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SEEMINGLY UNRELATED REGRESSION WITH DECOMPOSED VARIANCE-COVARIANCE MATRIX: A BAYESIAN APPROACH

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Abstract

This article considered estimation of seemingly unrelated regression equations when the variance-covariance is decomposed into upper and lower triangular matrices within the Bayesian context. The experiment was carried out using Gibbs sampling for sample sizes 50, 100, 200, 500 and 1000 replicated 10000 times in turn. The marginal and conditional posterior means tend to the true population parameters as sample size increases. The posterior standard deviations of the parameter estimates of the partitioned matrices were lower as the sample size increases. The estimates obtained using the upper triangular matrix are better than those obtained for the lower triangular matrix within the Bayesian context.

***Keywords*:** Bayesian Inference, Conditional posterior, Gibbs Sampling, Triangular Matrices, Seemingly Unrelated Regression.
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1.0 Introduction

The Seemingly Unrelated Regression (SUR) comprises multivariate regression model with a joint modelling system of equations [13]. It is a generalisation of the multivariate regression model to capture the effect of different covariates allowed in the regression equations while accounting for the correlated errors. The disturbances of several regression equations model contemporaneously correlated could account for unnoticeable or unquantifiable effects. Taking cognisance of such correlation leads to efficient estimates of the coefficients and standard errors [2]. The main motivation for use of SUR is to gain efficiency in estimation by combining information on different equations. Efficiency will be established if the disturbance terms are contemporaneously correlated and the explanatory variables in different equations are uncorrelated. When there is contemporaneous correlation of disturbances in the equation, definite gains are obtained for all sample sizes [1].

The SUR model has been studied extensively since it was popularised by [13]. Some other works on or involving the SUR model can be found in [4, 11, 12]. In [14], Bayesian inference in econometrics was popularised and SUR model within the context of Bayesian inference was described. There have been applications of Monte Carlo Markov Chain (MCMC) to SUR that has made the Bayesian estimation of SUR model more convenient and accessible [10, 14]. Other studies include [3, 6, 7, 10]. The Cholesky decomposition is a decomposition of a Hermitian positive definite matrix into the product of a lower triangular matrix and its conjugate transpose.

In this paper, we investigate the effects of decomposed variance-covariance matrix on the estimation of SUR model using the Bayesian (via MCMC) approach. The rest of the paper is organised as follows: Section 2 illustrates the theory behind the methodology followed by the design of the simulation experiment in section 3. Analysis and discussion of the results are presented in Sections 4 and 5, while Section 6 presented some concluding remarks provided.

2. Materials and Methods

2.1 The Model

Consider the SUR model of stacked system of m equations which can be expressed as a multiple linear regression model as follows:

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$$y_{nj} = X_{nj}\beta_j + \varepsilon_{ij}, \text{ i = 1, 2, …, m} \quad (1)$$

$$E[\varepsilon_i \varepsilon_j] = \{ \begin{matrix} \sigma_{ij}\, I(i \neq j) \\ \sigma_i^2\, I(i = j) \end{matrix}$$

where y_{nj} and ε_j are the $n \times 1$ vectors, X_{nj} is the $n \times p_j$ predetermined matrix and β_j is the p_j-dimensional vectors. The error terms in different equations are correlated and have different explanatory variables and variances. In matrix form, equation (1) can be expressed as:

$$\begin{pmatrix} y_{n1} \\ y_{n2} \\ \vdots \\ y_{nm} \end{pmatrix} = \begin{pmatrix} X_{n1} & 0 & \cdots & 0 \\ 0 & X_{n2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{nm} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{pmatrix} \quad (2)$$

That is:

$$y = X\beta + \varepsilon \quad (3)$$

where the distribution for ε is assumed to be distributed as $\varepsilon \sim N(0, \Sigma \otimes I)$; 0 is the zero matrix, $\otimes$ is the tensor product, Σ is the $m \times m$ matrix with the diagonal elements $\{\sigma_1^2, \cdots, \sigma_m^2\}$, and the off-diagonal ij^{th} elements are σ_{ij}. Ordinary Least Squares (OLS) estimator can be used to estimate the parameter in equation (1)

$$\hat{\beta}_{OLS} = (X'X)^{-1} X'y \quad (4)$$

A more efficient estimator is the Generalised Least Squares (GLS) estimator, which is used to estimate a $m \times m$ covariance matrix of the disturbances.

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}y) = (X'(\hat{\Sigma}^{-1} \otimes I_n)X)^{-1}(X'(\hat{\Sigma}^{-1} \otimes I_n)y) \quad (5)$$

2.2 Likelihood Function

The maximum likelihood estimates of β and ∑ are obtained by maximising the likelihood function:

$$f(Y_n / X_n, \beta, \Sigma) = \frac{1}{(2\pi)^{nm/2} |\Sigma|^{n/2}} \exp[-\frac{1}{2} \text{tr}\{R\Sigma^{-1}\}] \quad (6)$$

Where tr denotes the trace of matrix, $|\Sigma| = \det(\Sigma)$ and $R = (y_{ni} - X_{ni}\beta_i)^T (y_{nj} - X_{nj}\beta_j)$

