Journal of the Nigerian Association of Mathematical Physics
***Volume 52* (*July & Sept., 2019 Issue*), pp15 – 26**

EFFICIENCY OF NEW ESTIMATORS OF POPULATION MEAN UNDER TWO-PHASE SAMPLING

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Abstract

In this article the efficiency of some estimators of population mean under two-phase sampling is investigated. In some earlier researches when $\bar{X}$ is not attainable due to incomplete or insufficient information, $\bar{x}_1$ is used to estimate $\bar{X}$; that is by utilizing information obtained from an auxiliary variable. The bias and mean square (MSE) have been derived up to second degree to determine the performance of the new estimators using some published available datasets and the results obtained reveals that the proposed estimators are more efficient.

***Keywords*:** Ratio Estimator, Two-phase Sampling, Auxiliary Variable, Mean Squared Error, Efficiency

1.0 Introduction

In sampling theory, the auxiliary information is used in improving the precision of estimators of population mean of the main characteristic under study. The ratio, product, and regression estimators have been widely used when auxiliary information is available. In estimating the population means, a sample mean is usually used. Sample mean, although is unbiased, may possess a large amount of variation, hence the need to obtain an estimator with smaller variance (MSE) as compared to sample mean even it is asymptotically unbiased.

Two-phase sampling is a sampling method that makes use of auxiliary data where the auxiliary information is obtained through sampling. Two-phase sampling is generally employed when number of unit, required to give the desired precision on a different item is widely different. Many survey statisticians have used such technique to reduce the cost of sample survey when information of study variable is very expensive to obtain for all population units. The technique is employed to utilize the information collected at the first phase in order to improve the precision of the information to be collected at the second phase.

2. Sampling Strategy and Estimation Procedures

Let the finite population under study consists of N distinct and identifiable units. A sample of size *n* is drawn at random using simple random sampling without replacement (SRSWOR) technique. Let $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ and $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ respectively represent the population means of the study and the auxiliary variables, while $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ be the respective sample means. When the population mean of the auxiliary variable *X* is not known, double sampling or two-phase sampling is used to estimate the population mean $\bar{Y}$ of the study variable *y*. In double sampling scheme the following procedure is used for the selection of the required sample:

Case I: A large sample S' of size n' $(n' < N)$ is drawn from the population by SRSWOR and the observations are taken only on the auxiliary variable x to estimate the population mean $\bar{X}$ of the auxiliary variable.

Case II: A sample S of size n $(n < N)$ is drawn either from S' or directly from the population of size N to observe both the study variable and the auxiliary variable.

In estimating the population means, we use a sample mean given by

$$t_o = \bar{y} \tag{2.1}$$

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The variance of t_o , up to the first order of approximation, is given by

$$V(t_o) = \phi \bar{Y}^2 C_y^2 \tag{2.2}$$

Where,

$\phi = \frac{1}{n} - \frac{1}{N}$, $C_y = \frac{S_y}{\bar{Y}}$, and $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$

Cochran [1] proposed the classical ratio type estimator of population mean utilizing the auxiliary information under simple random sampling as

$$t_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \tag{2.3}$$

Kadilar [2] proposed a modified exponential type estimator for estimating $\bar{Y}$ as

$$\bar{y}_{PR} = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha} \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{2.4}$$

The MSE of $\bar{y}_{PR}$, up to the first order of approximation is

$$MSE(\bar{y}_{PR}) = \gamma \bar{Y}^2 \left(C_y^2 + C_x^2/4 + 2\alpha\rho C_x C_y + \rho C_x C_y + \alpha^2 C_x^2 + \alpha C_x^2\right) \tag{2.5}$$

Abdullahi and Yahaya [3] proposed a modified ratio-product estimator of population mean using some known parameters of the auxiliary variable as

$$T_{RP} = \left[\delta \bar{y}\left(\frac{\bar{X}C_x + M_d}{\bar{x}C_x + M_d}\right) + (1-\delta)\bar{y}\left(\frac{\bar{x}C_x + M_d}{\bar{X}C_x + M_d}\right)\right] \tag{2.6}$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ are unbiased estimators of the population means $(\bar{Y}, \bar{X})$ respectively, with C_x being the coefficient of variation, M_d is the median (are both known characteristics positive scalars of the auxiliary variable X) and δ is a real constant to be determined such that the MSE of T_{RP} is minimum.

The MSEof T_{RP} , up to the first order of approximation is

$$MSE(T_{RP}) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + \theta(1-2\delta)C_x^2\left\{(1-2\delta)\theta + 2K\right\}\right] \tag{2.7}$$

where $f = \frac{n}{N}$ and $\theta = \frac{\bar{X}C_x}{\bar{X}C_x + M_d}$

Gupta and Yadav [4] proposed the generalized estimator of the population mean using the information on size of the sample as

$$\tau_p = \bar{y}\left[\alpha + (1-\alpha)\left(\frac{\bar{X}+n}{\bar{x}+n}\right)\right] \tag{2.8}$$

where α is a suitably chosen constant to be defined such that the MSE of the proposed estimator is minimum.

