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NUMERICAL METHODS FOR SOLVING FIRST ORDER INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATION USING PICARD'S, MODIFIED EULER AND RUNGE KUTTA METHODS

[1]D.I. Lanlege, [2]S. O. Momoh and [3]A. Abubakar

[*1, 2] Department of Mathematical Science, Federal University Lokoja Kogi State, Nigeria.
[3] Department of Mathematics/Computer Science, Ibrahim Badamasi Babangida University Lapai Niger State Nigeria.

Abstract

This research work mainly present Picard's Method (PM), Modified Euler Method (MEM) and Runge Kutta Method (RKM) for solving first order initial value problems (IVP) in ordinary differential equations (ODEs). The three methods are practically well suited for solving numerical problems. In order to verify the efficiency and accuracy, we compare the numerical solutions with that of the exact solutions. Finally we investigate and compute the errors of the methods for different step size to examine superiority. The results show that the Runge Kutta Method is suitable for solving first order ordinary differential equation.

***Keywords*:** Numerical Analysis, Ordinary Differential Equations(ODE),Numerical Solution, Exact Solution, Picard's Method, Modified Euler Method and Runge Kutta Method.

1. Introduction

Numerical methods of solution to ordinary differential equation are methods that are generally used for solving mathematical problems formulated in sciences and engineering where it is difficult to obtain exact solution. In most cases, these numerical methods give appropriate results. Therefore whenever a numerical method is adopted there must be error in the solution, which may be as a result of rounding up or truncation of values in the computation. By error here, we mean the deviation of the obtained result from the exact solution. There are many methods for finding the solution of ordinary differential equations (ODEs), despite that there exist a large number of Ordinary Differential Equations whose solution cannot be obtain in close form by renowned analytical method may be due to their complexity or because of the boundary condition, hence we use the numerical methods to get the approximate solution of the differential equations under the prescribed initial condition or conditions. A well-known numerical technique is the Runge Kutta method.

The method is popular because it is quite stable, accurate and easy to program. It is differentiated by their order in the sense that they agree with Taylor's series solutions up to h^r where r is the order of the method. The method is a technique for approximating the solution of ordinary differential equations which was developed in the year 1900 by mathematicians Carl Runge and Wilhelm Kutta. Runge Kutta method is one of the most popular method because of its efficient and its use in computer programs, typically it produce accurate approximations even when the number of iteration is reasonably small. The order listed below is orders of Runge Kutta method:

Runge Kutta method of order one is called Euler's method. Runge Kutta method of order two is the same as modified Euler's or Heun's Method. The fourth order Runge Kutta method is called classical Runge Kutta method.

In this research work fourth order Runge Kutta will be used [1]. In contrast, Picard method is an iterative method that gives a sequence of approximation to the solution of differential equations such that the *nth* approximation is obtained from one or more previous approximation, the method involves starting with a differential equations and a given initial condition which is done by integrating both side with respect to one variable from a defined starting point to a defined terminating point that is $[x_n, x_{n+1}]$ for n = 0,1,2,.. and x_0 is the initial variable give , the result can be used to generate successive approximations of a solution to initial value problems.

Corresponding Author**:** Lanlege D.J., Email: loislanlege@gmail.com, Tel: +2348030528667, +2348131235684

The number of iterations is determined by how fast the series converge and how far away from the point of interest is the value given in the initial problem [2]. The Modified Euler Method is a simple Method that uses the average value of y'at both the left and right end point of $[x_n, x_{n+1}]$ for$n$ = 0,1,2,... as the slope of the line segment approximate to the solution over that interval. The resulting equations are;

$$y_1^{'*} = y_n + hy_n^{'} \quad (1)$$

$$y_{n+1} = y_n + \frac{h}{2}(y_n^{'} + y_n^{'*}) \quad (2)$$

Where the $y_n^{'}$ in the equation (1) is

$$y_n^{'} = f(x_n, y_n)$$

And $y_n^{'*}$ in equation (2) is

$$y_n^{'*} = f(x_{n+1}, y_{n+1}^{'*})$$

Equation (1) is the Method for solving Euler and equation (2) is Modified

Euler method used, which depend on the Euler Method to obtain it's solution [3]. This Method is of predictor-corrector like type, the predictor uses Euler method to get the solution to equation (1) while the corrector method is the equation (2) which depend on the predictor.

The solution of both ordinary differential equations (ODEs) and partial differential equations (PDEs) are computed by the use of numerical method. This research basically deals with ordinary differential equations. Observing that Modified Euler Method (MEM) involves differentiation while Picard's Method (PM) involve integration of function. Depending on the ease of operation, one can select the appropriate methods that involve Modified Euler, Picard and Runge Kutta Methods. The approximate solution is given in forms of mathematical expression. Finally examples on differential equation (DE) are given to verify the formulae.

2. Material and Method

Picard's method

Considering the first order differential equation of the form

$$y^{'} = \frac{dy}{dx} = f(x, y) \quad (3)$$

With initial condition:

$$y(x_0) = y_0$$

By integrating over the interval (x_0, x) in (x) gives:

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \quad (4)$$

Then we show that solution to (3) is equivalent to finding solution to equation (4)

Proof

In this method, the value of variable y is expressed as function of x. To show that (3) and (4) are equivalent let

$$\frac{dy}{dx} = f(x, y) \quad (5)$$

$$dy = f(x, y)dx \quad (6)$$

Since $y(x)$ is differentiable function in some neighbourhood of x_0. Then, $f(x, y)$ is a continuous function of x in some neighbourhood ofx_0. Thus if we integrate (4) between x_0 and x we have;

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx \quad (7)$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)dx \quad (8)$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \quad (9)$$

This is equivalent to (3). Since the information concerning the expression of y in terms of x is abstract, the integral on the right hand side of (13) cannot be evaluated. Therefore, we determine a sequence of approximation and obtain the first approximation to be;

$$y_1(x) = y_0 + \int_{x}^{x_0} f(x, y_0)dx \quad (10)$$

Where $y_1(x)$ is the corresponding value of y(x).to determine a better approximations, we replace y by y_2 as

$$y_2(x) = y_0 + \int_x^{x_0} f(x, y_1)dx \quad (11)$$

The following approximations of y is generated sequentially

$$y_3(x) = y_0 + \int_x^{x_0} f(x, y_2)dx \quad (12)$$

$\vdots \qquad \vdots$

$$y_n(x) = y_0 + \int_x^{x_0} f(x, y_{n-1})dx \quad (13)$$

For $n = 1,2,3 \cdots$ Hence (13) is the general nth proof of (4) for the Picard's method.

