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TVERBERG'S THEOREM ON CONVEX FUZZY SETS

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Abstract

In this paper, by considering convex fuzzy sets in R^d and extension of Carath´eodory's theorem on convex setsto convex fuzzy sets, we prove colorful Carath´eodory's theorem on convex fuzzy sets using fuzzy points. Furthermore, we extend the classical theorem of Tverberg on convex sets to convex fuzzy sets using fuzzy points.

***Keywords*: ** Fuzzy point, Convex fuzzy sets, Carath´ eodory's theorem, Colorful Carath´eodory's theorem, Tverberg's theorem

1. Introduction

Tverberg's theorem was first stated by H. Tverberg [1] in 1966 and it is among the important theorems both in discrete geometry and combinatorial convexity. The theorem states that, given a set X of at least $v_1,\cdots,v_m$points in R^d, m≥ (r−1)(d+1)+1, then, Xcan be partitioned into r disjoint subsets $X_1,\cdots,X_r$such that$\bigcap_{j=1}^{r} conv(v_i \in X_r) \neq \emptyset$.This is a generalisation of Radon's theorem [2] that needs only two disjoint subsets whose convex hulls intersect in at least one point instead of r subsets in Tverberg's theorem.

In 2009, Maruyama [3] established (lattice) L-valued Radon's, Helly's (finite and infinite) theorems using α-cut operators. He conjectured that, as a generalisation of L-valued Radon's theorem, L-fuzzy version of Tverberg's theorem can be obtained by exploiting the idea of fuzzy version of Carath$\acute{e}$odory's theorem established by Feiyue [4] in 1991. Furthermore, in 2018, Sangodapo and Ajayi [5] obtained the extensions of Radon's and Helly's theorems on convex sets to convex fuzzy sets using fuzzy points.

Motivated by the conjecture of Maruyama [3] and the earlier work of Sangodapo and Ajayi [5], we extend the classical theorem of Tverberg on convex sets to convex fuzzy sets in R^d using fuzzy points, by first extending the colorful Carath´ eodory's theorem to convex fuzzy sets.

2. Preliminaries

Throughout this paper, Idenotes the unit interval [0,1] and R^d denotes d-dimensional Euclidean space.The following definitions and results are from [4,6,7].

Definition 2.1. A *fuzzy set*σ on R^d is described by its membership function$\sigma : R^d \to I$.

Definition 2.2.A *fuzzy point* on R^d is a fuzzy set $a_\alpha$defined as,

$$a_\alpha(y) = \begin{cases} \alpha, \text{ if } y = a \\ \\ 0, \; otherwise \end{cases}$$

for all y∈R^d, where $a \in$R^d is called its support point and αits value.

Definition 2.3. Let σbe a fuzzy set on R^d, the *t-level* subset of σ, denoted by $\sigma_t$is defined as

$$\sigma_t = \left\{a_\alpha \in \mathbb{R}^d : \sigma(a_\alpha) \geq t\right\}.$$

Definition 2.4. Let σbe a fuzzy set on R^d,$a_{\alpha_1}, \cdots, a_{\alpha_d}$, be its fuzzy points,$\lambda_i$are non-negative,$\sum_i^d \lambda_i = 1$ then

$a_\alpha = \lambda_1 a_{\alpha_1} + \cdots + \lambda_d a_{\alpha_d}$

is called a *fuzzy convex combination* of the fuzzy points $a_{\alpha_i} \in \sigma$.

Definition 2.5.A fuzzy set σ on R^d is said to be a *convex fuzzy set* if

$\sigma[\lambda a_\alpha + (1-\lambda) b_\beta] \geq \sigma(a_\alpha) \wedge \sigma(b_\beta)$

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for every a_α, $b_\beta \in \sigma$, $\lambda \in$I.

