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# THE SUBSEMIGROUP GENERATED BY NILPOTENTS IN THE SEMIGROUP OF PARTIAL ONE-TO-ONE ORDER-PRESERVING CONTRACTION MAPPINGS

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***Abstract***

***Let $X_n = \{1, 2, \dots, n\}$. A partial one-to-one mapping $\alpha$ from $X_n$ to itself is called order-preserving if $x \le y \Rightarrow x\alpha \le y\alpha$ for all $x, y$ in $X_n$ and is called a contraction mapping if $|x\alpha - y\alpha| \le |x - y|$ for all $x, y$ in $X_n$. Let $OCI_n$ be the semigroup of all partial one-to-one order-preserving contraction mappings on $X_n$. In this paper, we obtained the subsemigroup generated by the nilpotent elements in $OCI_n$.***

## 1. Introduction and Preliminaries

If a finite semmigroup $S$ contains zero, then it contains nipotents, and so it is natural to ask for a description of the subsemigroup of $S$ generated by all nilpotents of $S$. In 1987, Gomes and Howie [**1**], and Sullivan [**2**] independently initiated the study of nilpotent generated subsemigroups of the semigroups of mappings on the set $X_n$ by considering $I_n$, the symmetric inverse semigroup and $P_n$, the semigroup of all partial mappings on $X_n$ respectively. In [**3**], and [**4**] Garba considered $IO_n$, the semigroup of all partial one-one order-preserving mappings and $PO_n$, semigroup of patial order-preserving mappings on $X_n$ respectively. Let

$OCI_n = \{\alpha \in IO_n : (\forall x, y \in dom(\alpha)) \ |x\alpha \le y\alpha| \le |x - y|\}$

a semigroup of partial one-to-one order-preserving contraction mappings. The Green's relations in $OCI_n$ have been characterised in [**5**]. Let $N$ be set a set of all nilpotents in $OCI_n$, and $\langle N \rangle$ the subsemigroup of $OCI_n$ generated by $N$. In section 2, we give a characterisation of the elements of $\langle N \rangle$.

For $1 \le i \le r$. An element $\alpha$ in $IO_n$ or $OCI_n$ is defined by

$\alpha = \begin{pmatrix} a_1 \ a_2 \dots a_r \\ b_1 \ b_2 \dots b_r \end{pmatrix}$

where $a_i$, $b_i \in X_n$ for $i = 1, 2, \dots, r$. We now state some existing results that shall be used in the subsequent section. Following [**3**], we say that $\alpha$ in $IO_n$ has an *upper jump* of length $k$ (a *lower jump* of length $k$) if there exists an $i$ such that

$a_{i+1} = a_i + k + 1$ ($b_{i+1} = b_i + k + 1$).

If $a_i = k + 1$ ($b_i = k + 1$) and $k \ge 1$, we say also that $\alpha$ has an upper jump of length $k$ (a lower jump of length $k$).

**Theorem 1.1** [**3**] For $n \ge 2$. Let $\alpha \in IO_n$. Then $\alpha$ is not a product of order-preserving nilpotents if and only if $\alpha$ satisfies the following:

(1) $a_1 = 1$, $a_r = n$ and all upper jumps are of length 1,

(2) $b_1 = 1$, $b_r = n$ and all lower jumps are of length 1

**Theorem 1.1** [**3**] Let $N$ be the set of all nilpotents in $IO_n$, $\langle N \rangle$ the subsemigroup of $IO_n$ generated by the nilpotents, and $\Delta(\langle N \rangle)$ the unique $k$ for which

$\langle N \rangle = N \cup N^2 \cup \dots \cup N^k, \langle N \rangle \ne N \cup N^2 \cup \dots \cup N^{k-1}$.

Then $\Delta(\langle N \rangle) = 3$ for all $n \ge 3$

**Lemma 1.3** [**5**] Let $\alpha$ be in $IO_n$. Then $\alpha$ is a contraction if and only if

$b_{i+1} - b_i \le a_{i+1} - a_i$ for each $1 \le i \le r - 1$

**Proposition 1.4** [**6**] Let $\alpha$ and $\beta$ be partial mappings.Then

$dom(\alpha.\beta) = (im\alpha \cap dom\beta)\alpha^{-1}$,

$im(\alpha.\beta) = (im\alpha \cap dom\beta)\beta$,

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and
$(\forall x \in dom(\alpha.\beta))\ \ x(\alpha.\beta) = (x\alpha)\beta.$

## 2. The Nilpotent Generated Subsemigroup

An element $\alpha \in OCI_n$ is called nilpotents if $\alpha^k = 0$ for some $k \geq 1$. First, we give in this investigation a characterisation of nilpotent elements in $OCI_n$.

**Lemma 2.1** Let $\alpha$ be in $OCI_n$. Then $\alpha$ is a contraction nilpotent if and only if $x\alpha \neq x$ for every $x \in dom(\alpha)$

**Proof.** If $\alpha = \emptyset$(empty map). Then the result is trivial. If $\alpha \neq \emptyset$. Then $\alpha$ cannot be nilpotent if $x\alpha = x$ for some $x \in dom(\alpha)$ for if $x\alpha = x$ for $x \in dom(\alpha)$ then $x = x\alpha = x\alpha^2 = \cdots$ Thus $\alpha^n \neq \emptyset$ for all$n \in N$. Hence $\alpha$ cannot be a contraction nilpotent.