Modified Euler's method

In other to develop the Modified Euler's method, different numerical methods can be use such as Taylor's series, Trapezoidal Rule etc. in this research work Trapezoidal Rule will be use.

Considering equation (3) we have

$$dy = f(x, y)dx \quad (14)$$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \quad (15)$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)\, dx \quad (16)$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \quad (17)$$

Then the integration of the Right Hand Side can be done using trapezoidal rule. Applying the trapezoidal rule with step size h which gives

$$\int_{x_0}^{x} f(x, y)dx \approx \frac{h}{2}[\, f(x_i, y_i) + f(x_{i+1}, y_{i+1}^*)] \quad (18)$$

Then the above integration becomes:

$$y_{i+1} = y_i + \frac{h}{2}[\, f(x_i, y_i) + f(x_{i+1}, y_{i+1}^*)] \quad (19)$$

$$h = (x_{i+1}, x_i) \text{ for } i = 0,1,2,3, \cdots, N \quad (20)$$

Where y_{i+1}^* correspond to Euler step size h starting from (x_i, y_i) which is of the form:

$$y_{i+1}^* = y_i + hf(x_i, y_i) \quad (21)$$

To integrate the first approximation we have:

$$y_1^* = y_0 + hf(x_0, y_0) \quad (22)$$

$$y_1 = y_0 + \frac{h}{2}[\, f(x_0, y_0) + f(x_1, y_1^*)] \quad (23)$$

Therefore, the above approximation of y is generated sequentially

$$y_2^* = y_0 + hf(x_0, y_0) \quad (24)$$

$$y_2 = y_0 + \frac{h}{2}[\, f(x_0, y_0) + f(x_1, y_2^*)] \quad (25)$$

$$y_{i+1}^* = y_i + hf(x_i, y_i) \quad (26)$$

$$y_{i+1} = y_i + \frac{h}{2}[\, f(x_i, y_i) + f(x_{i+1}, y_{i+1}^*)] \quad (27)$$

Hence equations (26) and (27) are the general ith proof of modified Euler's method.

Note From the proof, the Modified Euler's method can be interpreted as a predictor corrector like method where Euler's method is used as the predictor for the (implicit) trapezoidal rule, the iteration are continued until two successive approximation y_{i+1}^* and y_{i+1}coincide to the desired accuracy, we call equation (24) and (25) the trapezoidal scheme.

Runge Kutta Method

This Method named after [5] and [6] is a method based on the first- five terms of Taylor's series derived from equation (3). And by the use of the first- five terms of Taylor's series, he obtain the scheme.

$$\mathrm{y}_1 = \mathrm{y}_0 + \frac{1}{6}(\mathrm{k}_1 + 2\mathrm{k}_2 + 2\mathrm{k}_3 + \mathrm{k}_4) \quad (28)$$

Where:

$$k_1 = hf(x_0, y_0), k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right), k_3 = hf\left(x_0 + \frac{h}{2}, y_0 \frac{k_2}{2}\right) k_4 = hf(x_0 + h, y_0 + k_3) \quad (29)$$

Thus generating sequentially we have:

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (30)$$

Where:

$k_1 = hf(x_1, y_1), k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right), k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) k_4 = hf(x_1 + h, y_1 + k_3)$
(31)

$$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (32)$$

Where: $k_1 = hf(x_2, y_2)$,

$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$,

$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$,

$k_4 = hf(x_2 + h, y_2 + k_3)$ (33)

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (34)$$

Where:

$k_1 = hf(x_i, y_i), k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right), k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$,

$k_4 = hf(x_i + h, y_i + k_3)$ (35)

Hence (35) is the general ith proof of Runge Kutta Method.

Example 1

Use Picard's method, Modified Euler Method and Runge Kutta method to approximate the solution of the initial value problem $\frac{dy}{dx} = 2xy$with initial condition $y(0) = 1$ in the interval $0 \le x \le 1$ given step size $h = 0.2$ whose exact value is given by $y(x) = e^{x^2}$

Solution

By Picard's Method:

From equation (13),

Putting$n = 1$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$f(x, y) = 2xy$ Where$y_0 = 1$ and $x_0 = 0$

Putting the initial condition, we have

$$y_1(x) = 1 + \int_0^x 2x(1)dx$$

Integrate:

$y_1(x) = 1 + x^2$

Put $\boldsymbol{n = 2}$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$y_1 = 1 + x^2$

$$y_2(x) = 1 + \int_0^x 2x\,(1 + x^2)dx = 1 + \int_0^x (2x + 2x^3)\,dx$$

Integrate:

$$y_2(x) = 1 + x^2 + \frac{x^4}{2}$$

Putting $n = 3$ In equation (13)

$$y_3(x) = yo + \int_{xo}^{x} f(x, y_2)\,dx$$

$$y_3(x) = 1 + \int_0^x 2x\left(1 + x^2 + \frac{x^4}{2}\right)dx = 1 + \int_o^x (2x + 2x^3 + x^5)dx$$

Integrate:

$y3(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

Put $n = 4$

$y_4(x) = yo + \int_{xo}^{x} f(x, y_3)\, dx$

$y_4(x) = 1 + \int_0^x 2x(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6})dx = 1 + \int_0^x (2x + 2x^3 + x^5 + \frac{x^7}{3})dx$

$y_4(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}$

Put $n = 5$

$y_5(x) = yo + \int_{xo}^{x} f(x, y_4)\, dx$

$y_5(x) = 1 + \int_0^x [\![2x\left(1 + x^2 + \frac{x^4}{2}]\!] + \frac{x^6}{6} + \frac{x^8}{24}\right) dx = 1 + \int_0^x \left(2x + 2x^3 + x^5 + \frac{x^7}{3} + \frac{x^9}{12}\right) dx$

Integrate:

$y_5(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120}.$

We have: $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

$x_4 = x_3 + h = 0.6 + 0.2 = 0.8$

$x_5 = x_4 + h = 0.8 + 0.2 = 1.0$

Hence putting $x_1, x_2, \cdots, x_5$ in the values above we have

$y(0.0)$ = 1.0000000

$y_1(0.2)$ = 1.0400000000

$y_2(0.4)$ = 1.1728000000

$y_3(0.6)$= 1.4325760000

$y_4(0.8)$ = 1.8954811735

$y_5(1.0)$ = 2.7166666667

Table 1: Tabular Representation of Picard's Method For the step size *h = 0.2*

Value of x	Approx. value of y[y(x)]	Exact Solution of y[Y(x)]	Error Incurred[e_{PM}]
0.0	1.00000000	1.00000000	0.00000000
0.2	1.0400000000	1.040810774192388	8.10774192×10^{-4}
0.4	1.1728000000	1.173510870991810	7.10870992×10^{-4}
0.6	1.4325760000	1.43332941456034	7.53414561×10^{-4}
0.8	1.8954811735	1.896480879304952	9.997095971×10^{-4}
1.0	2.7166666667	2.718281828459046	1.61516146×10^{-3}