Definition 2.5 can also be defined in terms of Definition 2.4 as follows:

Definition 2.6. Let σbe a fuzzy set on R^d. Then, $\sigma$is called a *convex fuzzy set* if for all $a_{\alpha_1}, \cdots, a_{\alpha_d} \in \sigma$, $\lambda_1,\cdots,\lambda_d \in$Isuch that

$\sigma(\lambda_1 a a_{\alpha_1} + \cdots + \lambda_d a_{\alpha_d}) \geq \sigma(a_{\alpha_1}) \wedge \cdots \wedge \sigma(a_{\alpha_d})$

and $\sum_i^d \lambda_i = 1$

Definition 2.7. Let σbe a fuzzy set on R^d. Then, the *convex fuzzy hull* of $\sigma$is defined as the set of all fuzzy convex combinations of fuzzy points in σ. That is;

$$conv(\sigma)_f = \left\{ a_{\alpha_1}, \cdots, a_{\alpha_p} \in \sigma : \exists\ \lambda_i \in \mathbb{I},\ \sum_{i=1}^{p} \lambda_i = 1,\ a_\alpha = \sum_{i=1}^{p} \lambda_i a_{\alpha_i} \right\}$$

which is the smallest convex fuzzy set containing σ.

Theorem 2.1.*Let σ be a fuzzy set on* R^d*. Then σ is a convex fuzzy set if and only if it contains all fuzzy convex combinations of its fuzzy points.*

Theorem 2.2.*Let σ be a fuzzy set on* R^d*. Then,$conv(\sigma)_f$ is the set of all fuzzy convex combinations of fuzzy points in σ. That is;*

$$conv(\sigma)_f = \left\{ a_{\alpha_1}, \cdots, a_{\alpha_n} \in \sigma : \exists\ \lambda_i \in \mathbb{I},\ \sum_{i=1}^{n} \lambda_i = 1,\ a_\alpha = \sum_{i=1}^{p} \lambda_i a_{\alpha_i} \right\}$$

Thus, $conv(\sigma)_f$is the smallest convex fuzzy set containing σ.

3. Some Classical Theorems on Convex Sets in Euclidean Space, R^d and their Fuzzy Versions

This section presents the statements of both the classical theorems on convex sets and their fuzzy versions in R^d. The classical ones are as follows:

Theorem 3.1. (Radon's Theorem) [2]*: Let X be a subset of* R^d *having at least d* + 2 *points. Then, X can be partitioned into two disjoint subsets X_1, $X_2 \subset X$ such that*

conv$(X_1)\cap$conv$(X_2) \neq \emptyset$

Theorem 3.2. (Helly's Theorem) [8]*: For a family $K_1, K_2, \cdots, K_n$ $n \geq d+1$ of convex sets in* R^d*, if every d* + 1 *member have a common point, then all of members have a common point.*

Theorem 3.3. (Carathéodory's Theorem) [9]*: If Y is a set of n points in* R^d *and y* ∈*conv*(Y)*. Then, there is a subset X of Y such that x*∈*conv*(*X*) *with* |*X*| = *d*+1.

Theorem 3.4. (Colorful Carathéodory's Theorem) [10]: *Let $A_1,\cdots,A_{d+1}$be d* + 1 *sets in* R^d*. Suppose that*

x ∈*conv*$(A_1) \cap \cdots \cap$ *conv*(A_{d+1}).

Then, there is a set

$A = \{a_1 \in A_1, \cdots, a_{d+1} \in A_{d+1}\}$

such that x ∈ conv$(a_1, \cdots, a_{d+1})$

In order to state the fuzzy version of the theorems, we first define the notion of fuzzy Radon partition and fuzzy Radon point.

Definition 3.1.Given a fuzzy set φon R^d of (*d* + 2)- fuzzy points, $a_1,\cdots,a_{d+2}$, there is a fuzzy partition$\varphi = \tau_1 \vee \tau_2$if

$conv(\tau_1)_f \wedge conv(\tau_2)_f \neq \emptyset$.

Then, such fuzzy partition is said to be *fuzzy Radon partition*. Every fuzzy point in the intersection is said to be fuzzy Radon point.

Theorem 3.5. (Radon's Theorem on Fuzzy Convex Sets) [5] *Let φ be a fuzzy set of* (*d* + 2) *fuzzy points on* R^d*. Then, any t-level subset of φ,$\varphi_t$of at least d* + 2 *fuzzy points* (| φ_t|≥*d* + 2) *in* R^d *has a fuzzy Radon partition.*

Theorem 3.6. (Helly's Theorem on Fuzzy Convex Sets) [5] *Let $\sigma_1,\cdots,\sigma_r$in* R^n*, be convex fuzzy sets with r* ≥*n* + 1*. Assume that every n* + 1 *convex fuzzy sets has a nonempty intersection. Then, the intersection of all the convex fuzzy sets σ_r, k* = 1,⋯ ,*r is nonempty.*

Theorem 3.7. (Carathéodory's Theorem) [4] *Let A* = $A_0 \cup A_1$*be a set of fuzzy points and fuzzy directions, and let*C(*A*, *n* + 1)*denote the set of all convex combinations of n* + 1 *or fewer elements in A. Then, $conv(A)_f$*~C(*A*, *n* + 1).