The MSE of τ_p , up to the first order of approximation is

$$MSE(\tau_p) = \gamma \bar{Y}^2\left[C_y^2 + \delta^2 C_x^2 - 2\delta\rho C_x C_y + \alpha^2\delta^2 C_x^2 + 2\alpha\delta\rho C_x C_y - 2\alpha\delta^2 C_x^2\right] \tag{2.9}$$

Sukhatme [5] proposed the usual ratio estimator of population mean in two-phase sampling as

$$t_R^d = \bar{y}\left(\frac{\bar{x}_1}{\bar{x}}\right) \tag{2.10}$$

where $\bar{x}_1 = \frac{1}{n'}\sum_{i=1}^{n'} x_i$ is an unbiased estimator of population mean $\bar{X}$ of the auxiliary variable based on the sample s' of size n' .

The MSE of t_R^d , up to the first order of approximation, for Case-I and Case-II respectively are,

$$MSE\left(t_R^d\right)_{\text{I}} = \bar{Y}^2\left[\phi C_y^2 + \phi^{**} C_x^2 (1-2C)\right] \tag{2.11}$$

$$MSE\left(t_R^d\right)_{\text{II}} = \bar{Y}^2\left[\phi C_y^2 + \phi^{***} C_x^2 - 2\phi C C_x^2\right] \tag{2.12}$$

where, $\phi^{*} = \left(\frac{1}{n'} - \frac{1}{N}\right)$, $\phi^{**} = \left(\frac{1}{n} - \frac{1}{n'}\right)$, $\phi^{***} = \left(\phi + \phi^{*}\right)$, $C = \rho_{yx}\frac{C_y}{C_x}$, $C_x = \frac{S_x}{\bar{X}}$, $S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2$ and $\rho_{yx} = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{Y})^2(x_i - \bar{X})$

Kumar and Bahl [6] proposed the following dual to ratio estimator of population under two-phase sampling as;

$$t_{\mathrm{R}}^{*d} = \bar{y}\left(\frac{\bar{x}^{*d}}{\bar{x}_1}\right) \qquad (2.13)$$

The MSE of t_{R}^{*d}, up to the first order of approximation for both case-I and case-II respectively are,

$$MSE(t_{\mathrm{R}}^{*d})_I = \bar{Y}^2\left[\phi C_y^2 + g\phi^{**}C_x^2\left(g-2C\right)\right] \qquad (2.14)$$

$$MSE(t_{\mathrm{R}}^{*d})_{II} = \bar{Y}^2\left[\phi C_y^2 + gC_x^2\left(g\phi^{***}-2\phi C\right)\right] \qquad (2.15)$$

where, $g = \frac{n}{n_1 - n}$

Singh and Vishwakarma [7] proposed the exponential type ratio estimator of population mean of study variable in two-phase sampling as,

$$t_{\mathrm{Re}}^{d} = \bar{y}\exp\left(\frac{\bar{x}_1-\bar{x}}{\bar{x}_1+\bar{x}}\right) \qquad (2.16)$$

The MSE of t_{Re}^{d}, up to the first order of approximation for both case-I and case-II respectively are,

$$MSE(t_{\mathrm{Re}}^{d})_I = \bar{Y}^2\left[\phi C_y^2 + \phi^{**}C_x^2\left(\frac{1}{4}-C\right)\right] \qquad (2.17)$$

$$MSE(t_{\mathrm{Re}}^{d})_{II} = \bar{Y}^2\left[\phi C_y^2 + \frac{1}{4}\phi^{***}C_x^2 - \phi C_x^2 C\right] \qquad (2.18)$$

Kalita and Singh [8] proposed the following exponential dual to ratio estimator in two-phase sampling as,

$$t_{\mathrm{Re}}^{*d} = \bar{y}\exp\left(\frac{\bar{x}^{*d}-\bar{x}_1}{\bar{x}^{*d}+\bar{x}_1}\right) \qquad (2.19)$$

The MSE of t_{Re}^{*d}, up to the first order of approximation for both case-I and case-II respectively are,

$$MSE(t_{\mathrm{Re}}^{*d})_I = \bar{Y}^2\left[\phi C_y^2 + g\phi^{**}C_x^2\left(\frac{1}{4}g-C\right)\right] \qquad (2.20)$$

$$MSE(t_{\mathrm{Re}}^{*d})_{II} = \bar{Y}^2\left[\phi C_y^2 + \frac{1}{4}g^2\phi^{***}C_x^2 - \phi g C_x^2 C\right] \qquad (2.21)$$

3. Proposed Estimators

Using the estimators of Kadilar [2], Abdullahi and Yahaya [3] as well as Gupta and Yadav [4], we proposed the following type of estimators for the population mean in simple random sample under two-phase sampling as;

$$T_1^{(d)} = \bar{y}\left(\frac{\bar{x}}{\bar{x}_1}\right)^{\beta}\exp\left(\frac{\bar{x}_1-\bar{x}}{\bar{x}_1+\bar{x}}\right) \qquad (3.1)$$

$$T_2^{(d)} = \bar{y}\left[\lambda\left(\frac{\bar{x}_1C_x+M_d}{\bar{x}C_x+M_d}\right)+(1-\lambda)\left(\frac{\bar{x}C_x+M_d}{\bar{x}_1C_x+M_d}\right)\right] \qquad (3.2)$$

$$T_3^{(d)} = \bar{y}\left[\theta+(1-\theta)\left(\frac{\bar{x}_1+n}{\bar{x}+n}\right)\right] \qquad (3.3)$$

The Bias and MSE of the proposed estimators are obtained through the following two cases.