2.3 The Prior

Priors express the general information about a parameter. The principle of indifference is the simplest and oldest rule for determining a non-informative prior. One of the most widely used non-informative priors is Jeffreys' invariant prior, which is proportional to the square root of the determinant of the Fisher information matrix given as:

$$f(\beta, \Sigma) = f(\beta) f(\Sigma) \propto |\Sigma|^{-\frac{m+1}{2}} \quad (7)$$

2.4 Joint Posterior

Applying Bayes' theorem to the prior pdf in (7) and the likelihood function in (6) yields the joint posterior pdf for β and ∑ given as:

$$f(\beta, \Sigma / Y_n, X_n) \propto f(Y_n / X_n, \beta, \Sigma) f(\beta, \Sigma) \quad (8)$$

$$\propto |\Sigma|^{-(n+m+1)/2} \exp[-\frac{1}{2} \text{tr}\{R\Sigma^{-1}\}] \quad (9)$$

However, the prior distribution gives both the analytical conditional posterior densities of β and ∑, and the analytical joint posterior density.

2.5 Conditional Posterior

It is obvious from the form of the joint posterior density function $f(\beta, \Sigma | Y_n, X_n)$ that the conditional posteriors $f(\beta | Y_n, X_n, \Sigma)$ and $f(\Sigma | Y_n, X_n, \beta)$ are:

$$f(\beta | Y_n, X_n, \Sigma) = N(\hat{\beta}, \hat{\Omega}); \quad (10)$$

$$\hat{\beta} = \{X_n^T (\Sigma^{-1} \otimes I) X_n\}^{-1} X_n^T (\Sigma^{-1} \otimes I) y_n,$$

$$\hat{\Omega} = (X_n^T (\Sigma^{-1} \otimes I) X_n)^{-1}; \text{ and}$$

$$f(\beta | Y_n, X_n, \beta) = IW(R, n) \quad (11)$$

Where IW(.,.) denotes the Inverse Wishart distribution.

2.6 Marginal Posterior

The marginal posterior pdf for β and ∑ are given as:

$$f(\beta|y) \propto |R|^{-\frac{n}{2}} \tag{12}$$

And

$$f(\Sigma|y) = |X'(\Sigma^{-1} \otimes I_n)|^{-\frac{1}{2}} |\Sigma|^{-\frac{(n+m+1)}{2}} \exp[-\frac{1}{2} tr(\hat{R}\Sigma^{-1})] \tag{13}$$

We used the Gibbs sampling by starting from an initial value β(0)and Σ(0), through the following steps:

i. Update the coefficient vector β(j) by drawing a new value from the conditional posterior density $f(\beta | Y_n, X_n, \Sigma^{j-1})$ in (10).

ii. Update Σ(j) by drawing a new value from the conditional posterior density $f(\Sigma | Y_n, X_n, \beta^j)$ in (11)

The process is then repeated a large number of times j = 1, ⋯, 10000. The initial part generated was discarded as being unrepresentative of the posterior distribution. The remaining samples are then used for the posterior inference.

3. The Design of the Simulation Experiment

Simulation design of four equations SUR model with correlated errors are given as

$$\begin{aligned} y_1 &= 0.4 + 1.5X_{11} + 0.7X_{12} + 1.2X_{13} + 0.8X_{14} + \varepsilon_1 \\ y_2 &= 0.5 + 0.7X_{21} + 1.1X_{22} - 1.4X_{23} + 1.8X_{24} + \varepsilon_2 \\ y_3 &= 0.7 + 1.5X_{31} - 2.2X_{32} + \varepsilon_3 \\ y_4 &= 0.6 - 1.5X_{41} + 1.2X_{42} + \varepsilon_4 \end{aligned} \tag{14}$$

The data series were generated using the following steps.

Step 1: The vectors of X's regressors were drawn from uniform distribution U(-4, 4).

Step 2: The sequences (ε_{i1}, ε_{i2}, ε_{i3}, ε_{i4}) were generated from mutually independent N(0,1) and transformed to ensure that the disturbance terms are contemporaneously correlated and distributed as N(0, Σ). Let = (ε_{i1}, ε_{i2}, ε_{i3}, ε_{i4}), where E(ε) = 0. The positive definite (4 × 4) variance-covariance matrix is defined by

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{bmatrix} = \begin{bmatrix} 1.0 & 0.7 & 0.6 & 0.9 \\ 0.7 & 1.0 & 0.7 & 0.9 \\ 0.6 & 0.7 & 1.0 & 0.6 \\ 0.9 & 0.9 & 0.6 & 1.0 \end{bmatrix} \tag{15}$$