Conversely, suppose that $x\alpha \neq x$ for all $x \in dom(\alpha)$ then $im(\alpha) \neq dom(\alpha)$ and so $dom(\alpha^2) \subset dom(\alpha)$. We now show that $dom(\alpha^{k+1}) \subset dom(\alpha^k)$ for all $k \in N$. Now by way of contradiction suppose that
$dom(\alpha^k) \neq \emptyset$ and $dom(\alpha^{k+1}) = dom(\alpha^k)$.

Then by proposition 1.4
$dom(\alpha^k) = dom(\alpha^{k+1}) = dom(\alpha.\alpha^k) = (im(\alpha) \cap dom\alpha^k)\alpha^{-1}$
$\Rightarrow (dom(\alpha^k))\alpha = im(\alpha) \cap dom(\alpha^k)$ (2.2)
Since $\alpha$ is injective, $| dom(\alpha^k) | = | im(\alpha) \cap dom(\alpha^k) |$ since $n$ is finite $dom(\alpha^k) \subseteq im(\alpha)$. Then
$dom(\alpha^k) = im(\alpha) \cap dom(\alpha^k)$ (2.3)
From (2.2) and (2.3) we have
$im(\alpha^k) = (dom(\alpha^k))\alpha = im(\alpha) \cap dom(\alpha^k) = dom(\alpha^k)$
which implies $im(\alpha^k) = dom(\alpha^k)$. Since $\alpha$ is an order-preserving contraction so is $\alpha^k$, and so $x\alpha^k = x$ for all $x \in dom(\alpha^k)$. Now fix $x' \in dom(\alpha^k)$ such that $x'\alpha^k = x'$, then since $\alpha$ is an order-preserving contraction and $dom(\alpha^k) \subseteq dom(\alpha)$ we have $x'\alpha^{k+1} = x'$. And so,
$x' = x'\alpha^{k+1} = x'\alpha^k.\alpha = x'\alpha$
Therefore, there exists at least one $x'$ in $dom(\alpha)$ such that $x'\alpha = x'$ . This is contrary to the earlier hypothesis that $x\alpha \neq x$ for all $x \in dom(\alpha)$. Thus we have a proper inclusion
$\ldots \subset dom(\alpha^{k+1}) \subset dom(\alpha^k) \subset \cdots \subset dom(\alpha)$
which implies there exists an $m \geq 1$ such that $dom(\alpha^m) = \emptyset$. That is, $\alpha^m = 0$.

**Definition 2.4** Let $\alpha \in IO_n$. For $1 \leq i \leq r-1$, we define the length between $a_i$ and $a_{i+1,}$ as the number of missing points between $a_i$ and $a_{i+1}$ denoted by $\rho^i(\alpha)$ and the length between $b_i$ and $b_{i+1}$ as the number of missing points between $b_i$ and $b_{i+1}$ denoted by $\rho_i(\alpha)$. Let $\rho^{i^*}(\alpha)$ denotes any $\rho^i(\alpha) \geq 2$ whose length is atleast 2 greater than the corresponding $\rho_i(\alpha)$. We define $m(\rho(\alpha))$ as
$m(\rho(\alpha)) = max\{\rho^1(\alpha), \rho^2(\alpha), \ldots, \rho^{r-1}(\alpha)\}$
For example, let
$\alpha = \begin{pmatrix} 1 & 4 & 5 & 8 & 12 \\ 3 & 5 & 6 & 9 & 11 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 3 & 5 & 9 & 13 & 20 & 23 \\ 3 & 5 & 6 & 9 & 11 & 17 & 18 \end{pmatrix}$ Then
$\rho^1(\alpha) = 2,\ \rho^2(\alpha) = 0,\ \rho^3(\alpha) = 2,\ \rho^4(\alpha) = 3,\ \rho_1(\alpha) = 1,\ \rho_2(\alpha) = 0,\ \rho_3(\alpha) = 2,\ \rho_4(\alpha) = 1,\ m(\rho(\alpha)) = 3$
and
$\rho^1(\beta) = 1,\ \rho^2(\beta) = 1,\ \rho^3(\beta) = 3,\ \rho^{4^*}(\beta) = 3,\ \rho^5(\beta) = 6,\ \rho^{6^*}(\alpha) = 2,\ \rho_1(\beta) = 1,\ \rho_2(\beta) = 0,\ \rho_3(\beta) = 2,\ \rho_4(\beta) = 1,\ \rho_5(\beta) = 5,\ \rho_6(\beta) = 0,\ m(\rho(\alpha)) = 6,\ m(\rho(\beta)) = 6$

**Lemma 2.5** Let $\alpha$ be in $OCI_n$. If $\alpha$ satisfies any of the following:
(i) $a_1 = 1, a_r \neq n, b_1 \neq 1, b_r = n$
(ii) $a_1 \neq 1, a_r = n, b_1 = 1, b_r \neq n$
then $\alpha$ is a contraction nilpotent.