Case I: When the second phase sample of size n is a subsample of the first phase sample of size n'.

Case II: Here, the second phase sample of size n is drawn independently of the first phase sample of size n'.

Case I

$$E(e_o) = E(e_1) = E(e_2) = 0$$

$$E(e_0^2) = \left(\frac{1}{n}-\frac{1}{N}\right)C_y^2,\ E(e_1^2) = \left(\frac{1}{n}-\frac{1}{N}\right)C_x^2$$

$$E(e_2^2) = \left(\frac{1}{n_1}-\frac{1}{N}\right)C_x^2,\ E(e_oe_1) = \left(\frac{1}{n}-\frac{1}{N}\right)\rho C_yC_x$$

$$E(e_oe_2) = \left(\frac{1}{n_1}-\frac{1}{N}\right)\rho C_yC_x,\ E(e_1e_2) = \left(\frac{1}{n_1}-\frac{1}{N}\right)C_x^2$$

where, $\phi^{*}=\left(\frac{1}{n'}-\frac{1}{N}\right)$, $\phi^{**}=\left(\frac{1}{n}-\frac{1}{n'}\right)$, $\phi^{***}=\left(\phi+\phi^{*}\right)$, $C=\rho_{yx}\frac{C_y}{C_x}$, $C_x=\frac{S_x}{\overline{X}}$, $S_x^2=\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\overline{X})^2$ and $\rho_{yx}=\frac{1}{N}\sum_{i=1}^{N}(y_i-\overline{Y})^2(x_i-\overline{X})$

The expression for the bias of $T_1^{(d)}$ up to first order of approximation under case I is

$$B\left(T_1^{(d)}\right)_I=\overline{Y}\left[\frac{3}{8}\phi^{**}C_x^2-\beta\phi^{**}C_x^2+\frac{\beta^2}{2}\phi^{**}C_x^2-\frac{1}{2}\phi^{**}CC_x^2+\beta\phi^{**}CC_x^2\right] \quad (3.4)$$

The MSE of $T_1^{(d)}$ under case I to first order of approximation is

$$MSE\left(T_1^{(d)}\right)_I=\overline{Y}^2\left[\phi C_y^2+\left(\frac{1}{4}+\beta^2-\beta\right)\phi^{**}C_x^2+(2\beta-1)\phi^{**}CC_x^2\right] \quad (3.5)$$

To obtain expression for β that minimize $MSE(T_1^{(d)})_I$, we differentiate (3.5) with respect to β and equate the derivative to zero.

$$2\beta-1=-2C$$

$$\beta=\frac{1}{2}-C$$

Substituting the value of β in (3.5), we get the minimum value of $MSE_{\min}(T_1^{(d)})_I$ as

$$MSE_{\min}(T_1^{(d)})_I=\overline{Y}^2\left[\phi C_y^2-\phi^{**}C^2C_x^2\right] \quad (3.6)$$

Under Case II

$$E(e_o)=E(e_1)=E(e_2)=0$$

$$E(e_0^2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_y^2,\ E(e_1^2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_x^2$$

$$E(e_2^2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_x^2,\ E(e_oe_1)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho C_yC_x$$

$$E(e_oe_2)=0,\ E(e_1e_2)=0$$

The bias of $T_1^{(d)}$ under case II is obtained up to the first order of approximation, as

$$B(T_1^{(d)})_{II}=\overline{Y}\left[\left(\frac{4\beta^2-1}{8}\right)\phi^{*}C_x^2+\left(\frac{4\beta^2-8\beta+3}{8}\right)\phi C_x^2+\left(\frac{2\beta-1}{2}\right)\phi CC_x^2\right] \quad (3.7)$$

The MSE of $T_1^{(d)}$ under case II to first order of approximation as

$$MSE(T_1^{(d)})_{II}=\overline{Y}^2\left[\phi C_y^2+\left(\frac{1}{4}+\beta^2-\beta\right)\phi^{***}C_x^2+(2\beta-1)\phi CC_x^2\right] \quad (3.8)$$

To obtain expression for β that minimize $MSE(T_1^{(d)})_{II}$, we differentiate (3.8) with respect to β and equate the derivative to zero.