4. Results

In this section, we establish the extension of the classical theorem of Tverberg on convex sets to convex fuzzy sets. We shall first prove the fuzzy version of colorful Carathéodory's theorem.

Theorem 4.1. *Let $\sigma_1,\cdots,\sigma_{d+1}$be d* + 1 *fuzzy sets in* R^d *and suppose that*

$a_\alpha \in conv(\sigma_1)_f \cap \cdots \cap conv(\sigma_1)_f$

Then, there exists fuzzy points
$a_{\alpha_1} \in \sigma_1, a_{\alpha_1} \in \sigma_2, \cdots, a_{\alpha_{d+1}} \in \sigma_{g+1}$
such that
$a_\alpha \in conv(a_{\alpha_1}, \cdots, a_{\alpha_{d+1}})_f$.
Proof. Since $a_\alpha \in conv(\sigma_1)_f, \cdots, a_\alpha \in conv(\sigma_{d+1})_f$ we can assume that each fuzzy set is finite by *Theorem 3.4.* Consider the fuzzy convex hull of the form
$\sigma(a_{\alpha_1}, \cdots, a_{\alpha_{d+1}}) := conv(a_{\alpha_1}, \cdots, a_{\alpha_{d+1}})_f$.
Since there are only finitely many convex hulls of equation (1), we can assume that
$A := \sigma(a_{\alpha_1}, \cdots, a_{\alpha_{d+1}})$
is as close to a_α as possible. If $a_\alpha \in A$, the result follows.
Assume by contradiction that $a_\alpha \notin A$, this means that a_α cannot be expressed as a convex combination of the fuzzy points $a_{\alpha_1}, \cdots, a_{\alpha_{d+1}}$ and let b_β be the fuzzy point in A nearest to a_α. Let D be the open ball fixed at a_α having radius $r = d(a_\alpha, b_\beta)$. The interior of D and A is disjoint. Consider the tangent hyperplane to the ball at b_β. Because of the convexity of A and the open half-space containing a_α, they must also be disjoint. By *Theorem 3.3* for R^{d-1}, b_β can be expressed as a convex combination of at most d of a_{α_i}. If it is an affinely dependent system, then the statement is trivial. Otherwise b_β cannot be in the interior of A because it is the closest fuzzy point to a_α from A. Therefore, at least one of the fuzzy points from $a_{\alpha_1}, \cdots, a_{\alpha_{d+1}}$ must have a zero coefficient in the convex combination representing b_β. Suppose that, the first fuzzy point a_{α_1} is not used, then a_{α_1} can be replaced by any fuzzy point of σ_1 in such a way that the distance between a_α and A is not increased, because of the condition that a_α is in the convex combination of σ_1 there is a fuzzy point $(a_{\alpha 1})^* \in \sigma_1$ in the same open half-space as a_α. Thus, the distance from a_α to the $conv((a_{\alpha_1}),^* \cdots, a_{\alpha_{d+1}})_f$ is smaller than that to $A := \sigma(a_{\alpha_1}, \cdots, a_{\alpha_{d+1}}) := conv(a_{\alpha_1}, \cdots, a_{\alpha_{d+1}})_f$. This is a contradiction. Therefore,
$a_\alpha = \lambda_1 a a_{\alpha_1} + \cdots + \lambda_d a_{\alpha_{g+1}}$.
Hence,
$a_\alpha \in conv(a_{\alpha_1}, \cdots, a_{\alpha_{d+1}})_f$.
Next, we define fuzzy Tverberg partition and fuzzy Tverberg point.