Since Σ by definition is a positive matrix, there exist a non-singular triangular matrix P such that

$$\Sigma = PP' \tag{16}$$

The estimated variance-covariance matrix was obtained and partitioned through the Cholesky decomposition method into upper and lower non-singular triangular matrices such that

$$\hat{\Sigma} = PP'$$

$$PP' = \hat{\Sigma} = \begin{pmatrix} p_{11}^* & p_{12}^* & p_{13}^* & p_{14}^* \\ 0 & p_{22}^* & p_{23}^* & p_{24}^* \\ 0 & 0 & p_{33}^* & p_{34}^* \\ 0 & 0 & 0 & p_{44}^* \end{pmatrix} \begin{pmatrix} p_{11}^* & 0 & 0 & 0 \\ p_{21}^* & p_{22}^* & 0 & 0 \\ p_{31}^* & p_{32}^* & p_{33}^* & 0 \\ p_{41}^* & p_{42}^* & p_{43}^* & p_{44}^* \end{pmatrix} = \begin{pmatrix} 0.283 & -0.323 & 0.075 & 0.9 \\ 0 & 0.387 & 0.2 & 0.9 \\ 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.283 & 0 & 0 & 0 \\ -0.323 & 0.387 & 0 & 0 \\ 0.075 & 0.2 & 0.8 & 0 \\ 0.9 & 0.9 & 0.6 & 1 \end{pmatrix} \tag{17}$$

Step 3: We set the number of observations, n = 50, 100, 200, 500 and 1000; and chose the GLS estimates as the initial value of the parameters. The total number of MCMC iterations is chosen to be 10000. The posterior samples were generated using the Gibbs sampling approach.

4. Results

The performances of the lower and upper triangular matrices in SUR using the Gibbs sampler are assessed using the marginal posterior and conditional posterior means for β and Σ respectively. The marginal posterior means of β and Σ for the triangular matrices are presented in tables 1 and 9, and tables 2 and 10 respectively. The conditional posterior mean of β and Σ for the triangular matrices are presented in tables 3 and 11, and tables 4 and 12 respectively. The summary of the posterior means for β and Σ for the triangular matrices are presented in tables 5, 6, 13 and 14.

Tables 1 and 9 show the marginal posterior of β for upper and lower triangular matrices respectively. When n = 50, the marginal posterior are: 1.5004 and 1.5163, 0.7468 and 0.6623, 1.1532 and 1.2337, 0.7817 and 0.7801, 0.7191 and 0.6757, 1.0933 and 1.0847, -1.4472 and -1.3708, 0.7721 and 0.7955, 1.5584 and 1.5431, -2.2701 and -2.3104, -0.1273 and -0.0927, 0.0721 and 0.0678, -1.5831 and -1.5212, 1.2044 and 1.1039, -0.0442 and 0.0046 and 0.0304 and 0.0269. When n = 1000, the marginal posterior are: 1.5016 and 1.5024, 0.6997 and 0.6998, 1.1839 and 1.1815, 0.8067 and 0.8094, 0.7143 and 0.7222, 1.0900 and 1.0994, -1.4153 and -1.4144, 0.7932 and 0.7951, 1.5003 and 1.4993, -2.1699 and -2.1705, 0.0051 and 0.0089, -0.0156 and -0.0148, -1.4934 and -1.4962, 1.990 and 1.200, -0.0125 and -0.0220 and 0.0207 and 0.0256 respectively.

Tables 2 and 10 show the marginal posterior of Σ for upper and lower triangular matrix. At different sample sizes, the marginal posterior of σ_{11} for upper and lower triangular matrices are: 91.3649 and 89.4601, 95.3008 and 90.6577, 90.9053 and 86.5499, 83.9951 and 84.2386 and 88.2386 and 85.8934 respectively. At σ_{44}, the marginal posteriors are: 98.3797 and 96.0346, 100.7707 and 111.1439, 121.7306 and 122.1420, 104.8438 and 109.4006 and 104.0374 and 108.0358 respectively.

Tables 5 and 13 present the summary of the posterior means and convergence diagnostic test statistic of β for upper and lower triangular matrices at different sample sizes respectively. At n = 50, the posterior means are: 1.500 and 1.5159, 0.7549 and 0.6718, 1.1577 and 1.2385, 0.7734 and 0.7704, 0.7237 and 0.6804, 1.0985 and 1.0897, -1.4425 and -1.3662, 0.7844 and 0.8073, 1.5625 and 1.5480. The convergence statistic gives the following results: 1.9042 and 1.8802, 2.7394 and 2.7367, 1.8356 and 1.8173, -0.3789 and -0.3643, -1.0061 and -1.0062, -0.7994 and -0.7892, -1.8088 and -1.8153.

Tables 6 and 14 present the summary of the posterior means of β for upper and lower triangular matrices for different sample sizes. When n = 50, the posterior means are: 101.71 and 99.60, 14.749 and 7.222, -19.201 and -20.224, 69.17 and 49.839, 14.749 and 7.222, 143.82 and 137.74 The convergence statistic gives: 0.3410 and 0.4291, 0.5521 and 0.5639, -0.8212 and -0.3980, -0.0933 and 0.2333, 0.5521 and 0.5639, 1.0214 and 1.0025, 0.6539 and 0.5401.