$$2\beta-1=\frac{-2\phi C}{\phi^{***}}$$

$$\beta=\frac{1}{2}-\frac{\phi C}{\phi^{***}}$$

Substituting the value of β in (3.8), we get the minimum value of $MSE_{\min}(T_1^{(d)})_{II}$ as

$$MSE_{\min}(T_1^{(d)})_{II}=\overline{Y}^2\left[\phi C_y^2-\frac{\phi^2C^2C_x^2}{\phi^{***}}\right] \quad (3.9)$$

The bias of $T_2^{(d)}$ under case I is obtained up to the first order of approximation, as

$$B\left(T_2^{(d)}\right)_I=\overline{Y}\left[\lambda\tau^2\phi^{**}C_x^2+(1-2\lambda)\tau\phi^{**}CC_x^2\right] \quad (3.10)$$

The MSE of $T_2^{(d)}$ under case I to first order of approximation as

$$MSE\left(T_2^{(d)}\right)_I=\overline{Y}^2\left[\phi C_y^2+(1-2\lambda)^2\tau^2\phi^{**}C_x^2+2\tau(1-2\lambda)\phi^{**}CC_x^2\right] \quad (3.11)$$

To obtain expression for λ that minimize $MSE(T_2^{(d)})_I$, we differentiate (3.11) with respect to λ and equate the derivative to zero.

$$2\lambda - 1 = \frac{C}{\tau}$$

$$1 - 2\lambda = -\frac{C}{\tau}$$

Substituting the value of λ in (3.11), we get the minimum value of $MSE(T_2^{(d)})_I$ as

$$MSE_{\min}\left(T_2^{(d)}\right)_I = \bar{Y}^2\left[\phi C_y^2 - \phi^{**}C^2C_x^2\right] \qquad (3.12)$$

The bias of $T_2^{(d)}$ under case II is obtained up to the first order of approximation, as

$$B\left(T_2^{(d)}\right)_{II} = \bar{Y}\left[\lambda\tau^2\phi^{**}C_x^2 + \tau^2\phi^{*}C_x^2 + (\tau - 2\lambda\tau)\phi CC_x^2\right] \qquad (3.13)$$

The MSE of $T_2^{(d)}$ under case II to first order of approximation as

$$MSE\left(T_2^{(d)}\right)_{II} = \bar{Y}^2\left[\phi C_y^2 + \phi^{***}(1-2\lambda)^2\tau^2C_x^2 + 2\tau(1-2\lambda)\phi CC_x^2\right] \qquad (3.14)$$

To obtain expression for λ that minimize $MSE(T_2^{(d)})_{II}$, we differentiate (3.14) with respect to λ and equate the derivative to zero.

$$2\lambda - 1 = \frac{\phi C}{\tau\phi^{***}}$$

$$2\lambda = 1 + \frac{\phi C}{\tau\phi^{***}}$$

Substituting the value of λ in (3.14), we get the minimum value of $MSE(T_2^{(d)})_{II}$ as

$$MSE_{\min}\left(T_2^{(d)}\right)_{II} = \bar{Y}^2\left[\phi C_y^2 - \frac{\phi^2C^2C_x^2}{\phi^{***}}\right] \qquad (3.15)$$

The bias of $T_3^{(d)}$ under case I is obtained up to the first order of approximation, as

$$Bias(T_3^{(d)})_I = \bar{Y}\left[\gamma^2(1-\theta)\phi^{**}C_x^2 - \gamma(1-\theta)\phi^{**}CC_x^2\right] \qquad (3.16)$$

The MSE of $(T_3^{(d)})_I$, to first order of approximation as

$$MSE(T_3^{(d)})_I = \bar{Y}^2\left[\phi C_y^2 + \gamma^2\phi^{**}(1-\theta)^2C_x^2 - 2\gamma\phi^{**}(1-\theta)CC_x^2\right] \qquad (3.17)$$

To obtain expression for θ that minimize $MSE(T_3^{(d)})_I$, we differentiate (3.17) with respect to θ and equate the derivative to zero.

$$(1-\theta) = \frac{C}{\gamma}$$

$$\theta = 1 - \frac{C}{\gamma}$$

Substituting the value of θ in (3.17), we get the minimum value of $MSE(T_3^{(d)})_I$ as

$$MSE_{\min}(T_3^{(d)})_I = \bar{Y}^2\left[\phi C_y^2 - \phi^{**}C^2C_x^2\right] \qquad (3.18)$$

The bias of $T_3^{(d)}$ under case II is obtained up to the first order of approximation as

$$Bias(T_3^{(d)})_{II} = \bar{Y}\left[\gamma^2\phi(1-\theta)C_x^2 - \gamma(1-\theta)\phi CC_x^2\right] \qquad (3.19)$$

The MSE of $T_3^{(d)}$ under case II is

$$MSE(T_3^{(d)})_{II} = \bar{Y}^2\left[\phi C_y^2 + \phi^{***}\gamma^2(1-\theta)^2C_x^2 - 2\gamma(1-\theta)\phi CC_x^2\right] \qquad (3.20)$$

To obtain expression for θ that minimize $MSE(T_3^{(d)})_{II}$, we differentiate (3.20) with respect to θ and equate the derivative to zero.

$$\gamma\phi^{***}(1-\theta) = \phi C$$

$$1-\theta=\frac{\phi C}{\gamma\phi^{***}}$$

Substituting the value of θ in (3.20), we get the minimum value of $MSE(T_3^{(d)})_{II}$ as

$$MSE_{\min}(T_3^{(d)})_{II}=\bar{Y}^2\left[\phi C_y^2-\frac{\phi^2C^2C_x^2}{\phi^{***}}\right] \quad (3.21)$$

4. Efficiency Comparison

Theoretical comparison of the proposed estimators has been made with the competing estimators of population mean. The conditions under which the proposed estimators perform better than the competing estimators have also been given for, j= 1, 2, 3.