**Proof.** Suppose that $\alpha \in OCI_n$ and satisfies (i). Let's assume by way of contradiction that $\alpha$ is not a contraction nilpotent. Then by lemma 2.1 there exists at least an $i$ $(1 \leq i \leq r)$ such that $a_i = b_i$ for $a_i \in dom\alpha$, $b_i \in im\alpha$. Since $a_r < b_r$ we have
$b_r - b_i > a_r - a_i$ (2.6)
Since $\alpha$ is order-preserving we have
$| b_r - b_i | = b_r - b_i, | a_r - a_i | = a_r - a_i$ (2.7)
And so, from (2.6) and (2.7) we have
$| b_r - b_i | > | a_r - a_i |$

where $a_i, a_r \in dom\alpha$, $b_i, b_r \in im\alpha$ for some $i$ $(1 < i < r)$. Then by definition $\alpha$ is not a contraction. This is a contradiction. Thus, $\alpha$ must be a contraction nilpotent.
Suppose that $\alpha$ satisfies (ii). Since $\alpha$ is a contraction then for all $i$ $(1 \le i \le r-1)$ we have
$\rho_i(\alpha) \le \rho^i(\alpha)$
and so, $a_1 > b_1$, $a_r > b_r$ and $\alpha$ being order-preserving implies
$a_1 > b_1$, $a_{i+1} > b_{i+1}$,…, $a_r > b_r$ (2.8)
It is clear from (2.8) that $a_i\alpha \ne a_i$, $a_i \in dom\alpha$ for all $i$ $(1 \le i \le r)$. Thus by lemma 2.1 $\alpha$ is a contraction nilpotent.

**Lemma 2.9** Let α be in $IO_n$. Then for $1 \le i \le r-1$,
$\rho^i(\alpha)+1 = a_{i+1} - a_i$ and $\rho_i(\alpha) + 1 = b_{i+1} - b_i$
**Proof.** Since $\alpha$ is order-preserving it is clear that $a_{i+1} - a_i > 0$ and $b_{i+1} - b_i > 0$, and by definition 2.4 the result follows.
We give the following useful remark:
**Remark 2.10** Let $\alpha \in IO_n$ have an upper jump of length $k$ $a_{i+1} = a_i+k+1$ for some $(1 \le i \le r)$. Then $a_{i+1} - a_i = k+1$. And by lemma 2.9 we have $\rho^i(\alpha) = k$. This implies that the definition of upper jump between $a_i$ and $a_{i+1}$ in [**3**] and the definition of $\rho^i(\alpha)$ in definition 2.4 are equivalent and so, the two can be used interchangeably if need arises.

**Lemma 2.11** Let $\alpha$ be in $IO_n$. If $\rho_i(\alpha) > \rho^i(\alpha)$ for some $i$ $(1 \le i \le r-1)$, then α is not a contraction.
**Proof.** Suppose that $\rho_i(\alpha) > \rho^i(\alpha)$ for some $i$ $(1 \le i \le r-1)$. Since $\alpha$ is order-preserving, it is clear that for each $i$ $(1 \le i \le r-1)$ we have
$| a_{i+1} - a_i | = a_{i+1} - a_i$ (2.12)
$| a_{i+1}\alpha - a_i\alpha | = | b_{i+1} - b_i | = b_{i+1} - b_i$ (2.13)
Since $\rho_i(\alpha) > \rho^i(\alpha)$ for some $i$ $(1 \le i \le r-1)$
then by (2.12), (2.13) and lemma 2.9 we have
$| a_{i+1}\alpha - a_i\alpha | = b_{i+1} - b_i = \rho_i(\alpha) + 1 > \rho^i(\alpha)+1 = a_{i+1} - a_i = | a_{i+1} - a_i |$
Where $a_i, a_{i+1} \in dom(a)$ for some $i$ $(1 \le i \le r-1)$. Then by definition $\alpha$ is not a contraction, hence the result.
**Lemma 2.14** Let α be in $OCI_n$. If $\alpha$ is such that $b_1 = 1$, $b_r = n$. Then $a_i = b_i$ for all $i$ $(1 \le i \le r)$
**Proof.** Suppose $\alpha \in \mathrm{OCI_n}$ is such that $b_1 = 1$, $b_r = n$. Then if $r = 2$ the result is obvious. Suppose that $r > 2$. By way of contradiction let α be such that there exists at least an
$i$ $(1 \le i \le r)$ such that $a_i \ne b_i$, $a_i \in dom\alpha$, $b_i \in im\alpha$. Then for $b_1 = 1$, $b_r = n$, $\alpha$ must have a case where $\rho_i(\alpha) > \rho^i(\alpha)$ for some $i$ $(1 \le i \le r-1)$ and so, since $\alpha$ is order-preserving by lemma 2.11 it is not a contraction. This is a contradiction. Hence we must have $a_i = b_i$ for all $i$ $(1 \le i \le r)$.
**Theorem 2.15** For $n \ge 2$. Let $\alpha \in OCI_n$ Then $\alpha$ is not expressible as a product of contraction nilpotents if and only if it satisfies any of the following:
(1) $a_1 = 1$, $a_r = n$ and $\rho^{i^*}(\alpha)$ does not exist,
(2) $b_1 = 1$, $b_r = n$

**Proof.** Suppose that $\alpha$ satisfies neither (1) nor (2). We shall consider four different cases.
**Case 1.** $a_1 \ne 1$, $b_1 \ne 1$. We look for a set $A = \{c_1, c_2, \dots, c_r\}$ such that $c_i < c_{i+1}$ for
$1 \le i \le r-1$ and $a_i \ne c_i$, $c_i \ne \boldsymbol{b_i}$, for all $i$. Now define

$$c_i = \begin{cases} max\{a_i, b_i\} + 1, if\ a_r \ne n, b_r \ne n \\ min\{a_i, b_i\} - 1, if\ a_r = n\ or\ b_r = n\ or\ \ a_r = b_r = n \end{cases}$$