Table 1: Marginal Posterior Pdf for β (Upper Triangular Matrix)

n	β_{11}	β_{12}	β_{13}	β_{14}	β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	β_{41}	β_{42}	β_{43}	β_{44}
50	1.5004	0.7468	1.1532	0.7817	0.7191	1.0933	-1.4472	0.7721	1.5584	-2.2701	-0.1273	0.0721	-1.5831	1.2044	-0.0442	0.0304
100	1.4294	0.6254	1.2601	0.8104	0.4891	0.9203	-1.4709	0.9515	1.5736	-2.1128	0.1355	0.0625	-1.3654	1.2333	0.0150	0.0273
200	1.4945	0.6572	1.2346	0.8381	0.7079	1.0903	-1.4080	0.7889	1.5047	-2.1653	-0.0510	-0.0547	-1.5541	1.1954	-0.0241	-0.0036
500	1.4713	0.7166	1.1810	0.7885	0.7002	1.0860	-1.3715	0.8033	1.4768	-2.1991	-0.0129	-0.0561	-1.4940	1.2101	-0.0172	0.0132
1000	1.5016	0.6997	1.1839	0.8067	0.7143	1.0900	-1.4153	0.7932	1.5003	-2.1699	0.0051	-0.0156	-1.4934	1.1990	-0.0125	0.0207

Table 2: Marginal Posterior Pdf for Σ (Upper Triangular Matrix)

n	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{21}	σ_{22}	σ_{23}	σ_{24}	σ_{31}	σ_{32}	σ_{33}	σ_{34}	σ_{41}	σ_{42}	σ_{43}	σ_{44}
50	91.3649	13.2620	-17.4509	62.1563	13.2620	129.5534	44.4267	51.6592	-17.4509	44.4267	199.0280	19.0825	62.1563	51.6592	19.0825	98.3797
100	95.3008	-21.6650	-68.9038	-2.7345	-21.6650	135.0925	-85.6147	-39.7339	-68.9038	-85.6147	184.6219	-6.6282	-2.7345	-39.7339	-6.6282	100.7707
200	90.9053	31.9526	58.0108	44.7892	31.9526	116.3517	-9.8997	33.0321	58.0108	-9.8997	192.1919	30.1246	44.7892	33.0320	30.1246	121.7306
500	83.9951	17.6386	7.3201	8.9595	17.6386	109.6828	-10.3971	5.2506	7.3201	-10.3971	197.0793	-6.9068	8.9595	5.2506	-6.9068	104.8438
1000	88.1870	4.3668	23.2655	22.8874	4.3668	116.7374	14.5248	-19.9988	23.2655	14.5248	197.2992	4.8273	22.8874	-19.9988	4.8273	104.0374

Table 3: Conditional Posterior Pdf for β (Upper Triangular Matrix)

n	β_{11}	β_{12}	β_{13}	β_{14}	β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	β_{41}	β_{42}	β_{43}	β_{44}
50	1.0164	0.4316	1.1941	0.4895	0.3156	1.5507	-1.2212	0.6913	2.4447	-1.2160	-1.0638	-0.6128	-2.4222	1.0130	0.4218	0.8486
100	1.3857	0.5972	1.0236	0.7215	-0.0018	0.8979	-1.3599	1.5428	1.7700	-2.2035	0.4365	0.3300	-1.2363	1.3447	-0.2742	0.3731
200	1.4106	0.4748	0.8296	0.7320	0.7122	0.9929	-1.4804	1.0191	1.0039	-2.5721	-0.4314	0.0364	-2.0292	1.5122	-0.3725	0.3075
500	1.6420	0.9014	1.3634	0.5871	0.7704	1.0166	-1.6172	0.7387	1.5531	-2.2831	0.0408	0.6390	-1.2904	1.3228	-0.1025	0.1825
1000	1.4194	0.7358	1.0026	0.6899	0.7137	0.9627	-1.3589	0.6236	1.6151	-2.4800	0.1241	-0.0548	-1.4357	1.0790	-0.0482	0.2477

Table 4: Conditional Posterior Pdf for Σ (Upper Triangular Matrix)

n	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{21}	σ_{22}	σ_{23}	σ_{24}	σ_{31}	σ_{32}	σ_{33}	σ_{34}	σ_{41}	σ_{42}	σ_{43}	σ_{44}
50	122.9684	-7.4269	-79.2194	74.5231	-7.4269	129.1995	44.7436	61.2490	-79.2194	44.7436	270.5602	-13.7210	74.5231	61.2490	-13.7210	115.4587
100	109.1533	-43.0029	-70.3375	14.3648	-43.0029	170.0093	-99.5972	-38.5792	-70.3375	-99.5972	203.1657	-18.2607	14.3648	-38.5792	-18.2607	96.0406
200	85.9788	28.4162	54.7221	43.2272	28.4162	122.4140	-20.6546	28.750	54.7221	-20.6546	167.7092	32.2611	43.2272	28.750	32.2611	115.3277
500	83.2864	29.1816	12.2320	6.7784	29.1816	113.5674	-14.1614	7.0253	12.2320	-14.1614	214.5322	-8.9962	6.7784	7.0253	-8.9962	103.0075
1000	87.1187	10.5901	21.9435	25.0956	10.5901	110.2133	16.1642	-13.6030	21.9435	16.1642	193.3752	2.1127	25.0956	-13.6030	2.1127	98.2908

Table 5: Posterior means for β and [5] convergence diagnostic test statistic (Upper Triangular Matrix).