Under Case I

The minimum mean square error for the three proposed estimators under case I can be expressed as;

$$MSE_{\min}(T_j^{(d)})_I=\bar{Y}^2\left[\phi C_y^2-\phi^{**}C^2C_x^2\right] \quad (4.1)$$

where, j= 1, 2, 3.

From equation (2.2) and equation (4.1), we have,

$$v(t_o)-MSE_{\min}(T_j^{(d)})_I>0\text{ , if }\phi^{**}CC_x^2>0 \quad (4.2)$$

Thus the proposed estimators are better than the usual mean per unit estimator of population mean.

From equation (2.11) and equation (4.1), we have,

$$MSE(t_R^d)_I-MSE_{\min}(T_j^{(d)})_I>0\text{, if }\phi^{**}C_x^2(1+2C)-(2C-1)>0 \quad (4.3)$$

Under the above condition, the proposed estimators are better than the usual ratio estimator of Sukhatme [5] if $2C-1>0$.

From equation (2.14) and equation (4.1), we have,

$$MSE(t_R^{*(d)})_I-MSE_{\min}(T_j^{(d)})_I>0\text{ , if }g^2+(1-2g)C>0 \quad (4.4)$$

Under the above condition, the proposed estimators perform better than the dual ratio estimator given by Kumar and Bahl [6], if $1-2g>0$.

From equation (2.17) and equation (4.1), we have,

$$MSE(t_{\text{Re}}^{(d)})_I-MSE_{\min}(T_j^{(d)})_I>0\text{, if }\phi^{**}\left(\tfrac{1}{4}-C\right)+\phi^{**}C>0 \quad (4.5)$$

Under the above condition, proposed estimators perform better than the exponential type ratio estimator given by Singh and Vishwakarma [7], if $\tfrac{1}{4}-C>0$.

From equation (2.20) and equation (4.1), we have,

$$MSE(t_{\text{Re}}^{*(d)})_I-MSE_{\min}(T_j^{(d)})_I>0\text{, if }g^2>0 \quad (4.6)$$

Under the above condition, proposed estimators perform better than the exponential dual to ratio estimator given by Kalita and Singh [8].

Under Case II

The minimum mean square error for the three proposed estimators under case II can be expressed as;

$$MSE_{\min}(T_j^{(d)})_{II}=\bar{Y}^2\left[\phi C_y^2-\frac{\phi^2C^2C_x^2}{\phi^{***}}\right] \quad (4.7)$$

where, j= 1, 2, 3.

From equation (2.2) and equation (4.7), we have,

$$v(t_o)-MSE_{\min}(T_j^{(d)})_{II}>0\text{ , if }\phi^2C^2C_x^2>0 \quad (4.8)$$

Thus the proposed estimators are better than the usual mean per unit estimator of population mean.

From equation (2.12) and equation (4.7), we have,

$$MSE(t_R^d)_{II}-MSE_{\min}(T_j^{(d)})_{II}>0\text{, if }\phi^{***}\left(\phi^{***}-2\phi C\right)+\phi^2C^2>0 \quad (4.9)$$

Under the above condition, the proposed estimator is better than the usual ratio estimator of Sukhatme [5], if $\phi^{***}-2\phi C>0$.

From equation (2.15) and equation (4.7), we have,

$$MSE(t_R^{*(d)})_{II}-MSE_{\min}(T_j^{(d)})_{II}>0\text{ , if }g(g\phi^{***}-2\phi C)+\frac{\phi C^2}{\phi^{***}}>0 \quad (4.10)$$

Under the above condition, the proposed estimators perform better than the dual ratio estimator given by Kumar and Bahl [6], if $g\phi^{***} - 2\phi C > 0$.

From equation (2.18) and equation (4.7), we have,

$$MSE(t_{\text{Re}}^{(d)})_{II} - MSE_{\min}(T_3^{(d)})_{II} > 0, \text{ if } \left(\tfrac{1}{4}\phi^{***} - \phi C\right) + \frac{\phi^2 C^2}{\phi^{***}} > 0 \quad (4.11)$$

Under the above condition, proposed estimators perform better than the exponential type ratio estimator given by Singh and Vishwakarma [7], if $\frac{1}{4}\phi^{***} - \phi C > 0$

From equation (2.21) and equation (4.7), we have,

$$MSE(t_{\text{Re}}^{*(d)})_{II} - MSE_{\min}(T_j^{(d)})_{II} > 0, \text{ if } g\left(\tfrac{1}{4} g\phi^{***} - \phi C\right) + \frac{\phi^2 C^2}{\phi^{***}} > 0 \quad (4.12)$$

Under the above condition, proposed estimators perform better than the exponential dual to ratio estimator given by Kalita and Singh [8], if $\frac{1}{4} g\phi^{***} - \phi C > 0$

5. Numerical Illustration

To examine the merits of the suggested estimators we are going to consider some natural population data sets. The description of the populations is given below.