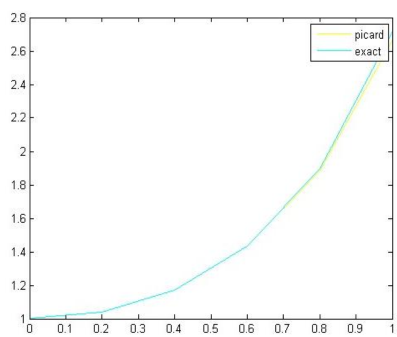

Figure 1: Graphical Illustration of Picard Solution and Exact Solution for *h = 0.2*

Modified Euler Method:
From equation (26) and (27) , we have
$y_{i+1}^{*} = y_i + hf(x_i, y_i)$
And:
$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{*})]$
For $i = 0$
$y_1^{*} = y_0 + hf(x_0, y_0)$
Where $x_0 = 0, y_0 = 1$
$f(x_0, y_0) = 2x_0$ and $h = 0.2$
$y_1^{*} = 1 + 0.2(2 * 0 * 1) = 1 + 0$
$y_1^{*} = 1.0000000000$
Putting i and y_1^{*} in y_{i+1} we have
$y_1 = y_0 + \frac{0.2}{2}[f(x_0, y_0) + f(x_1, y_1^{*})]$
$x_1 = x_0 + h = 0 + 0.2 = 0.2$
$y_1 = 1 + \frac{0.2}{2}[(2 * 0 * 1) + (2 * 0.2 * 1.0000000000)]$
$y_1 = 1 + 0.40000000$
$y_1 = 1.0400000000$
When$x_1 = 0.2$, $y_1(x) = 1.0400000000$
For $i = 1$
$y_2^{*} = y_1 + hf(x_1, y_1)$
$y_2^{*} = 1.0400000000 + 0.2(2 * 0.2 * 1.0400000000)$
$y_2^{*} = 1.1232000000$
Putting i and y_2^{*} in y_{i+1} we have
$y_2 = y_1 + \frac{0.2}{2}[f(x_1, y_1) + f(x_2, y_2^{*})]$
$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$
$y_2 = 1.0400000000 + \frac{0.2}{2}[(2 * 0.2 * 1.0400000000) + (2 * 0.4 * 1.1232000000)]$
$y_2 = 1.0400000000 + 0.1314560000$
$y_2 = 1.1714560000$
When $x_2 = 0.4, y_2(x) = 1.1714560000$
For $i = 2$
$y_3^{*} = y_2 + hf(x_2, y_2)$
$y_3^{*} = 1.1714560000 + 0.2(2 * 0.4 * 1.1714560000)$
$y_3^{*} = 1.3588888888$
Putting i and y_3^{*} in y_{i+1} we have
$y_3 = y_2 + \frac{0.2}{2}[f(x_2, y_2) + f(x_3, y_3^{*})]$
$x_3 = x_2 + h = 0.4 + 0.2 = 0.6$
$y_3 = 1.1714560000 + \frac{0.2}{2}[(2 * 0.4 * 1.1714560000) + (2 * 0.6 * 1.3588888888)]$
$y_3 = 1.4282391552000$
When $x_3 = 0.6, y_3(x) = 1.4282391552$
For $i = 3$
$y_4^{*} = y_3 + hf(x_3, y_3)$
$y_4^{*} = 1.4282391552 + 0.2(2 * 0.6 * 1.4282391552)$
$y_4^{*} = 1.7710165524$
Putting i and y_4^{*} in y_{i+1} we have
$y_4 = y_3 + \frac{0.2}{2}[f(x_3, y_3) + f(x_4, y_4^{*})]$
$x_4 = x_3 + h = 0.6 + 0.2 = 0.8$
$y_4 = 1.4282391552 + \frac{0.2}{2}[(2 * 0.6 * 1.4282391552) + (2 * 0.8 * 1.7710165524)]$

$y_4 = 1.882990502215680$
When $x_4 = 0.8, y_4(x) = 1.882990502215680$
For $i = 4$
$y_5^* = y_4 + hf(x_4, y_4)$
$y_5^* = 1.882990502215680 + 0.2(2 * 0.8 * 1.882990502215680)$
$y_5^* = 2.48554746292498$
Putting i and y_5^* in y_{i+1} we have
$y_5 = y_4 + \frac{0.2}{2}[f(x_4, y_4) + f(x_5, y_5^*)]$
$x_5 = x_4 + h = 0.8 + 0.2 = 1.0$
$y_5 = 1.882990502215680 + \frac{0.2}{2}[(2 * 0.8 * 1.882990502215680) + (2 * 1.0 * 2.48554746292498)]$
$y_5 = 2.681378475155129$
When $x_5 = 1.0, y_5(x) = 2.681378475155129$

Table 2: Tabular Representation of Modified Euler Method for the step size *h = 0.2*

Value of x	Approx value of y[y(x)]	Exact Solution of y[Y(x)]	Error Incurred[e_{MEM}]
0.0	1.0	1.0	0.0000000
0.2	1.04000000000	1.040810774192388	8.10774192×10^{-4}
0.4	1.171456000000	1.173510870991810	2.05487099×10^{-3}
0.6	1.4282391552000	1.43332941456034	5.09025936×10^{-3}
0.8	1.882990502215680	1.896480879304952	0.01349037709
1.0	2.681378475155129	2.718281828459046	0.1044970769

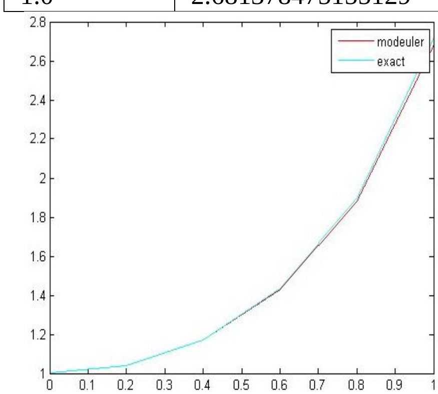

Figure 2: Graphical Illustration of Modified Euler Solution and Exact Solution for *h = 0.2* by Runge Kutta Method:
From equation (34), we have:
$y_{i+1} = y_i + \frac{1}{6}(\mathrm{k}_1 + 2\mathrm{k}_2 + 2\mathrm{k}_3 + \mathrm{k}_4)$
Where $k_1 = hf(x_i, y_i), k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right), k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right),$
$k_4 = hf(x_i + h, y_i + k_3)$
For $i = 0$
$k_1 = hf(x_i, y_i) = hf(x_0, y_0) = 0.2(2 * 0 * 1)$
$k_1 = 0$
$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$
$k_2 = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0}{2}\right) = 0.2f(0.1,1)$