Definition 4.1. Given a fuzzy set σ on R^d of $n \geq (r-1)(d+1)+1$ fuzzy points, $a_{\alpha_1}, \cdots, a_{\alpha_n}$, there is a fuzzy partition $\sigma = \sigma_1 \vee \cdots \vee \sigma_t$ such that their fuzzy convex hulls have a fuzzy point in common, that is, if $\bigwedge_{i=1}^{r} conv(\sigma_i)_f \neq \emptyset$ then, the fuzzy partition is called a *fuzzy Tverberg partition*. Every fuzzy point in this nonempty intersection is called a *fuzzy Tverberg point*.

Theorem 4.2. *For a fuzzy set σ on R^d of $(r-1)(d+1)+1$ fuzzy points, $a_{\alpha_1}, \cdots, a_{\alpha_n}$, $n \geq (r-1)(d+1)+1$. Then, any t-level subset, σ_t of at least n fuzzy points, $|\sigma_t| \geq n \in R^d$ has a fuzzy Tverberg partition.*

Proof. Suppose that n = (d + 1) (r - 1) +1. Consider the fuzzy points $a_{\alpha_1}, \cdots, a_{\alpha_n} \in R^{d+1}$ by adding a new coordinate to each fuzzy point such that the sum of the coordinates is one. Consider another set of fuzzy points $b_{\beta_1}, \cdots, b_{\beta_r}$ such that
$$b_{\beta_1} + \cdots + b_{\beta_r} = 0 \qquad (1)$$
is the only linear relation among b_{β_i}'s.
Define the tensor product $a_{\alpha_i} \otimes b_{\beta_j}$ for $i = 1, \cdots, d+1$, $j = r-1, \cdots, r$ as a fuzzy matrix
$$a_{\alpha_i} \otimes b_{\beta_j} = \begin{bmatrix} a_{\alpha_i}^1 b_{\beta_j}^1 & a_{\alpha_i}^1 b_{\beta_j}^2 & \cdots & a_{\alpha_i}^1 b_{\beta_j}^{r-1} \\ a_{\alpha_i}^2 b_{\beta_j}^1 & a_{\alpha_i}^2 b_{\beta_j}^2 & \cdots & a_{\alpha_i}^2 b_{\beta_j}^{r-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{\alpha_i}^{d+1} b_{\beta_j}^1 & a_{\alpha_i}^{d+1} b_{\beta_j}^2 & \cdots & a_{\alpha_i}^{d+1} b_{\beta_j}^{r-1} \end{bmatrix}_{(d+1)(r-1)}.$$
This can be considered as an element in R^m where m = (d + 1) (r - 1).
Let
$\Phi_1 = \{a_{\alpha_1} \otimes b_{\beta_j} | j = 1, \cdots, r\}$
$\Phi_2 = \{a_{\alpha_2} \otimes b_{\beta_j} | j = 1, \cdots, r\}$
$\vdots$
$\Phi_m = \{a_n \otimes b_{\beta_j} | j = 1, \cdots, r\}$.
Note that, 0 is in the fuzzy convex hull of every Φ_i that is

$0 = \sum_{j=1}^{r} a_{\alpha_i} \otimes b_{\beta_j} = a_{\alpha_i} \otimes \sum_{j=1}^{r} b_{\beta_j}$

Indeed, it is the sum of the fuzzy points in Φ_j.Since $n = m + 1$, by *Theorem 4.1,*

$0 = \sum_{k=1}^{n} \lambda_k d_{\delta_k}, \sum_{k=1}^{n} \lambda_k = 1$ (2)

where $d_{\delta_k} \in \Phi_k$ for any index k.Thus, the fuzzy Tverberg partition is realized in the following way:

$\sigma_1 = \{a_{\alpha_k} \mid d_{\delta_k} = a_{\alpha_k} \otimes b_{\beta_1}\}$

$\sigma_1 = \{a_{\alpha_k} \mid d_{\delta_k} = a_{\alpha_k} \otimes b_{\beta_2}\}$

$\vdots$

$\sigma_r = \{a_{\alpha_k} \mid d_{\delta_k} = a_{\alpha_k} \otimes b_{\beta_r}\}$

equation (2) can be rewritten as

$0 = \sum_{a_{\alpha_k} \in \sigma_1} \lambda_k a_{\alpha_k} \otimes b_{\beta_1} + \sum_{a_{\alpha_k} \in \sigma_2} \lambda_k a_{\alpha_k} \otimes b_{\beta_2} + \cdots + \sum_{a_{\alpha_k} \in \sigma_r} \lambda_k a_{\alpha_k} \otimes b_{\beta_r}$ (3)