Population I: Singh and Chaudary [9]

Y: Output

X: Fixed capital

$N = 80$, $n_1 = 70$, $n = 20$, $\bar{y} = 51.8264$, $\bar{x} = 11.7646$, $C_y = 0.3542$, $C_x = 0.7507$, $M_d = 7.575$, $\rho_{yx} = 0.9414$

Population II: Murthy [10]

Y: Output

X: Number of Workers

$N = 80$, $n_1 = 36$, $n = 20$, $\bar{y} = 51.8264$, $\bar{x} = 2.8513$, $C_y = 0.3504$, $C_x = 0.9484$, $M_d = 1.48$, $\rho_{yx} = 0.9150$

Population III: Cochran [1]

Y: Total number of inhabitants in the 196 cities in 1920

X: Total number of inhabitants in the 196 cities in 1930

$N = 49$, $n_1 = 25$, $n = 20$, $\bar{y} = 116.1633$, $\bar{x} = 98.6765$, $C_y = 0.8508$, $C_x = 1.0435$, $M_d = 64$, $\rho_{yx} = 0.6904$

Population I:

Table 1: MSE and Percentage Relative Efficiency(PRE) of Proposed and Some Existing Related Estimators over Sample Mean

Estimators	**Case I**		**Case II**	
	MSE	**PRE**	**MSE**	**PRE**
t_o	12.63661	100	12.63661	100
t_R^d	18.69258	67.60227	21.69838	58.23756
t_{Re}^d	2.149566	587.8679	2.300965	549.1873
t_R^{*d}	2.08457	606.1974	1.989449	635.1814
t_{Re}^{*d}	5.198184	243.0966	4.934383	256.0929
$T_1^{(d)}$	**1.979964**	**638.2242**	**1.955744**	**646.1278**
$T_2^{(d)}$	**1.979964**	**638.2242**	**1.955744**	**646.1278**
$T_3^{(d)}$	**1.979964**	**638.2242**	**1.955744**	**646.1278**

Population II:

Table 2: MSE and PRE of Proposed and Some Existing Related Estimators over Sample Mean

Estimators	**Case I**		**Case II**	
	MSE	**PRE**	**MSE**	**PRE**
t_o	12.63661	100	12.63661	100
t_R^d	29.63125	42.64622	78.22518	16.15414
t_{Re}^d	7.712069	163.8549	13.55398	93.23172
t_R^{*d}	50.65723	24.94531	134.4685	9.39745
t_{Re}^{*d}	10.67527	118.3728	23.74486	53.21829
$T_1^{(d)}$	**6.367165**	**198.4652**	**5.119463**	**246.8346**
$T_2^{(d)}$	**6.367165**	**198.4652**	**5.119463**	**246.8346**
$T_3^{(d)}$	**6.367165**	**198.4652**	**5.119463**	**246.8346**

Population III:

Table 3: MSE and PRE of Proposed and Some Existing Related Estimators over Sample Mean

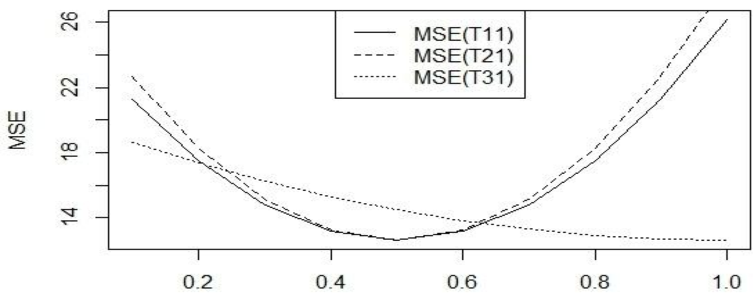
Estimators	**Case I**		**Case II**	
	MSE	**PRE**	**MSE**	**PRE**
t_o	289.0445	100	289.0445	100
t_R^d	270.5585	106.8326	522.2119	55.35005
t_{Re}^d	243.068	118.9151	224.9592	128.4875
t_R^{*d}	1978.31	14.61068	9893.828	2.921463
t_{Re}^{*d}	545.9408	52.9443	2200.732	13.13402
$T_1^{(d)}$	**242.4865**	**119.2002**	**206.1516**	**140.2097**
$T_2^{(d)}$	**242.4865**	**119.2002**	**206.1516**	**140.2097**
$T_3^{(d)}$	**242.4865**	**119.2002**	**206.1516**	**140.2097**

The results for the MSE of the three population being the same gives the minimum MSE that each of the three estimators can attain after minimizing. Hence, using the three populations as a yardstick for the adequacy test we can attest that the proposed estimators are favourable and are more efficient than the existing estimators.