$k_2 = 0.2(2 * 0.1 * 1) = 0.0400000000$
Therefore $k_2 = 0.0400000000$
$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$
$k_3 = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.0400000000}{2}\right) = 0.2f(0.1,1.020000000)$
$k_3 = 0.2(2 * 0.1 * 0.0200000000) = 0.2(0.2040000000)$
Therefore $k_3 = 0.0408000000$
$k_4 = hf(x_0 + h, y_0 + k_3)$
$k_4 = 0.2f(0 + 0.2,1 + 0.0408000000) = 0.2f(0.2,1.04080000000)$
$k_4 = 0.2(2 * 0.2 * 1.04080000000)$
Therefore $k_4 = 0.0832640000$
Putting i,k_1,k_2,k_3 and k_4 in equation (33)
$y_1 = y_0 + \frac{1}{6}(\mathrm{k}_1 + 2\mathrm{k}_2 + 2\mathrm{k}_3 + \mathrm{k}_4)$
$y_1 = 1 + \frac{1}{6}(0 + 2 * 0.0400000000 + 2 * 0.04080000000 + 0.0832640000)$
$y_1 = 1 + 0.0408166667$
$y_1 = 1.040810666667$
For $i = 1$ and $x_1 = x_0 + h = 0 + 0.2$
$x_1 = 0.2$
$k_1 = hf(x_1, y_1) = 0.2(2 * 0.2 * 1.040810666667)$
$k_{1=}0.0832648533333$
$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$
$k_2 = 0.2f\left(0.2 + \frac{0.2}{2}, 1.040810666667 + \frac{0.0832648533333}{2}\right) = 0.2f(0.3, 1.082443093333)$
$k_2 = 0.2(2 * 0.3 * 1.082443093333)$
Therefore $k_2 = 0.12989317120000$
$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$
$k_3 = 0.2f\left(0.2 + \frac{0.2}{2}, 1.04081666667 + \frac{0.12989317120000}{2}\right)$
$= 0.2f(0.4, 1.173507536)$
$k_3 = 0.2(2 * 0.3 * 1.1057632520)$
Therefore $k_3 = 0.132690870272000$
$k_4 = hf(x_1 + h, y_1 + k_3)$
$k_4 = 0.2f(0.2 + 0.2,1.04081666667 + 0.132690870272000) = 0.2(0.4,1.173507536)$
$k_4 = 0.2(2 * 0.4 * 1.173507536)$
Therefore $k_4 = 0.187760245910187$
Putting$i$,k_1,k_2,k_3 and k_4 in equation(34)
$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$y_2 = 1.04081666667 + \frac{1}{6}(0.0832648533333 + 2 * 0.12989317120000 + 2 * 0.132690870272000 + 0.187760245910187)$
$y_2 = 1.173509530364587$
For $i = 2$ and $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$
$k_1 = hf(x_2, y_2) = 0.2(2 * 0.4 * 1.173509530364587)$
$k_1 = 0.187761524858334$
$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$
$k_2 = 0.2f\left(0.4 + \frac{0.2}{2}, 1.173509530364587 + \frac{0.187761524858334}{2}\right) = 0.2f(0.5,1.267390293200)$
$k_2 = 0.2(2 * 0.5 * 1.267390293200) = 0.25347805855871$

Therefore $k_2 = 0.25347805855871$

$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$

$k_3 = 0.2f\left(0.4 + \frac{0.2}{2}, 1.173509530364587 + \frac{0.25347805855871}{2}\right) = 0.2f(0.5,1.30024855900)$

$k_3 = 0.2(2 * 0.5 * 1.30024855900)$

Therefore $k_3 = 0.2600497111928792$

$k_4 = hf(x_2 + h, y_2 + k_3)$

$k_4 = 0.2f(0.4 + 0.2,1.173509530364597 + 0.2600497111928792) = 0.2f(0.6,1.4335592141)$

$k_4 = 0.2(2 * 0.6 * 1.4335592141)$

Therefore $k_4 = 0.34405421810411$

Putting$i$,k_1,k_2,k_3 and k_4 in equation(34)

$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_3 = 1.1735096100 + \frac{1}{6}(0.187761524858334 + 2 * 0.25347805855971 + 2 * 0.2600497111928792 + 0.34405421810411)$$

$y_3 = 1.433321411028559$

For $i = 3$ and $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

$k_1 = hf(x_3, y_3) = 0.2(2 * 0.6 * 1.433321411028559)$

$k_1 = 0.343997138646854$

$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right)$

$k_2 = 0.2f\left(0.6 + \frac{0.2}{2}, 1.433321411028559 + \frac{0.343997138646854}{2}\right) = 0.2f(0.7,1.06053199800)$

$k_2 = 0.2(2 * 0.7 * 1.06053199800) = 0.4494896243$

Therefore $k_2 = 0.4494896243$

$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right)$

$k_3 = 0.2f\left(0.6 + \frac{0.2}{2}, 1.43331411028559 + \frac{0.4494896243}{2}\right) = 0.2f(0.7,1.658066208)$

$k_3 = 0.2(2 * 0.7 * 1.658066208)$

Therefore $k_3 = 0.464258538317794$

$k_4 = hf(x_3 + h, y_3 + k_3)$

$$k_4 = 0.2f(0.6 + 0.2,1.433321411028559 + 0.0.6 + 0.2,1.433321411028559 + 0.464258538317794) = 0.2f(0.8,1.897579949)$$

$k_4 = 0.2(2 * 0.8 * 1.897579949) = 0.2(3.036084)$

Therefore $k_4 = 0.607225583790833$

Putting$i$,k_1,k_2,k_3 and k_4 in equation(34)

$y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_4 = 1.433321411028559 + \frac{1}{6}(0.343997138646854 + 2 * 0.449489594498556 + 2 * 0.464258538317794 + 0.607225583790833)$$