Since $a_i < a_{i+1}$, $b_i < b_{i+1}$ for $1 \le i \le r-1$ then
$max\{a_i, b_i\} < max\{a_{i+1}, b_{i+1}\}, min\{a_i, b_i\} < min\{a_{i+1}, b_{i+1}\}$
which implies
$max\{a_i, b_i\} + 1 < max\{a_{i+1}, b_{i+1}\} + 1, min\{a_i, b_i\} - 1 < min\{a_{i+1}, b_{i+1}\} - 1.$
And so, for all $i$ $(1 \le i \le r-1)$ we have $c_i < c_{i+1}$. From the definition of $c_i$, it is clear that $a_i \ne \boldsymbol{c_i}$, $c_i \ne \boldsymbol{b_i}$, for all $i$. Thus, $\alpha$ is expressible as product of order-preserving nilpotents, that is,

$$\alpha = \begin{pmatrix} \alpha_1\ \alpha_2 \dots \alpha_r \\ c_1\ c_2 \dots c_r \end{pmatrix} \begin{pmatrix} c_1\ c_2 \dots c_r \\ b_1\ b_2 \dots b_r \end{pmatrix} = n_1 n_2.$$

Since $\alpha$ is an order-preserving contraction then by the definition of $c_i$ it is clear that for $(1 \le i \le r-1)$,
$c_{i+1} - c_i \le a_{i+1} - a_i$, $b_{i+1} - b_i \le c_{i+1} - c_i$.

And so, by lemma 1.3 $n_1$ and $n_2$ are contractions. Thus $\alpha$ is expressible as a product of contraction nilpotents.

**Case 2.** $a_1 = 1$, $b_1 \neq 1$. There are two subcases to be considered here.
**Case 2.1.** $a_r \neq n$. We require a set $A = \{c_1, c_2, \dots, c_r\}$ such that for $1 \leq i \leq r-1$ we will have $c_i < c_{i+1}$ and $a_i \neq c_i$, $c_i \neq b_i$, for all $i$. Now if $b_r = n$ then by lemma 2.5(i) $\alpha$ is a contraction nilpotent. Thus there is nothing to prove, and so we consider $b_r \neq n$. We now define
$c_i = max\{a_i, b_i\} + 1$.
Since $a_i < a_{i+1}$, $b_i < b_{i+1}$ for $i$ $(1 \leq i \leq r-1)$ then
$max\{a_i, b_i\} < max\{a_{i+1}, b_{i+1}\}$
which implies
$max\{a_i, b_i\} + 1 < max\{a_{i+1}, b_{i+1}\} + 1$
Therefore, for all $i$ $(1 \leq i \leq r-1)$ we have $c_i < c_{i+1}$. And by the definition of $c_i$ it is easy to see that $a_i \neq c_i$, $c_i \neq b_i$, for all $i$. Thus $\alpha$ is expressible as a product of order-preserving nilpotents, that is,
$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \dots \alpha_r \\ c_1 & c_2 \dots c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 \dots c_r \\ b_1 & b_2 \dots b_r \end{pmatrix} = n_1 n_2$.
Since $\alpha$ is an order-preserving contraction then by the definition of $c_i$ it is clear that for $i$ $(1 \leq i \leq r-1)$,
$c_{i+1} - c_i \leq a_{i+1} - a_i$, $b_{i+1} - b_i \leq c_{i+1} - c_i$.
And so, by lemma 1.3 $n_1$ and $n_2$ are contractions. Thus $\alpha$ is expressible as a product of contraction nilpotents.

**Case 2.2**. $a_r = n$. This case gives rise to another two subcases.
**Case 2.2.1**. $b_r \neq n$. Here we require two sets $A = \{c_1, c_2, \dots, c_r\}$ and $B = \{d_1, d_2, \dots, d_r\}$ such that for all $i$ $(1 \leq i \leq r-1)$ $c_i < c_{i+1}$, $d_i < d_{i+1}$ and $a_i \neq c_i$, $c_i \neq d_i$, $d_i \neq b_i$, for all $i$. Now since $a_1 = 1$, $a_r = n$, then it is clear that $\rho^{i^*}(\alpha)$ exists. Suppose $\rho^{i^*}(\alpha)$ occurs between $a_l$ and $a_{l+1}$. Define $c_i = \begin{cases} a_l + 1, & if\ i \leq l \\ a_l - 1, & if\ i > l \end{cases}$ and $d_i = max\{a_i, b_i\} + 1$.
Then
$c_{l+1} = a_{l+1} - 1 \geq (a_l + 3) - 1 = a_l + 2 > c_l$.
And so for all $i$ $(1 \leq i \leq r-1)$ we have $c_i < c_{i+1}$. Since for $i$ $(1 \leq i \leq r-1)$,
$c_i \leq c_{i+1}$, $b_i \leq b_{i+1}$ then
$max\{c_i, b_i\} < max\{c_{i+1}, b_{i+1}\}$
which implies
$max\{c_i, b_i\} + 1 < max\{c_{i+1}, b_{i+1}\} + 1$
Therefore $d_i < d_{i+1}$ for all $i$ $(1 \leq i \leq r-1)$. And by the definitions of $c_i$ and $d_i$ we can easily see that $a_i \neq c_i$, $c_i \neq d_i$, $d_i \neq b_i$ for all $i$. Thus so $\alpha$ is expressible as a product of three order-preserving nilpotents. That is,
$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \dots \alpha_r \\ c_1 & c_2 \dots c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 \dots c_r \\ d_1 & d_2 \dots d_r \end{pmatrix} \begin{pmatrix} d_1 & d_2 \dots d_r \\ b_1 & b_2 \dots b_r \end{pmatrix} = n_1 n_2 n_3$.
Since $\alpha$ is an order-preserving contraction, then by the definitions of $c_i$ and $d_i$ it is clear that for all$i$ $(1 \leq i \leq r-1)$,
$c_{i+1} - c_i \leq a_{i+1} - a_i$, $d_{i+1} - d_i \leq c_{i+1} - c_i$, $b_{i+1} - b_i \leq d_{i+1} - d_i$
Then by lemma 1.3 $n_1$, $n_2$, and $n_3$ are contractions. Thus $\alpha$ is expressible as a product of contraction nilpotents.