Furthermore, we can assign some constant say β, λ and θ, to check whether MSE of the proposed estimators differ significantly before minimizing taking values say, $\beta, \lambda, \theta = 0.1, 0.2, 0.3 ... 1.0$

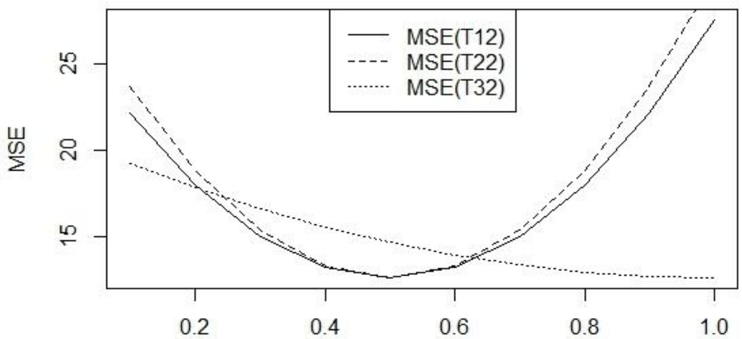
Table 4: MSE of the proposed estimators for values of constant β, λ and θ using population I

CASE I										
β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{11})	21.28453	17.50075	14.79817	13.17678	12.63661	13.17763	14.79986	17.50328	21.28791	26.15375
λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{21})	22.26519	18.05302	15.04420	13.23873	12.63661	13.23784	15.04241	18.05034	22.26162	27.67625
θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{31})	18.31968	17.12680	16.07428	15.16211	14.39030	13.75885	13.26775	12.91702	12.70663	12.63661
CASE II										
β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{12})	22.14941	17.98723	15.01436	13.23082	12.63661	13.23171	15.01614	17.98989	22.15296	27.50535
λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{22})	23.22796	18.59459	15.28491	13.23873	13.29892	15.28304	15.28304	18.59178	23.22421	29.18032
θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{32})	18.88805	17.57588	16.41810	15.41471	14.56571	13.87110	13.33089	12.94507	12.71364	12.63661



Range of Parameters (β , λ and θ **)**

Figure 1: Plot for Population I under Case I

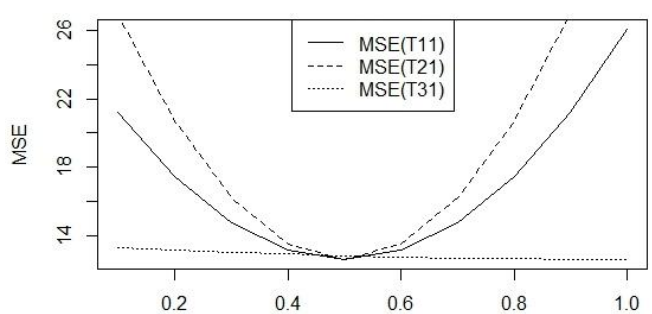


Range of Parameters (β , λ and θ **)**

Figure 2: Plot of Population I under Case II

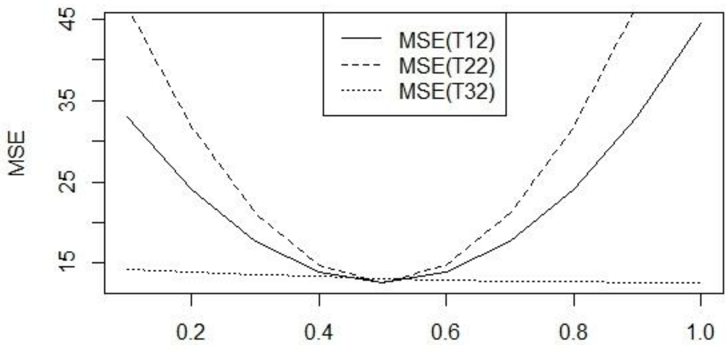
Table 5: MSE of the proposed estimators for values of constant β , λ and θ using population II

CASE I										
β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{11})	21.22492	17.46722	14.78326	13.17306	12.63661	13.17390	14.78494	17.46974	21.22828	26.06056
λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{21})	26.99044	20.71105	16.22561	13.53413	12.63661	13.53304	16.22344	20.70779	26.98610	35.05837
θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{31})	13.31319	13.17114	13.04581	12.93720	12.84531	12.77014	12.71168	12.66994	12.64491	12.63661
CASE II										
β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{12})	33.03500	24.11017	17.73550	13.91097	12.63661	13.91239	17.73833	24.11442	33.04067	44.51706
λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{22})	46.72548	31.81228	21.15974	14.76785	12.63661	14.76602	17.73833	31.80679	46.71815	65.89016
θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{32})	14.24381	13.90642	13.60873	13.35074	13.13246	12.95388	12.81501	12.71584	12.65637	12.63661