$y_4 = 1.433321411028559 + 0.46311983200$

$y_4 = 1.896441242373623$

For $i = 4$ and $x_4 = x_3 + h = 0.6 + 0.2 = 0.8$

$k_1 = hf(x_4, y_4) = 0.2(2 * 0.8 * 1.896441242373623)$

$k_1 = 0.606861197559559$

$k_2 = hf\left(x_4 + \frac{h}{2}, y_4 + \frac{k_1}{2}\right)$

$k_2 = 0.2f\left(0.8 + \frac{0.2}{2}, 1.896441242373623 + \frac{0.606861197559559}{2}\right) = 0.2f(0.9,2.1998723)$

$k_2 = 0.2(2 * 0.9 * 2.1998723) = 0.791953862815225$

Therefore $k_2 = 0.791953862815225$

$k_3 = hf\left(x_4 + \frac{h}{2}, y_4 + \frac{k_2}{2}\right)$

$k_3 = 0.2f\left(0.8 + \frac{0.2}{2}, 1.896441242373623 + \frac{0.791953862815225}{2}\right) = 0.2f(0.9, 2.292418173)$

$k_3 = 0.2(2 * 0.9 * 2.292418173)$

Therefore $k_3 = 0.825270542561245$

$k_4 = hf(x_4 + h, y_4 + k_3)$

$k_4 = 0.2f(0.8 + 0.2, 1.896441242373623 + 0.825270542561245)$

$k_4 = 0.2f(1.0, 2.721711785)$

$k_4 = 0.2(2 * 1.0 * 2.721711785)$

Therefore $k_4 = 1.088684713973947$

Putting$i$,k_1,k_2,k_3 and k_4 in equation(34)

$$y_5 = y_4 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_5 = 1.896441368 + \frac{1}{6}(0.606861197559559 + 2 * 0.791953862815225 + 2 * 0.82527054256125 + 1.088684713973947)$$

$y_5 = 2.718107029421364$

Table 3: Tabular Representation of Runge Kutta Method for the step size *h = 0.2*

Value of x	Approx value of y[y(x)]	Exact Solution of y[Y(x)]	Error Incurred[e_{RK4M}]
0.0	1.000000	1.000000	00000000
0.2	1.040810666667	1.040810774192388	1.07525388×10^{-7}
0.4	1.173509530364587	1.173510870991810	1.34062722×10^{-6}
0.6	1.43332141102856	1.43332941456034	8.00353178×10^{-6}
0.8	1.896441242373623	1.896480879304952	3.96369313×10^{-5}
1.0	2.718107029421364	2.718281828459046	1.74799038×10^{-4}

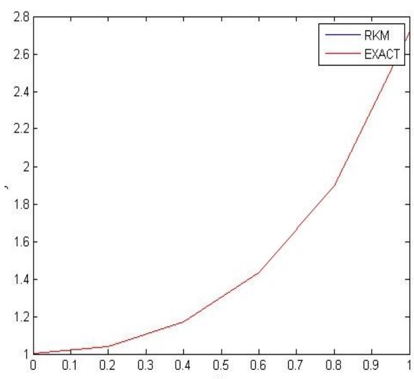

Figure 3: Graphical Illustration of Runge Kutta Solution and Exact Solution for *h = 0.2*

However, the exact value of $y(x)$ is given as:

$$y(x) = e^{x^2} \quad (36)$$

Then we generate the solution of the exact value to be:

$y(0.0)$ = 1.00000000

$y(0.2)$ = 1.040810774192388

$y(0.4)$ = 1.173510870991810

$y(0.6)$ = 1.43332941456034

$y(0.8)$ = 1.896480879304952

$y(1.0)$ = 2.718281828459046

COMPARATIVE RESULT OF EXAMPLE 1

Table 4: The Comparative Result of Picard method, Modified, Euler Method and Runge Kutta Method for the step size *h = 0.2*

Value of x	$[y(x)_{PM}]$	$[y(x)_{MEM}]$	$[y(x)RK4M]$	Exact Solution of y[Y(x)]
0.0	1.000000000	1.00000000	1.00000000	1.00000000
0.2	1.0400000000	1.040000000000	1.040810666667	1.040810774192
0.4	1.1728000000	1.171456000000	1.173509530365	1.173510870992
0.6	1.4325760000	1.428239155200	1.433321411029	1.4333294145603
0.8	1.8954811735	1.882990502216	1.896441242374	1.8964808793049
1.0	2.7166666667	2.681378475155	2.718107029421	2.7182818284590

ERROR INCURED IN EXAMPLE 1

Table 5: Error Incurred from Picard's Method, Modified Euler Method and Runge Kutta Method

Value of x	$[Y(x) - y(x) = e_{PM}]$	$[Y(x) - y(x) = e_{MEM}]$	$[Y(x) - y(x) = e_{RK4M}]$
0.0	0.0000	0.0000	0.0000
0.2	8.107741924×10^{-4}	8.10774192×10^{-4}	1.07525388×10^{-7}
0.4	7.10870992×10^{-4}	2.05487099×10^{-3}	1.34062722×10^{-6}
0.6	7.53414561×10^{-4}	5.09025936×10^{-3}	8.00353178×10^{-6}
0.8	9.997095971×10^{-4}	0.01349037709	3.96369313×10^{-5}
1.0	1.61516146×10^{-3}	0.1044970769	1.74799038×10^{-4}

Example 2

Use Picard's method, Modified Euler Method and Runge Kutta method to solve a linear first order initial value problem of the form $y' = y - x, y(0) = 2$ on the interval $0 \leq x \leq 1$ given step size $h = 0.1$ whose exact value is given by $y(x) = e^x + x + 1$

Solution

Picard's Method:

From equation (13), putting n=1

Where $x_0 = 0, y_0 = 2$

$$y_1(x) = 2 + \int_0^x (y_0 - x)\,dx = 2 + \int_0^x (2 - x)dx$$

Integrate:

$$y_1(x) = 2 + 2x - \frac{x^2}{2}$$

For $x_1 = x_0 + hx_1 = 0 + 0.1 = 0.1$

Put x_1 into $y_1(x)$ we have

$$y_1(0.1) = 2 + 2(0.1) - \frac{(0.1)^2}{2}$$

$$y_1(0.1) = 2.1952000000$$

Put $n = 2$

$$y_2(x) = y_0 + \int_{x_0}^x f(x, y_1)dx$$

$$y_2(x) = 2 + \int_0^x (2 + 2x - \frac{x^2}{2} - x)dx$$

Integrate:

$$y_2(x) = 2 + 2x + \frac{x^2}{2} - \frac{x^3}{6}$$

For $x_2 = x_1 + h, x_2 = 0.1 + 0.1 = 0.2$

$$y_2(0.2) = 2 + 2(0.2) + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{6}$$

$$y_2(0.2) = 2.4186666667$$

Put $n = 3$

$$y_3(x) = y_0 + \int_{x_0}^x f(x, y_2)dx$$

$$y_3(x) = 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} - \frac{x^3}{6} - x \right) dx$$