**Case 2.2.2**. $b_r = n$. Again we require two sets
$A = \{c_1, c_2, \dots, c_r\}$ and $B = \{d_1, d_2, \dots, d_r\}$ such that for all $i$ $(1 \leq i \leq r-1)$
$c_i < c_{i+1}$, $d_i < d_{i+1}$ and $a_i \neq c_i$, $c_i \neq d_i$, $d_i \neq b_i$, for all $i$. Define
$c_i = \begin{cases} a_l + 1, & if\ i \leq l \\ a_l - 1, & if\ i > l \end{cases}$ and $d_i = min\{a_i, b_i\} - 1$.
Following a similar argument as in case 2.2.1 we see that,
$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \dots \alpha_r \\ c_1 & c_2 \dots c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 \dots c_r \\ d_1 & d_2 \dots d_r \end{pmatrix} \begin{pmatrix} d_1 & d_2 \dots d_r \\ b_1 & b_2 \dots b_r \end{pmatrix} = n_1 n_2 n_3$
and $n_1$, $n_2$, and $n_3$ are contractions.
**Case 3.** $a_1 \neq 1$, $b_1 = 1$. If $a_r = n$ then by lemma 2.5(ii) $\alpha$ is a contraction nilpotent. Thus there is nothing to prove, and so we consider the case where $a_r \neq n$. Here we require a set $A = \{c_1, c_2, \dots, c_r\}$ such that for all $i$ $(1 \leq i \leq r-1)$ $c_i < c_{i+1}$ and $a_i \neq c_i$, $c_i \neq b_i$ for all $i$. Define

$c_i = max\{a_i, b_i\} + 1$

Following the same argument as in case 2.1 we see that

$$\alpha = \begin{pmatrix} \alpha_1\ \alpha_2 \ldots \alpha_r \\ c_1\ c_2 \ldots c_r \end{pmatrix} \begin{pmatrix} c_1\ c_2 \ldots c_r \\ b_1\ b_2 \ldots b_r \end{pmatrix} = n_1 n_2$$

and $n_1$ and $n_2$ are contractions.

**Case 4**. $a_1 = 1$, $b_1 = 1$. This case gives rise to another two subcases.

**Case 4.1**. $a_r \neq n$. We look for a set $A = \{c_1, c_2, \ldots, c_r\}$ such that for all $i$ $(1 \leq i \leq r-1)$ $c_i < c_{i+1}$ and $a_i \neq c_i$, $c_i \neq b_i$ for all $i$. Define

$c_i = max\{a_i, b_i\} + 1$

Following the same argument as in case 2.1 we see that

$$\alpha = \begin{pmatrix} \alpha_1\ \alpha_2 \ldots \alpha_r \\ c_1\ c_2 \ldots c_r \end{pmatrix} \begin{pmatrix} c_1\ c_2 \ldots c_r \\ b_1\ b_2 \ldots b_r \end{pmatrix} = n_1 n_2$$

and $n_1$ and $n_2$ are contractions.

**Case 4.2**. $a_r = n$. We look for two sets $A = \{c_1, c_2, \ldots, c_r\}$ and $B = \{d_1, d_2, \ldots, d_r\}$ such that for all $i$ $(1 \leq i \leq r-1)$, $c_i < c_{i+1}$, $d_i < d_{i+1}$ and $a_i \neq c_i$, $c_i \neq d_i$, $d_i \neq b_i$ for all $i$. Since $a_1 = 1$, $a_r = n$, then clearly $\rho^{i^*}(\alpha)$ exists. Suppose $\rho^{i^*}(\alpha)$ occurs between $a_l$ and $a_{l+1}$. Define

$$c_i = \begin{cases} a_l + 1, & if\ i \leq l \\ a_l - 1, & if\ i > l \end{cases} \quad \text{and} \quad d_i = max\{a_i, b_i\} + 1.$$

Following the same argument as in case 2.2.1 we see that

$$\alpha = \begin{pmatrix} \alpha_1\ \alpha_2 \ldots \alpha_r \\ c_1\ c_2 \ldots c_r \end{pmatrix} \begin{pmatrix} c_1\ c_2 \ldots c_r \\ d_1\ d_2 \ldots d_r \end{pmatrix} \begin{pmatrix} d_1\ d_2 \ldots d_r \\ b_1\ b_2 \ldots b_r \end{pmatrix} = n_1 n_2 n_3$$

and $n_1$, $n_2$, and $n_3$ are contractions.