Beta	β_{11}	β_{12}	β_{13}	β_{14}	β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	β_{41}	β_{42}	β_{43}	β_{44}
n = 50																
mean	1.500	0.7549	1.1577	0.7734	0.7237	1.0985	-1.4425	0.7844	1.5625	-2.2801	-0.1278	0.0691	-1.5827	1.2069	-0.0374	0.0465
CD	1.9042	2.7394	1.8356	-0.3789	-1.0061	-0.7994	-1.8088	-0.1171	0.2655	1.3371	0.5406	0.2128	-0.9329	-0.0742	0.3416	0.5437
n = 100																
mean	1.4341	0.6259	1.2635	0.8081	0.4901	0.9134	-1.4706	0.9592	1.5727	-2.115	0.1372	0.0629	-1.3642	1.2312	0.0199	0.0353
CD	0.9167	0.7462	0.3808	2.0046	0.1093	-1.0443	-0.2333	-1.6684	0.2599	0.3129	0.7398	-1.6674	0.0077	-0.0130	0.1207	0.0896
n = 200																
mean	1.4954	0.6576	1.2333	0.8389	0.7096	1.0887	-1.4086	0.7935	1.5012	-2.1653	-0.0504	-0.0514	-1.5489	1.1913	-0.0236	-0.0033
CD	0.3639	0.3606	0.1336	0.6248	-0.6477	-0.0217	1.4850	-1.1113	0.4664	0.7192	-1.8658	0.8513	-0.6936	0.4056	0.5271	0.3046
n = 500																
mean	0.9934	0.4845	0.7932	0.5348	0.4731	0.7315	-0.9226	0.5398	0.9905	-1.485	-0.0085	-0.0378	-1.005	0.8171	-0.0125	0.0067
CD	1.7933	1.8260	2.0075	1.7220	1.6316	1.7268	-1.7267	1.6842	1.7054	-1.8633	-1.6557	-3.8582	-1.8975	1.8100	-0.7604	1.1903
n = 1000																
mean	0.9934	0.4845	0.7932	0.5348	0.47314	0.7315	-0.9226	0.5398	0.9905	-1.485	-0.0085	-0.0378	-1.005	0.8171	-0.0125	0.0067
CD	0.4520	0.1162	-0.2907	-0.6731	-0.4587	1.2129	2.1017	-0.8622	1.5535	-0.1853	-1.5468	-0.7724	0.2585	-1.0570	1.2025	-1.2864

Table 6: Posterior means for Σ and [5] convergence diagnostic test statistic (Upper Triangular Matrix).

Sigma	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{21}	σ_{22}	σ_{23}	σ_{24}	σ_{31}	σ_{32}	σ_{33}	σ_{34}	σ_{41}	σ_{42}	σ_{43}	σ_{44}
n = 50																
mean	101.71	14.749	-19.201	69.17	14.749	143.82	49.74	57.53	-19.201	49.74	221.8	21.499	69.17	57.53	21.499	109.28
CD	0.3410	0.5521	-0.8212	-0.0933	0.5521	1.0214	0.6539	0.4588	-0.8212	0.6539	1.6993	-1.3808	-0.0933	0.4588	-1.3808	-1.0482
n = 100																
mean	100.52	-22.89	-72.49	-3.029	-22.89	142.28	-90.13	-41.812	-72.49	-90.13	194.1	-6.841	-3.029	-41.812	-6.841	106.06
CD	-0.2196	2.0822	-1.2694	-0.2812	2.0822	-1.7817	0.0792	0.0248	-1.2694	0.0792	0.5745	0.7834	-0.2812	0.0248	0.7834	0.5058
n = 200																
mean	93.17	32.8402	59.51	45.91	32.8402	119.46	-10.043	33.870	59.51	-10.043	197.3	31.09	45.91	33.870	31.09	124.80
CD	-0.1030	0.1836	0.7763	2.1171	0.1836	0.8468	-0.1607	1.7660	0.7763	-0.1607	1.0461	0.9900	2.1171	1.7660	0.9900	2.2620
n = 500																
mean	57.19	12.02	4.994	6.106	10.02	74.67	-7.003	3.609	4.994	-7.003	134.2	-4.572	6.106	3.609	-4.572	71.49
CD	1.8416	1.8319	2.9026	2.0641	1.8319	1.7561	-2.2525	2.9324	2.9026	-2.2525	1.8224	-2.7433	2.0641	2.9324	-2.7433	1.7908
n = 1000																
mean	88.71	4.562	23.37	23.07	4.562	117.3	14.835	-19.936	23.37	14.835	198.4	4.809	23.07	-19.936	4.809	104.47
CD	1.0625	-0.3494	1.5899	-1.2458	-0.3494	1.7160	-1.2551	-0.7154	1.5899	-1.2551	1.7926	-0.4669	-1.2458	-0.7154	-0.4669	-0.0785

Table 7: Standard Deviations (SDs) for β (Upper Triangular Matrix)

n	β_{11}	β_{12}	β_{13}	β_{14}	β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	β_{41}	β_{42}	β_{43}	β_{44}
50	0.4027	0.3923	0.3975	0.4111	0.6116	0.5412	0.5934	0.5774	0.9264	0.8136	0.7868	0.9177	0.4283	0.4274	0.4797	0.4653
100	0.2187	0.2163	0.2313	0.2444	0.2746	0.2688	0.2700	0.2731	0.2621	0.2843	0.2866	0.2815	0.3475	0.3539	0.3689	0.3389
200	0.2136	0.2256	0.2319	0.2428	0.3005	0.3114	0.2898	0.3153	0.3776	0.3614	0.3768	0.3855	0.2880	0.3201	0.3305	0.2951
500	0.1709	0.1758	0.1666	0.1679	0.2009	0.1985	0.1941	0.2024	0.2761	0.2678	0.2745	0.2799	0.1925	0.199	0.1988	0.1880
1000	0.1605	0.1298	0.1483	0.1339	0.1313	0.1454	0.1575	0.1342	0.1618	0.1877	0.1661	0.172	0.1604	0.1488	0.1202	0.1256