Integrate:

$$y_3(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$$

For $x_3 = x_2 + h, x_3 = 0.2 + 0.1 = 0.3$

Put x_3 into $y_3(x)$ we have

$$y_3(0.3) = 2 + 2(0.3) + \frac{(0.3)^2}{2} + \frac{(0.3)^3}{6} - \frac{(0.3)^4}{24}$$

$$y_3(0.3) = 2.6491625000$$

Put $n = 4$

$$y_4(x) = y_0 + \int_{x_0}^x f(x, y_3)dx$$

$$y_4(x) = 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} - x \right) dx$$

Integrate:

$$y_4(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

For $x_4 = x_3 + h, x_4 = 0.3 + 0.1 = 0.4$

Put x_4 into $y_4(x)$ we have:

$$y_4(0.4) = 2 + 2(0.4) + \frac{(0.4)^2}{2} + \frac{(0.4)^3}{6} + \frac{(0.4)^4}{24} - \frac{(0.4)^5}{120}$$

$$y_4(0.4) = 2.8916480001$$

Put $n = 5$

$$y_5(x) = y_0 + \int_{x_0}^x f(x, y_4)dx$$

$$y_5(x) = 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720} - x \right) dx$$

Integrate:

$$y_5(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720}$$

For $x_5 = x_4 + h, x_5 = 0.4 + 0.1 = 0.5$

$$y_5(0.5) = 2 + 2(0.5) + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6} + \frac{(0.5)^4}{24} + \frac{(0.5)^5}{120} - \frac{(0.5)^6}{720}$$

$$y_5(0.5) = 3.1486762150$$

Put $n = 6$

$$y_6(x) = y_0 + \int_{x_0}^x f(x, y_5)dx$$

$$y_6(x) = 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720} - x\right) dx$$

Integrate:

$$y_6(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040}$$

For $x_6 = x_5 + h, x_6 = 0.5 + 0.1 = 0.6$

$$y_6(0.6) = 2 + 2(0.6) + \frac{(0.6)^2}{2} + \frac{(0.6)^3}{6} + \frac{(0.6)^4}{24} + \frac{(0.6)^5}{120} + \frac{(0.6)^6}{720} - \frac{(0.6)^7}{5040}$$

$y_6(0.6) = 3.4221072458$

Put $n = 7$

$$y_7(x) = y_0 + \int_{x_0}^x f(x, y_6) dx$$

$$y_7(x) = 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} - x\right) dx$$

Integrate:

$$y_7(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} - \frac{x^8}{40320}$$

For $x_7 = x_6 + h, x_6 = 0.6 + 0.1 = 0.7$

$$y_7(0.7) = 2 + 2(0.7) + \frac{(0.7)^2}{2} + \frac{(0.7)^3}{6} + \frac{(0.7)^4}{24} + \frac{(0.7)^5}{120} + \frac{(0.7)^6}{720} + \frac{(0.7)^7}{5040} - \frac{(0.7)^8}{40320}$$

$y_7(0.7) = 3.7137497278$

Put $n = 8$

$$y_8(x) = y_0 + \int_{x_0}^x f(x, y_7) dx$$

$$y_8(x) = 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} - \frac{x^8}{40320} - x\right) dx$$

Integrate:

$$y_8(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} - \frac{x^9}{362880}$$

For $x_8 = x_7 + h, x_8 = 0.7 + 0.1 = 0.8$

$$y_8(0.8) = 2 + 2(0.8) + \frac{(0.8)^2}{2} + \frac{(0.8)^3}{6} + \frac{(0.8)^4}{24} + \frac{(0.8)^5}{120} + \frac{(0.8)^6}{720} + \frac{(0.8)^7}{5040} + \frac{(0.8)^8}{40320} - \frac{(0.8)^9}{362880}$$

$y_8(0.8) = 4.0255401567$

Put $n = 9$

$$y_9(x) = y_0 + \int_{x_0}^x f(x, y_8) dx$$

$$y_9(x) = 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} - \frac{x^9}{362880} - x\right) dx$$

Integrate:

$$y_9(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} - \frac{x^{10}}{3628800}$$

For $x_9 = x_8 + h, x_9 = 0.8 + 0.1 = 0.9$

$$y_9(0.9) = 2 + 2(0.9) + \frac{(0.9)^2}{2} + \frac{(0.9)^3}{6} + \frac{(0.9)^4}{24} + \frac{(0.9)^5}{120} + \frac{(0.9)^6}{720} + \frac{(0.9)^7}{5040} + \frac{(0.9)^8}{40320} + \frac{(0.9)^9}{362880} - \frac{(0.9)^{10}}{3628800}$$

$y_9(0.9) = 4.3596031030$

Put $n = 10$

$$y_{10}(x) = y_0 + \int_{x_0}^x f(x, y_9) dx$$

$$y_{10}(x) = 2 + \int_0^x \left(2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} - \frac{x^{10}}{3628800} - x\right) dx$$

Integrate:

$y_{10}(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} - \frac{x^{11}}{39916300}$

For $x_{10} = x_9 + h, x_{10} = 0.9 + 0.1 = 1.0$

$y_{10}(1.0) = 2 + 2(1.0) + \frac{(1.0)^2}{2} + \frac{(1.0)^3}{6} + \frac{(1.0)^4}{24} + \frac{(1.0)^5}{120} + \frac{(1.0)^6}{720} + \frac{(1.0)^7}{5040} + \frac{(1.0)^8}{40320} + \frac{(1.0)^9}{362880} + \frac{(1.0)^{10}}{3628800} - \frac{(1.0)^{11}}{39916800}$

$y_{10}(1.0) = 4.7182817776000$

Table 6: Tabular Representation of Picard's Method for the step size *h = 0.1*

Value of x	Approx value of y[y(x)]	Exact Solution of y[Y(x)]	Error Incurred[e_{PM}]
0.0	2.00000000	2.00000000	0.00000000
0.1	2.1952000000	2.20517091807565	9.970918×10^{-3}
0.2	2.4186666667	2.42140275816017	2.736092×10^{-3}
0.3	2.6491625000	2.64985880757600	6.963070×10^{-4}
0.4	2.8916480001	2.89182469764127	1.766970×10^{-4}
0.5	3.1486762157	3.14872127070013	4.505500×10^{-5}
0.6	3.4221072460	3.42211880039051	1.155540×10^{-5}
0.7	3.7137497278	3.71375270747048	2.98000×10^{-6}
0.8	4.0255341567	4.02554092849247	6.77200×10^{-6}
0.9	4.3596012103	4.35960311115695	1.90100×10^{-6}
1.0	4.7182757778	4.71828182845905	6.05100×10^{-6}