Range of Parameters (β , λ and θ **)**

Figure 3: Plot for Population II under Case I

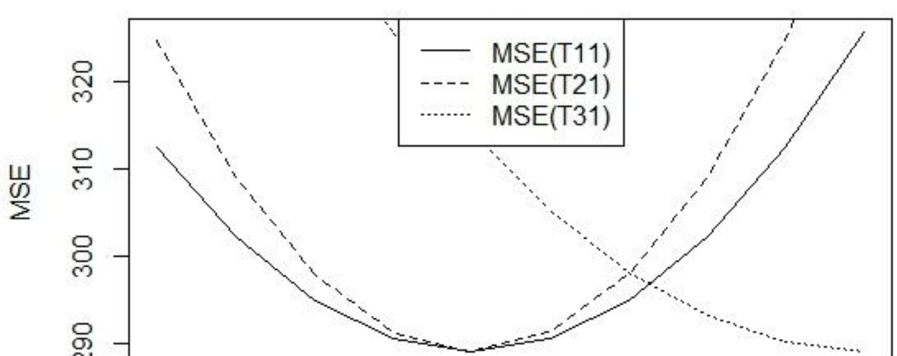


Range of Parameters (β **,** λ and θ **)**

Figure 4: Plot for Population II under Case I

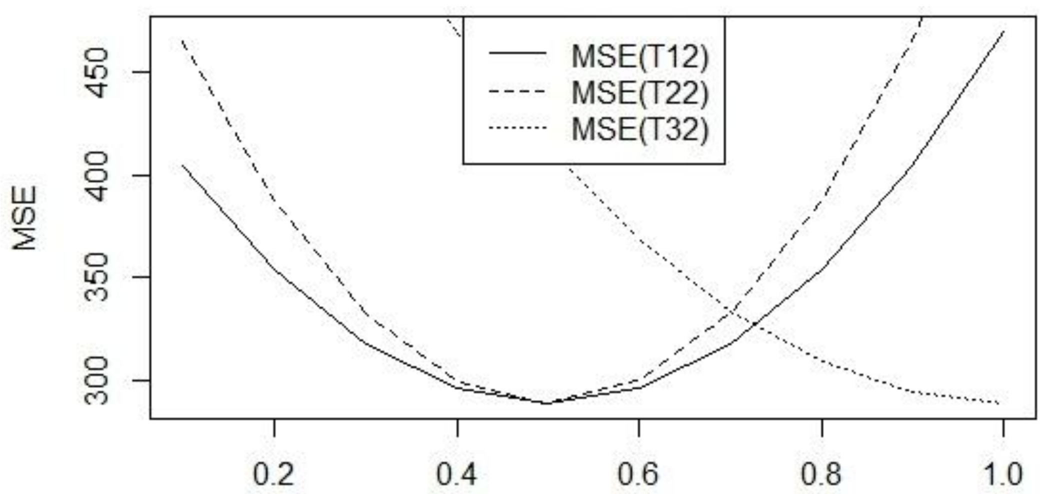
Table 6: MSE of the proposed estimators for values of constant β, λ and θ using population III

					CASE I					
β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{11})	312.5494	302.2652	294.9196	290.5127	289.0445	290.5150	294.9242	302.2721	312.5586	325.7838
λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{21})	324.8140	309.1659	297.9883	291.2812	289.0445	291.2784	297.9827	309.1574	324.8027	344.9184
θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{31})	371.3181	354.0500	338.8135	325.6087	314.4355	305.2940	298.1841	293.1059	290.0594	289.0445
					CASE II					
β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{12})	404.6591	354.0752	317.9448	296.2679	296.2747	317.9584	317.9584	354.0956	404.6863	469.7306
λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{22})	464.9609	388.0006	333.0278	300.0425	289.0445	300.0341	333.0110	387.9755	464.9273	563.8667
θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
MSE(T_{32})	693.7131	608.7802	533.8396	468.8916	413.9360	368.9728	334.0021	309.0238	294.0379	289.0445



Range of Parameters (β **,** λ and θ **)**

Figure 5: Plot for Population III under Case I



Range of Parameters (β , λ and θ)

Figure 6: Plot for Population III under Case II

5.2 Conclusion

In this article, we proposed three new ratio estimators of the population mean in two-phase sampling when the population mean of the auxiliary variable is unknown. However, the MSE of the proposed estimators were derived using first order approximation of the Taylor series method. Also, conditions for efficiency of the proposed estimators over some related existing estimators were derived and established. An empirical study is also carried out to evaluate the performance of the proposed estimators over some related existing estimators under two phase sampling using three populations. In the three populations the study variable and the auxiliary variable are positively correlated. However, in table 1, table 2 and table 3, we can clearly see that the mean squared error of the proposed estimators $T_1^{(d)}$, $T_2^{(d)}$ and $T_3^{(d)}$ respectively are smaller than the other estimators discussed. Hence, the proposed estimators may be preferred over the existing estimators.

Hence, using the plots from the three populations for both cases, we can conclude that the third proposed estimator among the proposed estimators produce a better result using their MSE as compared to the other two estimators

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