Equation (3) is a linear relation between $(d + 1)$ by $(r - 1)$ matrices; for the first row, we have

$0 = (\sum_{a^1_{\alpha_k} \in \sigma_1} \lambda_k a^1_{\alpha_k})\ b_{\beta_1} + (\sum_{a^1_{\alpha_k} \in \sigma_2} \lambda_k a^1_{\alpha_k})\ b_{\beta_2} \cdots + (\sum_{a^1_{\alpha_k} \in \sigma_r} \lambda_k a^1_{\alpha_k})\ b_{\beta_r}$

second row, we have

$0 = (\sum_{a^2_{\alpha_k} \in \sigma_1} \lambda_k a^2_{\alpha_k})\ b_{\beta_1} + (\sum_{a^2_{\alpha_k} \in \sigma_2} \lambda_k a^2_{\alpha_k})\ b_{\beta_2} \cdots + (\sum_{a^2_{\alpha_k} \in \sigma_r} \lambda_k a^2_{\alpha_k})\ b_{\beta_r}$

For the $d + 1$ row, we have

$0 = (\sum_{a^{d+1}_{\alpha_k} \in \sigma_1} \lambda_k a^{d+1}_{\alpha_k})\ b_{\beta_1} + (\sum_{a^{a+1}_{\alpha_k} \in \sigma_2} \lambda_k a^{d+1}_{\alpha_k})\ b_{\beta_2} \cdots + (\sum_{a^{d+1}_{\alpha_k} \in \sigma_r} \lambda_k a^{d+1}_{\alpha_k})\ b_{\beta_r}$

This means that, by equation (1)

$a^j_{\alpha_j} \coloneqq \sum_{a_{\alpha_k} \in \sigma_1} \lambda_k a^j_{\alpha_k} = \sum_{a_{\alpha_k} \in \sigma_2} \lambda_k a^j_{\alpha_k} = \cdots = \sum_{a_{\alpha_k} \in \sigma_r} \lambda_k a^j_{\alpha_k}$

for $j = 1, 2, \cdots r$.

Therefore,

$a_\alpha := \sum_{a_{\alpha_k} \in \sigma_1} \lambda_k a_{\alpha_k} = \sum_{a_{\alpha_k} \in \sigma_2} \lambda_k a_{\alpha_k} = \cdots = \sum_{a_{\alpha_k} \in \sigma_r} \lambda_k a_{\alpha_k}$

Since the sums of the coordinates of $a_{\alpha_1}, \cdots, a_{\alpha_{d+1}}$ are 1, we have that

$a_{\alpha_1} + \cdots + a_{\alpha_{d+1}} =: \lambda = \sum_{a_{\alpha_k} \in \sigma_1} \lambda_k = \sum_{a_{\alpha_k} \in \sigma_2} \lambda_k = \cdots = \sum_{a_{\alpha_k} \in \sigma_r} \lambda_k$.

On the other hand,

$1 = \sum_{a_{\alpha_k} \in \sigma_1} \lambda_k + \sum_{a_{\alpha_k} \in \sigma_2} \lambda_k + \cdots + \sum_{a_{\alpha_k} \in \sigma_r} \lambda_k$

and consequently,

Define

$ra_\alpha := \sum_{a_{\alpha_k} \in \sigma_1} (r\lambda_k)\ a_{\alpha_k} = \sum_{a_{\alpha_k} \in \sigma_2} (r\lambda_k)\ a_{\alpha_k} = \cdots = \sum_{a_{\alpha_k} \in \sigma_r} (r\lambda_k)\ a_{\alpha_k}$.

Hence,

$ra_\alpha \in conv(\sigma_1)_f \wedge \cdots \wedge conv(\sigma_r)_f$.

5. Conclusion

Tverberg's theorem is a core theorem in discrete geometry and convexity theory which claims that many points in d-dimensional Euclidean space can be partitioned into disjoint subsets with intersecting convex hulls. The extension of this classical theorem on convex sets to convex fuzzy sets was established in this paper using fuzzy points. In the process, weproved the colorful Carath´eodory's theorem on convex fuzzy sets also using fuzzy points.

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