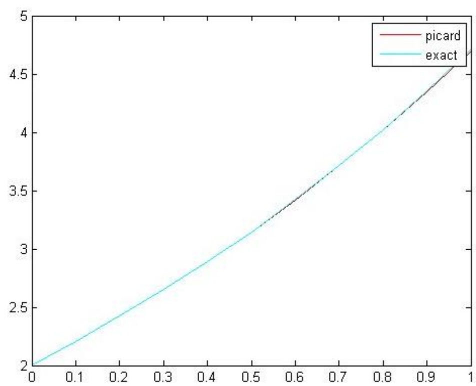

Figure 4: Graphical Illustration of Picard Solution and Exact Solution for *h = 0.1*

Modified Euler Method:

From equation (26) and (27), we have

$y^*_{i+1} = y_i + hf(x_i, y_i)$

And $y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y^*_{i+1})]$

For $i = 0$

$y^*_1 = y_0 + hf(x_0, y_0)$Where$x_0 = 0, y_0 = 2$

$f(x_0, y_0) = (y_0 - x_0)$ and $h = 0.1$

$y^*_1 = 2 + 0.1(2 - 0) = 2 + 0.1$

$y^*_1 = 2.20000000000000$

Putting i and y^*_1 in y_{i+1}we have

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y^*_1)]$$

$y_1 = y_0 + \frac{0.1}{2}[f(x_{0,}y_0) + f(x_{1,}y_1^*)]$
$x_1 = x_0 + h = 0 + 0.1 = 0.1$
$y_1 = 2 + \frac{0.1}{2}[(2 - 0) + (2.20000000000000 - 0.1)]$
$y_1 = 2 + 0.05(2 + 2.10000000000)$
$y_1 = 2.20500000000000$
When $x_1 = 0.1, y_1 = 2.20500000000000$
For $i = 1$
$y_2^* = y_1 + hf(x_{1,}y_1)$
$y_2^* = 2.20500000000000 + 0.1(2.20500000000000 - 0.1)$
$y_2^* = 2.41550000000000$
Putting i and y_1^* in y_{i+1}we have
$y_2 = y_1 + \frac{0.1}{2}[f(x_{1,}y_1) + f(x_{2,}y_2^*)]$
$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$
$y_2 = 2.2050000000 + \frac{0.1}{2}[(2.2050000000 - 0.1) + (2.4155000000 - 0.2)]$
$y_2 = 2.2050000000 + 0.05(2.10500000 + 2.21550000)$
$y_2 = 2.42102500000000$
When $x_2 = 0.2, y_2 = 2.42102500000000$
For $i = 2$
$y_3^* = y_2 + hf(x_{2,}y_2)$
$y_3^* = 2.42102500000000 + 0.1(2.42102500000000 - 0.2)$
$y_3^* = 2.64312750000000$
Putting i and y_3^* in y_{i+1} we have
$y_3 = y_2 + \frac{0.1}{2}[f(x_{2,}y_2) + f(x_{3,}y_3^*)]$
$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$
$y_3 = 2.42102500000 + \frac{0.1}{2}[(2.42102500000 - 0.2) + (2.64312750000 - 0.3)]$
$y_3 = 2.42102500000 + 0.05(2.221025 + 2.343127)$
$y_3 = 2.64923262500000$
When $x_3 = 0.3, y_2 = 2.64923262500000$
For $i = 3$
$y_4^* = y_3 + hf(x_3y_3)$
$y_4^* = 2.64923262500000 + 0.1(2.64923262500000 - 0.3)$
$y_4^* = 2.88415588750000$
Putting i and y_4^* in y_{i+1} we have
$y_4 = y_3 + \frac{0.1}{2}[f(x_{3,}y_3) + f(x_{4,}y_4^*)]$
$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$
$y_4 = 2.64923262500000 + \frac{0.1}{2}[(2.64923262500000 - 0.3) + (2.88415588750000 - 0.4)]$
$y_4 = 2.64923262500000 + 0.05(2.34923262500 + 2.484155887500)$
$y_4 = 2.89090205062500$
When $x_4 = 0.4, y_4 = 2.89090205062500$

For $i = 4$
$y_5^* = y_4 + hf(x_4y_4)$
$y_5^* = 2.89090205062500 + 0.1(2.89090205062500 - 0.4)$
$y_5^* = 3.13999225587500$
Putting i and y_5^* in y_{i+1} we have
$y_5 = y_4 + \frac{0.1}{2}[f(x_{4,}y_4) + f(x_{5,}y_5^*)]$

$x_5 = x_4 + h = 0.4 + 0.1 = 0.5$
$y_5 = 2.89090205062500 + \frac{0.1}{2}[(2.89090205062500 - 0.4) + (3.13999225587500 - 0.5)]$
$y_5 = 2.89090205062500 + 0.05(2.49090205 + 2.639992255) = 2.42131 + 0.11259225$
$y_5 = 3.14744676594063$
When $x_5 = 0.5, y_5 = 3.14744676594063$
For $i = 5$
$y_6^* = y_5 + hf(x_{5,}y_5)$
$y_6^* = 3.147446765940623 + 0.1(3.147446765940623 - 0.5)$
$y_6^* = 3.41219144253469$
Putting i and y_6^* in y_{i+1} we have
$y_6 = y_5 + \frac{0.1}{2}[f(x_{5,}y_5) + f(x_{6,}y_6^*)]$
$x_6 = x_5 + h = 0.5 + 0.1 = 0.6$
$y_6 = 3.14744676594063 + \frac{0.1}{2}[(3.14744676594063 - 0.5) + (3.41219144253469 - 0.6)]$
$y_6 = 3.14744676594063 + 0.05(2.647446765 + 2.812191445) = 2.53390 + 0.115800$
$y_6 = 3.42042867636439$
When $x_6 = 0.6, y_6 = 3.42042867636439$
For $i = 6$
$y_7^* = y_6 + hf(x_{6,}y_6)$
$y_7^* = 3.42042867636439 + 0.1(3.42042867636439 - 0.6)$
$y_7^* = 3.70247154400083$
Putting i and y_7^* in y_{i+1} we have
$y_7 = y_6 + \frac{0.1}{2}[f(x_{6,}y_6) + f(x_{7,}y_7^*)]$
$x_7 = x_6 + h = 0.6 + 0.1 = 0.7$
$y_7 = 3.42042867636439 + \frac{0.1}{2}[(3.42042867636439 - 0.6) + (3.70247154400083 - 0.7)]$
$y_7 = 3.42042867636439 + 0.05(2.82042867636439 + 3.00247154400083)$
$y_7 = 3.71157368738265$
When $x_7 = 0.7, y_7 = 3.71157368738265$
For $i = 7$
$y_8^* = y_7 + hf(x_{7,}y_7)$
$y_8^* = 3.71157368738265 + 0.1(3.71157368738265 - 0.7)$
$y_8^* = 4.01273105612092$
Putting i and y_8^* in y_{i+1} we have
$y_8 = y_7 + \frac{0.1}{2}[f(x_{7,}y_7) + f(x_{8,}y_8^*)]$
$x_8 = x_7 + h = 0.7 + 0.1 = 0.8$
$y_8 = 3.71157368738265 + \frac{0.1}{2}[(3.71157368738265 - 0.7) + (4.01273105612092 - 0.8)]$
$y_8 = 3.71157368738265 + 0.05(3.01157368738265 + 3.21273105612092)$
$y_8 = 4.02278892455783$
When $x_8 = 0.8, y_8 = 4.02278892455783$
For $i = 8$
$y_9^* = y_8 + hf(x_{8,}y_8)$
$y_9^* = 4.02278892455783 + 0.1(4.02278892455783 - 0.8)$
$y_9^* = 4.34506781701361$
Putting i and y_9^* in y_{i+1} we have
$y_9 = y_8 + \frac{0.1}{2}[f(x_{8,}y_8) + f(x_{9,}y_9^*)]$
$x_9 = x_8 + h = 0.8 + 0.1 = 0.9$