Conversely, suppose that $\alpha$ satisfies (1) and $m(\rho(\alpha)) \leq 1$. Then $\alpha$ is such that $a_1 = 1$, $a_r = n$ and $\rho^i(\alpha) \leq 1$ for all $i$ $(1 \leq i \leq r-1)$. But $\alpha \in OCI_n \subset IO_n$, and so by remark 2.10 and theorem 1.1 $\alpha$ is expressible as a product of neither order-preserving nilpotents nor contraction nilpotents. Suppose that $\alpha$ satisfies (1) and $m(\rho(\alpha)) \geq 2$. Now since $\alpha \in OCI_n \subset IO_n$ and $\alpha$ is such that $m(\rho(\alpha)) \geq 2$, then we have $\rho^i(\alpha) \geq 2$ for some $i$ $(1 \leq i \leq r-1)$, and so by remark 2.10 and theorem 1.1 $\alpha$ is expressible as a product of order-preserving nilpotents. Thus by theorem 1.2 either $\alpha$ is a product of two order-preserving nilpotents or a product of three order-preserving nilpotents. Suppose that $\alpha$ is a product of two order-preserving nilpotents, that is,

$$\alpha = \begin{pmatrix} \alpha_1\ \alpha_2 \ldots \alpha_r \\ c_1\ c_2 \ldots c_r \end{pmatrix} \begin{pmatrix} c_1\ c_2 \ldots c_r \\ b_1\ b_2 \ldots b_r \end{pmatrix} = n_1 n_2$$

and the set $\{c_1, c_2, \ldots, c_r\}$ is defined by

$$c_i = \begin{cases} a_l + 1, & if\ i \leq l \\ a_l - 1, & if\ i > l \end{cases}$$

where $\rho^i(\alpha) \geq 2$ occurs between $a_l$ and $a_{l+1}$. Then, clearly we have $a_l < c_l$ and $c_{l+1} < a_{l+1}$. And it is easy to see that $a_{l+1} - a_l$ is at least 2 greater than $c_{l+1} - c_l$, for $c_l$ is atleast 1 greater than $a_l$ and $a_{l+1}$ is atleast 1 greater than $c_{l+1}$. So using lemma 2.9 $\rho^i(n_1) + 1$ is atleast 2 greater than $\rho_i(n_1)$ +1 which implies $\rho^i(n_1)$ is at least 2 greater than $\rho_i(n_1)$ and so, $\rho^{i^*}(n_1)$ exists in $n_1$ by definition 2.4. So, since $\rho^{i^*}(\alpha)$ does not exist in $\alpha$, it follows that $b_{l+1} - b_l > c_{l+1} - c_l$ in $n_2$. Then by lemma 2.9 we have

$\rho_i(n_2) + 1 = b_{l+1} - b_l > c_{l+1} - c_l = \rho^i(n_2) + 1$

$\Rightarrow \rho_i(n_2) > \rho^i(n_2)$ for some $i$ $(1 \leq i \leq r-1)$.

Thus by lemma 2.11 $n_2$ is not a contraction. Suppose that the set $\{c_1, c_2, \ldots, c_r\}$ is defined otherwise. Then since $a_1 = 1$, $a_r = n$ and $\rho^{i^*}(\alpha)$ does not exist, it implies $\rho^i(\alpha)$ is at most 1 greater than $\rho_i(\alpha)$ for all $i$ $(1 \leq i \leq r-1)$. Thus, for some $i$ $(1 \leq i \leq r-1)$ we must have either

$\rho_i(n_1) > \rho^i(n_1)$, $\rho_i(n_2) > \rho^i(n_2)$ or $c_r \geq (n+1) \notin X_n$

But since $\alpha$ is expressible as a product of order-preserving nilpotents the case $c_r \geq (n+1) \notin X_n$ does not exist. So, the only possible case is either

$\rho_i(n_1) > \rho^i(n_1)$ or $\rho_i(n_2) > \rho^i(n_2)$

for some $i$ $(1 \leq i \leq r-1)$. Then by lemma 2.11 $n_1$ or $n_2$ is not a contractions as the case may be. Suppose now that $\alpha$ is a product of three order-preserving nilpotents, that is,