Table 8: Standard Deviations (SDs) for Σ (Upper Triangular Matrix)

N	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{21}	σ_{22}	σ_{23}	σ_{24}	σ_{31}	σ_{32}	σ_{33}	σ_{34}	σ_{41}	σ_{42}	σ_{43}	σ_{44}
50	21.8414	19.1829	23.2936	19.0238	19.1829	30.6789	27.2238	20.9519	23.2936	27.2238	46.9105	23.5199	19.0238	20.9519	23.5199	23.2379
100	14.4681	12.1760	15.3552	10.5838	12.1760	21.4428	20.0678	13.8396	15.3552	20.0678	27.8004	15.2814	10.5838	13.8396	15.2814	15.6084
200	9.9661	8.1464	11.0367	8.5742	8.1464	12.3724	11.0165	9.3081	11.0367	11.0165	20.2989	11.8238	8.5742	9.3081	11.8238	12.5754
500	5.4714	4.6190	6.0058	4.2998	4.6190	7.2671	6.5933	5.0356	6.0058	6.5933	12.6740	6.5976	4.2998	5.0356	6.5976	6.5320
1000	5.3404	3.4088	4.0432	3.5330	3.4088	6.8185	5.1985	3.5545	4.0432	5.1985	9.4315	4.5649	3.5334	3.5545	1.5649	1.6388

Table 9: Marginal Posterior Pdf for β (Lower Triangular Matrix)

N	β_{11}	β_{12}	β_{13}	β_{14}	β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	β_{41}	β_{42}	β_{43}	β_{44}
50	1.5163	0.6623	1.2337	0.7801	0.6757	1.0847	-1.3708	0.7955	1.5431	-2.3104	-0.0927	0.0678	-1.5212	1.1039	0.0046	0.0269
100	1.4945	0.7061	1.1909	0.78500	0.6434	1.1424	-1.3753	0.7354	1.5240	-2.1504	0.0619	0.0411	-1.4477	1.1905	0.0034	0.0443
200	1.4987	0.6806	1.1931	0.8034	0.6886	1.0879	-1.3948	0.7821	1.4870	-2.1290	-0.0181	-0.0098	-1.5211	1.2098	-0.0826	-0.0044
500	1.4940	0.7038	1.1966	0.7921	0.7153	1.0754	-1.4007	0.8174	1.4941	-2.2043	-0.01126	-0.0455	-1.5183	1.1742	-0.0404	0.0070
1000	1.5024	0.6998	1.1815	0.8094	0.7222	1.0994	-1.4144	0.7951	1.4993	-2.1705	0.0089	-0.0148	-1.4962	1.2000	-0.0220	0.0256

Table 10: Marginal Posterior pdf for Σ (Lower Triangular Matrix)

N	σ11	σ12	σ13	σ14	σ21	σ22	σ23	σ24	σ31	σ32	σ33	σ34	σ41	σ42	σ43	σ44
50	89.4601	6.5000	-18.3507	44.7263	6.5000	124.0593	41.6893	54.8917	-18.3507	41.6893	200.5814	16.7718	44.7263	54.8917	16.7718	96.0546
100	90.6577	-1.5721	7.2137	-5.7636	-1.5721	109.2565	-11.7895	-15.9379	7.2137	-11.7895	207.2113	63.1442	-5.7636	-15.9379	63.1442	111.1439
200	86.5499	25.3573	48.0956	39.2172	25.3573	114.0929	-23.5846	39.5283	48.0956	-23.5846	190.2084	44.9426	39.2172	39.5283	44.9426	122.1420
500	84.2386	20.4599	10.9195	4.7468	20.4599	112.1008	-12.5755	-4.8159	10.9195	-12.5755	198.9247	5.2477	4.7468	-4.8159	5.2477	109.4006
1000	85.8934	-0.8546	18.7347	17.0598	-0.8546	116.870	10.6932	-30.6749	18.7347	10.6932	196.8909	2.8065	17.0598	-30.6749	2.8065	108.0358

Table 11: Conditional Posterior pdf for β (Lower Triangular Matrix)

N	β_{11}	β_{12}	β_{13}	β_{14}	β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	β_{41}	β_{42}	β_{43}	β_{44}
50	0.9360	0.2922	1.2382	0.4672	0.2938	1.5244	-1.1640	0.7247	2.4481	-1.2261	-1.0636	-0.6204	-2.4670	0.8966	0.5444	0.9509
100	1.4140	0.6711	0.7028	0.7119	0.0354	1.1162	-1.1634	1.6246	1.8962	-2.3113	0.6220	0.5104	-1.2740	1.2660	-0.3807	0.5084
200	1.4089	0.4817	0.7650	0.6913	0.6983	0.9986	-1.4675	1.0031	0.9962	-2.5289	-0.3869	0.0672	-1.9799	1.5142	-0.4254	0.3015
500	1.1309	0.7053	0.8138	0.7635	1.0240	1.0237	-1.2824	0.9284	1.1697	-2.0057	-0.0106	-0.2756	-1.5487	1.4156	0.0291	0.3048
1000	1.4915	0.4446	1.0620	0.9248	0.8795	1.0831	-1.4215	0.9443	1.4795	-2.1299	0.3774	0.3073	-1.6969	1.2656	-0.0951	0.1001

