+1}(S+1-q)p(q) - \sum_{q=0}^{S}(S-q)p(q)\right] = h\sum_{q=0}^{S}p(q) = hP(S). \text{…………………(30)}$$

Similarly, the penalty cost component on the left of (29) is

$$\pi\left[\sum_{q=S+2}^{\infty}(q-S-1)p(q) - \sum_{q=S+1}^{\infty}(q-S)p(q)\right] = \pi\left[-\sum_{y>S}^{\infty}p(q)\right] = -\pi[1-P(s)]. \text{……..(31)}$$

Adding the rightmost terms in (30) and (31), the inequality in (29) becomes

$(h+\pi)P(S)-\pi \geq -c$……………………………………………………..(32)
which simplifies to (5).
Consider the linear holding and penalty cost case in which *p(q)* is a probability density function, the variable *y* is continuous, and

$$L(y) = \begin{cases} \int_0^y h.(y-q)p(q)\ dp + \int_y^\infty \pi(q-y)p(q)\,dq & \text{for} \quad y \geq 0. \\ \int_0^\infty \pi.(q-y)p(q)\ dq & \text{for} \quad y < 0. \end{cases} \quad \text{……………}(33)$$

Then if *dL(y)/dy=L'(y) exists, the y=S that minimizes [cy +L(y)] must satisfy*

$$c + L'(y) = 0 \text{……………………………………..}(34)$$

It can be shown by advanced calculus that, for $y \geq 0$.

$$L'(y) = h(y-y)p(y) + \int_y^\infty hp(q)\ dq - \pi(y-y)p(y) - \int_y^\infty \pi p(q)\ dq$$

$$= (h+\pi)\int_0^S p(q)\ dq - \pi. \text{…………………………………...}(35)$$

Therefore from (34) and (35), *S* satisfies

$$P(S) \equiv \int_0^y p(q)dq = \frac{\pi - c}{\pi + h}. \text{……..…………………………………..}(36)$$

The value of *s* is found by solving (8) expressed as equality.
Throughout this section, the holding cost formulas have been based on the value of inventory at the end of the horizon. If the holding cost is linear and assessed on the value of *y*, then

$$R = \frac{\pi - (c+h)}{\pi} \text{……………………………………………………..}(37)$$

If the holding cost is linear and assessed on the expected average value of inventory, namely [.5*y* + .5(*y* - *q*)], then

$$R \equiv \frac{\pi - (c+.5h)}{\pi + .5h} \text{………………………….………………………..}(38)$$

5.1 Conclusion

Inventory models presented in this article considers cases where demand is assumed to be probabilistic. The formulas derived are based on evaluating the holding cost of inventory at the end of a given horizon. This includes assessing the value of the inventory and its expected average for linear holding functions are cost.

References

[1] Wagner, H.M. (2001) Principles of Operations Research, Second Edition, Pretence-Hall of India, New Delhi

[2] Hillier, F.S. and Lieberman, G.J. (2007) Operations Research, Eight Edition. Tata McGraw-Hill Publishing Company Limited, New Delhi.

[3] Lieberman, M.B. and Demeester, L. (1999). Inventory Reduction and Productivity Growth: Linkages in the Japanese Automotive Industry. Management Science, Vol. 45, issue4, pp. 466-476.

[4] George, P. V (1972), "Inventory Behaviour and Efficacy of Credit Control", An- vesak, No.2, Vol. II, 1972, pp. 168-175.

[5] Lambrix, R.J and Singhvi, S.S (1979), "Managing the Working Capital Cycle", Financial Executive, June 1979, pp. 32-41.

[6] Lal, A.B (1981), "Inventory Models and Problems of Price Fluctuation", Shree Publishing House, New Delhi, 1981.

[7] Anichebe, N.A. and Agu O. A. (2013). Effects of Inventory Management on Organizational Effectiveness. Information and Knowledge Management. Vol.3 Issue 8 pp. 92-100.

[8] Panigrahi, A.K. (2013). Relationship Between Inventory management and profitability: An Empirical Analysis of Indian Cement Company. Asia Pacific Journal of Marketing Management Review. Vol.2 Issue 7 pp. 107-120.

[9] Gaur, V., Fisher, M. and Raman, A. (2005)." An Econometric Analysis of Inventory Turnover Performance in Retail Services". Management Science, vol.5, iss.2, pp.181–194.

[10] Singh, S. (2006). Inventory control practices in IFFCO. The Management Accountant, vol.41,iss.7, pp.577–582.

[11] Capkun, V., Hameri, A. and Weiss, L. A. (2009). On the Relationship Between Inventory and Financial Performance in Manufacturing. International Journal of Operations and Production Management, vol.29, iss.8, pp.789–806.

[12] Gaur, J. and Bhattacharya, S. (2011). The Relationship Between Financial and Inventory Performance of Manufacturing Firms in Indian context. California Journal of Operations Management, vol. 9, iss.2, pp.70–77.

[13] Eneje, B. C., Nweze, A .U. and Udeh, A. (2012). Effect of Efficient Inventory Management on Profitability: Evidence from Selected Brewery Firms in Nigeria. International Journal of Current Research, vol.4, iss.11, pp.350-354.

[14] Madishetti, S and Kibona, D. (2013). Impact of inventory management on the profitability of SMEs in Tanzania. International Journal of Research in Commerce & Management, vol.4,iss.2, pp.1–6.

[15] Sitienei, E. and Memba, F. (2015-16) ‘‘The Effect of Inventory Management on Profitability of Cement Manufacturing Companies in Kenya: A Case Study of Listed Cement Manufacturing Companies in Kenya" International Journal of Management and Commerce Innovations Vol. 3, Iss. 2, pp. 111- 119.