$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \ldots \alpha_r \\ c_1 & c_2 \ldots c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 \ldots c_r \\ d_1 & d_2 \ldots d_r \end{pmatrix} \begin{pmatrix} d_1 & d_2 \ldots d_r \\ b_1 & b_2 \ldots b_r \end{pmatrix} = n_1 n_2 n_3.$
and the set $\{c_1, c_2, \ldots, c_r\}$ is defined by
$c_i = \begin{cases} a_l + 1, \ if\ i \le l \\ a_l - 1, \ if\ i > l \end{cases}$
where $\rho^i(\alpha) \ge 2$ occurs between $a_l$ and $a_{l+1}$. Thus, we have $a_l < c_l$ and $c_{l+1} < a_{l+1}$. And so, $a_{l+1} - a_l$ is at least 2 greater than $c_{l+1} - c_l$. So using lemma 2.9 $\rho^i(n_1) + 1$ is at least 2 greater than $\rho_i(n_1)$ +1 which implies $\rho^i(n_1)$ is at least 2 greater than $\rho_i(n_1)$ and so $\rho^{i^*}(n_1)$ exists in $n_1$ by definition 2.4. Since $\rho^{i^*}(\alpha)$ does not exist in $\alpha$, then it follows that
$b_{l+1} - b_l > c_{l+1} - c_l,$
which implies
$c_{l+1} - c_l < b_{l+1} - b_l$ (2.16)
We shall consider three cases
Case 1. $c_{l+1} - c_l < d_{l+1} - d_l$. Then by lemma 2.9
$\rho_i(n_2) + 1 = c_{l+1} - c_l > d_{l+1} - d_l = \rho^i(n_2) + 1$
$\Rightarrow \rho_i(n_2) > \rho^i(n_2)$ for some $i$ $(1 \le i \le r-1)$.
Thus by lemma 2.11 $n_2$ is not a contraction
Case 2. $d_{l+1} - d_l < c_{l+1} - c_l$. Then by (2.16) we have $d_{l+1} - d_l < b_{l+1} - b_l$.
And by lemma 2.9
$\rho_i(n_3) + 1 = d_{l+1} - d_l > b_{l+1} - b_l = \rho^i(n_2) + 1$
$\Rightarrow \rho_i(n_3) > \rho^i(n_3)$ for some $i$ $(1 \le i \le r-1)$.
Then by lemma 2.11 $n_3$ is not a contraction.
Case 3. $d_{l+1} - d_l = c_{l+1} - c_l$. Applying (2.16) we have $d_{l+1} - d_l < b_{l+1} - b_l$.
Then it follows from Case 2 that $n_3$ is not a contraction.
Suppose that the set $\{c_1, c_2, \ldots, c_r\}$ is defined otherwise. Then $\rho^i(\alpha)$ is at most 1 greater than $\rho_i(\alpha)$ for all $i$ $(1 \le i \le r-1)$ since $\rho^{i^*}(\alpha)$ does not exist. Thus, for $a_1 = 1$, $a_r = n$ and $n_1$ a nilpotent we have either
$c_{i+1} - c_i < b_{i+1} - b_i$ or $c_r \ge (n+1) \notin X_n$
for some $i$ $(1 \le i \le r-1)$. But since $\alpha$ is expressible as a product of order-preserving nilpotents the case $c_r \ge (n+1) \notin X_n$ does not exist. So, only the following case is possible:
$c_{i+1} - c_i < b_{i+1} - b_i$ (2.17)
for some $i$ $(1 \le i \le r-1)$. We shall again consider three cases
Case 1. $c_{i+1} - c_l < d_{i+1} - d_i$ for some $i (1 \le i \le r-1)$.

Then by lemma 2.9
$\rho_i(n_2) + 1 = c_{i+1} - c_i > d_{i+1} - d_i = \rho^i(n_2) + 1$
$\Rightarrow \rho_i(n_2) > \rho^i(n_2)$ for some $i$ $(1 \le i \le r-1)$.
Then by lemma 2.11 $n_2$ is not a contraction.
Case 2. $d_{i+1} - d_i < c_{i+1} - c_i$ for some $i$ $(1 \le i \le r-1)$.
Then applying (2.17) we have $d_{i+1} - d_i < b_{i+1} - b_i$ for some $i$ $(1 \le i \le r-1)$,
and by lemma 2.9
$\rho_i(n_3) + 1 = d_{i+1} - d_i > b_{i+1} - b_l = \rho^i(n_2) + 1$
$\Rightarrow \rho_i(n_3) > \rho^i(n_3)$ for some $i$ $(1 \le i \le r-1)$.
Then by lemma 2.11 $n_3$ is not a contraction.
Case 3. $d_{i+1} - d_i = c_{i+1} - c_l$ for some $i$ $(1 \le i \le r-1)$.
Applying (2.17) we ha $d_{l+1} - d_l < b_{l+1} - b_l$ for some $i$ $(1 \le i \le r-1)$. Then it follows from Case 2 that $n_3$ is not a contraction.
Suppose now that $\alpha$ satisfies (2) and $m(\rho(\alpha)) \le 1$. Then by lemma 2.14 $a_i = b_i$ for all $(1 \le i \le r)$ and so $a_1 = 1$, $a_r = n$. But $\alpha \in OCI_n \subset IO_n$ and $m(\rho(\alpha)) \le 1$ implies $\rho^i(\alpha) \le 1$ for all $(1 \le i \le r)$ then by remark 2.10 and theorem 1.1 $\alpha$ is expressible as a product of neither order-preserving nilpotents nor contraction nilpotents. Suppose that $\alpha$ satisfies (2) and $m(\rho(\alpha)) \ge 2$. If $\alpha$ satisfies (2) then by lemma 2.14 $a_i = b_i$ for all $i$ $(1 \le i \le r)$ and so $a_1 = 1$, $a_r = n$. Then since