Table 12: Conditional Posterior pdf for Σ (Lower Triangular Matrix)

N	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{21}	σ_{22}	σ_{23}	σ_{24}	σ_{31}	σ_{32}	σ_{33}	σ_{34}	σ_{41}	σ_{42}	σ_{43}	σ_{44}
50	120.1009	-18.5067	-78.6491	52.8236	-18.5067	125.0206	39.2865	65.1644	-78.6491	39.2865	274.4650	-16.2154	52.8236	65.1644	-16.2154	112.7300
100	90.7584	-11.8498	13.9941	8.4274	-11.8498	141.0753	-26.1837	-22.9480	13.9941	-26.1837	207.5887	48.0824	8.4274	-22.9480	48.0824	105.9268
200	81.8377	22.5967	46.8922	37.7845	22.5967	120.5958	-30.9169	34.6069	46.8922	-30.9169	167.6191	46.1930	37.7845	34.6069	46.1926	115.7174
500	91.0220	24.8708	11.6895	6.6692	24.8708	102.5765	-8.0923	-5.9030	11.6895	-8.0923	192.3267	9.3253	6.6692	-5.9030	9.3253	99.1566
1000	84.7738	4.6100	24.1194	14.1469	4.6103	111.4439	16.1961	-28.0950	24.1194	16.1961	205.1251	2.1976	14.1470	-28.0950	2.1976	112.2737

Table 13: Posterior means for β and [5] convergence diagnostic test statistic (Lower Triangular Matrix).

n = 50																
Beta	β_{11}	β_{12}	β_{13}	β_{14}	β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	β_{41}	β_{42}	β_{43}	β_{44}
mean	1.5159	0.6718	1.2385	0.7704	0.6804	1.0897	-1.3662	0.8073	1.5480	-2.3204	-0.0921	0.0649	-1.5204	1.1066	0.0120	0.0448
CD	1.8802	2.7367	1.8173	-0.3643	-1.0062	-0.7892	-1.8153	-0.1085	0.2548	1.3354	0.5300	0.2139	-0.8718	-0.1368	0.3938	0.5109
n = 100																
mean	1.5035	0.7073	1.1966	0.7801	0.6476	1.1297	-1.3731	0.7487	1.5205	-2.1529	0.0657	0.0428	-1.4465	1.1874	0.0093	0.0543
CD	1.1893	0.7074	0.3522	1.9850	-0.4576	-1.0359	-0.1766	-1.8093	0.5257	0.5990	0.8682	-1.7556	-0.0495	0.0367	0.0392	0.0752
n = 200																
mean	1.4996	0.6811	1.1918	0.8042	0.6902	1.0863	-1.3953	0.7866	1.4836	2.129	-0.0176	-0.00647	-1.5161	1.2058	0.0821	-0.0041
CD	0.3656	0.3363	0.1152	0.6251	-0.6572	0.00095	1.4870	-1.1033	0.4703	0.7308	-1.8542	0.8400	-0.6813	0.3909	0.5678	0.2721
n = 500																
mean	1.4979	0.7082	1.1965	0.7901	0.7157	1.0833	-1.3998	0.8207	1.4934	-2.204	-0.0143	-0.0429	-1.5138	1.1784	-0.0423	0.0100
CD	-1.8757	0.1862	-0.8294	0.8565	0.1915	-1.3510	0.0335	-0.5034	0.3336	-2.4286	0.0827	-0.8756	-0.6710	0.3610	0.7047	-4.1566
n = 1000																
mean	1.498	0.6989	1.1788	0.8100	0.7227	1.1003	-1.408	0.7987	1.4839	-2.171	-0.00083	-0.0234	-1.493	1.1978	-0.0249	0.0350
CD	0.0122	0.4024	0.8695	-1.1952	-0.2784	-0.2562	1.2282	-1.3187	1.4657	0.0624	0.1294	-1.2178	-0.2282	-1.7177	-0.1442	-1.2196

Table 14: Posterior means for Σ and [5] convergence diagnostic test statistic (Lower Triangular Matrix).

n = 50																
Sigma	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{21}	σ_{22}	σ_{23}	σ_{24}	σ_{31}	σ_{32}	σ_{33}	σ_{34}	σ_{41}	σ_{42}	σ_{43}	σ_{44}
mean	99.60	7.222	-20.224	49.839	7.222	137.74	46.69	61.115	-20.224	46.69	223.6	18.930	49.839	61.115	18.930	106.69
CD	0.4291	0.5639	-0.3980	0.2333	0.5639	1.0025	0.5401	0.3619	-0.3980	0.5401	1.7382	-1.3595	0.2333	0.3619	-1.3595	-1.0482
n = 100																
mean	95.53	-1.814	7.406	-6.146	-1.814	115.10	-12.427	-16.686	7.406	-12.427	218.1	66.599	-6.146	-16.686	66.599	117.0
CD	0.0124	0.2842	0.3893	1.4800	0.2842	-2.2616	1.2854	0.8494	0.3893	1.2854	1.0327	0.9463	1.4800	0.8494	0.9463	0.5058
n = 200																
mean	88.70	26.059	49.310	40.18	26.059	117.10	-24.09	40.51	49.310	-24.09	195.3	46.2769	40.18	40.51	46.2769	125.22
CD	-0.4019	0.1912	0.8292	1.9801	0.1912	1.1040	-0.0564	1.7734	0.8292	-0.0564	1.1812	1.1731	1.9801	1.7734	1.1731	2.2620
n = 500																
mean	85.00	20.51	11.035	5.130	20.51	113.33	-12.890	-4.728	11.035	-12.890	201.2	5.608	5.130	-4.728	5.608	111.02
CD	-0.4023	-0.9267	-0.1629	-0.5189	-0.9267	-0.7316	0.8779	-0.6189	-0.1629	0.8779	-1.5813	-0.5417	-0.5189	-0.6189	-0.5417	0.1595
n = 1000																
mean	86.30	-0.7075	18.746	17.081	-0.7075	117.2	10.996	-30.76	18.746	10.996	198.5	2.7535	17.081	-30.76	2.7535	108.49
CD	1.9560	-0.2534	1.7958	1.0663	-0.2534	0.5450	-1.6046	0.5813	1.7958	-1.6046	-0.6754	0.4528	1.0663	0.5813	0.4528	-0.4373

Table 15: Standard Deviations (SDs) for β (Lower Triangular Matrix)

N	β_{11}	β_{12}	β_{13}	β_{14}	β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	β_{41}	β_{42}	β_{43}	β_{44}
50	0.4679	0.4589	0.4587	0.4792	0.5896	0.5183	0.5674	0.5531	0.9451	0.8280	0.8020	0.9354	0.4793	0.4832	0.5389	0.5197
100	0.4038	0.2440	0.4170	0.3225	0.3142	0.4155	0.4252	0.4188	0.4414	0.4472	0.3228	0.3945	0.6231	0.5329	0.4246	0.4530
200	0.2247	0.2368	0.2439	0.2552	0.2863	0.2964	0.2757	0.2999	0.3679	0.3528	0.3674	0.3760	0.2777	0.3087	0.3190	0.2842
500	0.1702	0.1750	0.1659	0.1671	0.2015	0.1989	0.1945	0.2029	0.2765	0.2682	0.2749	0.2804	0.1973	0.2039	0.2037	0.1928
1000	0.1722	0.1349	0.1573	0.1401	0.1363	0.1538	0.1682	0.1399	0.1728	1.1049	0.1153	0.1158	0.1723	0.1581	0.1096	0.1121

Table 16: Standard Deviations (SDs) for Σ (Lower Triangular Matrix)

	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{21}	σ_{22}	σ_{23}	σ_{24}	σ_{31}	σ_{32}	σ_{33}	σ_{34}	σ_{41}	σ_{42}	σ_{43}	σ_{44}
50	21.5141	18.4758	23.2987	17.2953	18.4758	29.3679	26.6388	20.6091	23.2987	26.6388	47.2259	23.2844	17.2953	20.6091	23.2844	22.6887
100	14.0581	11.1060	15.0052	10.3858	12.6170	20.4008	20.0608	13.3698	15.2535	20.0478	26.9006	15.1284	10.3859	13.7369	15.2184	15.4084
200	9.4306	7.7359	10.5546	8.2490	7.7359	12.1568	11.1012	9.3452	10.5546	11.1012	20.2082	12.0372	8.2490	9.3452	12.0372	12.6179
500	5.4642	4.6772	6.0814	4.3775	4.6772	7.3871	6.7026	5.1916	6.0814	6.7026	12.8625	6.7879	4.3775	5.1916	6.7879	6.8159
1000	1.2675	2.5745	3.5893	3.5933	2.5745	6.1401	6.3920	4.9527	3.5893	6.3920	5.0388	4.1274	3.5933	4.9527	4.1274	1.9209

5. Discussion

Marginal posterior means of β for upper and lower triangular matrices converge to the true population parameter as the sample size increases. The conditional posterior of β for the decomposed partitioned matrices is not as close to the true population value as the marginal posterior mean estimates. In addition, the marginal and conditional posterior means of Σ for both matrices decreases as the sample size increases but not consistent. Results from summary posterior means of β for the partitioned matrices show that upper triangular matrix decreases as the sample size increases but experienced some fluctuations in the degree of orderliness at n = 100 and 200. However, the values are constant at n = 500 and 1000, while in most case, the lower triangular matrix posterior means decrease as the sample size increases. The posterior standard deviations of β and Σ of the lower and triangular matrices decrease as the sample size increases.

6. Conclusion

The posterior means converge to the population parameter as sample size increases. Also, the posterior standard deviations of the parameter estimates of the triangular matrices are lower as the sample size increases. For the partitioned variance-covariance matrix, the upper triangular matrix outperformed the lower triangular matrix.

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