$\alpha \in OCI_n \subset IO_n$ is such that $m(\rho(\alpha)) \geq 2$ which implies $\rho^i(\alpha) \geq 2$ for some $i$ $(1 \leq i \leq r)$, by remark 2.10 and theorem 1.1 $\alpha$ is expressible as a product of order-preserving nilpotents. Thus $\alpha$ is expressible as a product of either two order-preserving nilpotents or three order-preserving nilpotents by theorem 1.2. We now show that $\alpha$ is expressible as a product of at most two order-preserving nilpotents. Since $\alpha$ is expressible as a product of two order-preserving nilpotents, there must exist a set $\{c_1, c_2, \ldots, c_r\}$ where $c_i < c_{i+1}$ for all $i$ $(1 \leq i \leq r-1)$ such that the mapping $\begin{pmatrix} \alpha_1\ \alpha_2 \ldots \alpha_r \\ c_1\ c_2 \ldots c_r \end{pmatrix}$ is an order-preserving nilpotent. Then the mapping $\begin{pmatrix} c_1\ c_2 \ldots c_r \\ b_1\ b_2 \ldots b_r \end{pmatrix}$ is also an order-preserving nilpotent since by lemma 2.14 $a_i = b_i$ for all $i$ $(1 \leq i \leq r)$**.** Let $n_1 = \begin{pmatrix} \alpha_1\ \alpha_2 \ldots \alpha_r \\ c_1\ c_2 \ldots c_r \end{pmatrix}$, $n_2 = \begin{pmatrix} c_1\ c_2 \ldots c_r \\ b_1\ b_2 \ldots b_r \end{pmatrix}$

Since $n_1, n_2 \in IO_n$ and $im(n_1 = dom(n_2)$, then we can write,

$$\alpha = \begin{pmatrix} \alpha_1\ \alpha_2 \ldots \alpha_r \\ c_1\ c_2 \ldots c_r \end{pmatrix} \begin{pmatrix} c_1\ c_2 \ldots c_r \\ b_1\ b_2 \ldots b_r \end{pmatrix}$$

Thus $\alpha$ is expressible as a product of at most two order-preserving nilpotents. Suppose that the set $\{c_1, c_2, \ldots, c_r\}$ is define by

$$c_i = \begin{cases} a_l + 1, & if\ i \leq l \\ a_l - 1, & if\ i > l \end{cases}$$

where $\rho^i(\alpha) \geq 2$ occurs between $a_l$ and $a_{l+1}$. Then we have $a_l < c_l$ and $c_{l+1} < a_{l+1}$ and so $a_{l+1} - a_l$ is at least 2 greater than $c_{l+1} - c_l$ which implies $c_{l+1} - c_l < a_{l+1} - a_l$. Then $c_{l+1} - c_l < b_{l+1} - b_l$ since $a_i = b_i$ for all $\boldsymbol{i}$ $(1 \leq i \leq r)$. By lemma 2.9 we have

$\rho^i(n_2) + 1 \ = c_{l+1} - c_l < b_{l+1} - b_l = \rho_i(n_1) + 1$

$\Rightarrow \rho_i(n_2) > \rho^i(n_2)$ for some $i$ $(1 \leq i \leq r-1)$.

Then by lemma 2.11 $n_2$ is not a contraction. Suppose that the set $\{c_1, c_2, \ldots, c_r\}$ is defined otherwise. Then since by lemma 2.14 $\alpha$ is such that $a_i = b_i$ for all $\boldsymbol{i}$ $(1 \leq i \leq r)$, we must have either

$\rho_i(n_1) < \rho^i(n_1)$ or $c_r \geq (n+1) \notin X_n$ for some $i$ $(1 \leq i \leq r-1)$.

But since $\alpha$ is expressible as a product of order-preserving nilpotents the case $c_r \geq (n+1) \notin X_n$ does not exist. So, the only possible case is

$\rho_i(n_1) < \rho^i(n_1)$ for some $i$ $(1 \leq i \leq r-1)$

This implies $\rho_i(n_2) > \rho^i(n_2)$ for some $i$ $(1 \leq i \leq r-1)$ since $a_i = b_i$ for all $i$ $(1 \leq i \leq r)$. Then by lemma 2.11 $n_1$ or $n_2$ not contractions as the case may be. Hence $\alpha$ is not expressible as a product of contraction nilpotents if it satisfies (1) or (2).

**Corollary 2.18** For $n \geq 2$. Let $\alpha \in OCI_n$**.** Then $\alpha$ is expressible as a product of contraction nilpotents if and only if:

(1) $a_1 \neq 1$, $b_1 \neq 1$;

(2) $a_1 = 1$, $b_1 \neq 1$, $a_r \neq n$, $b_r \neq n$;

(3) $a_1 = 1$, $b_1 \neq 1$, $a_r = n$, $b_r \neq n$ and $\rho^{i^*}(\alpha)$ exists;

(4) $a_1 = 1$, $b_1 \neq 1$, $a_r = n$, $b_r = n$ and $\rho^{i^*}(\alpha)$ exists;

(5) $a_1 \neq 1$, $b_1 = 1$;

(6) $a_1 = 1$, $b_1 = 1, a_r \neq n$;

(7) $a_1 = 1$, $b_1 = 1, a_r = n$ and $\rho^{i^*}(\alpha)$ exists;

(8) $a_1 \neq 1$, $b_1 \neq 1$, $a_r \neq n$, $b_r \neq n$.

**Remark 2.20** From remark 2.19 and thereom 1.1 it is easy to see that for $n \geq 2$ and $\alpha \in OCI_n$ is such that $m(\rho(\alpha)) \geq 2$ and $\rho^{i^*}(\alpha)$ does not exist, then $\alpha$ is expressible as a product of order-preserving nilpotents where at least one of the nilpotents is not a contraction if and only if it satisfies any of the following:

(1) $a_1 = 1$, $a_r = n$,

(2) $b_1 = 1$, $b_r = n$

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