$y_9 = 4.02278892455783 + \frac{0.1}{2}[(4.02278892455783 - 0.8) + (4.34506781701361 \ - 0.9)]$
$y_9 = 4.02278892455783 + 0.05(3.22278892455783 + 3.44506781701361)$
$y_9 = 4.35618176163640$
When $x_9 = 0.9, y_9 = 4.35618176163640$
For $i = 9$
$y_{10}^* = y_9 + hf(x_{9,}y_9)$
$y_{10}^* = 4.35618176163640 + 0.1(4.35618176163640 - 0.9)$
$y_{10}^* = 4.701799937800004$
Putting i and y_{10}^* in y_{i+1} we have
$y_{10} = y_9 + \frac{0.1}{2}[f(x_{9,}y_9) + f(x_{10,}y_{10}^*)]$
$x_{10} = x_9 + h = 0.9 + 0.1 = 1.0$
$y_{10} = 4.35618176163640 + \frac{0.1}{2}[(4.35618176163640 - 0.9) + (4.70179993780004 \ - 1.0)]$
$y_{10} = 4.35618176163640 + 0.05(3.55618176163640 + 4.70179993780004)$
$y_{10} = 4.71408084660822$
When $x_{10} = 1.0, y_{10} = 4.71408084660822$

Table 7: Tabular Representation of Modified Euler Method for the step size *h = 0.1*

Value of x	Approx value of y[y(x)]	Exact Solution of y[Y(x)]	Error Incurred[e_{MEM}]
0.0	2.00000000	2.00000000	0.00000000
0.1	2.20500000000000	2.20517091807565	1.70918×10^{-4}
0.2	2.42102500000000	2.42140275816017	3.77758×10^{-4}
0.3	2.64923262500000	2.64985880757600	6.26182×10^{-4}
0.4	2.89090205062500	2.89182469764127	9.22647×10^{-4}
0.5	3.14744676594063	3.14872127070013	1.274505×10^{-3}
0.6	3.42042867636439	3.42211880039051	1.690124×10^{-3}
0.7	3.71157368738265	3.71375270747048	2.179020×10^{-3}
0.8	4.02278892455783	4.02554092849247	2.752004×10^{-3}
0.9	4.35618176163640	4.35960311115695	3.421350×10^{-3}
1.0	4.71408084660822	4.71828182845905	4.200982×10^{-3}

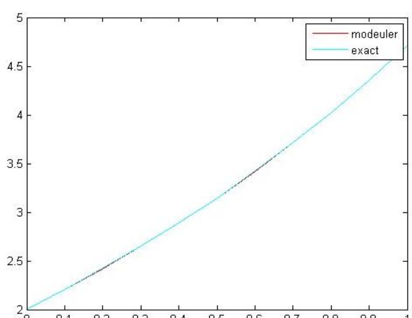

Figure 5: Graphical Illustration of Modified Euler Solution and Exact Solution for *h = 0.1*
By Runge Kutta Method:
From equation (27), we have
$y_{i+1} = y_i + \frac{1}{6}(\mathrm{k}_1 + 2\mathrm{k}_2 + 2\mathrm{k}_3 + \mathrm{k}_4)$

Where $k_1 = hf(x_i, y_i), k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$,
$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$,
$k_4 = hf(x_i + h, y_i + k_3)$

For $i = 0 k_1 = hf(x_i, y_i) = hf(x_0, y_0) = 0.1(2-0)$
$k_1 = 0.200000000000$
$K_2 = 0.1f\left(0 + \frac{0.1}{2}, 2 + \frac{0.200000000000}{2}\right) = 0.05f(0.05, 2.100000000000)$
$k_2 = 0.1(2.100000000000 - 0.05) = 0.20500000000000$
Therefore $k_2 = 0.20500000000000$
$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$
$k_3 = 0.1f\left(0 + \frac{0.1}{2}, 2 + \frac{0.20500000000000}{2}\right) = 0.1f(0.05, 2.10250000000)$
$k_3 = 0.1(2.10250000000 - 0.05) = 0.20525000000000$
Therefore $k_3 = 0.20525000000000$
$k_4 = hf(x_0 + h, y_0 + k_3)$
$k_4 = 0.1f(0 + 0.1, 2 + 0.20525000000000) = 0.1f(0.1, 0.20525000000000)$
$k_4 = 0.1(0.20525000000000 - 0.1) = 0.21052500000000$
Therefore $k_4 = 0.21052500000000$
Putting i, k_1, k_2, k_3 and k_4 in the equation below:
$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$$y_1 = 2 + \frac{1}{6}(0.200000000000 + 2 * 0.20500000000000 + 2 * 0.20525000000000 + 0.21052500000000)$$
$y_1 = 2.20517083333333$
For $i = 1$ and $x_1 = x_0 + h = 0 + 0.1 = 0.1$
$k_1 = hf(x_1, y_1) = 0.1(2.20517083333333 - 0.1)$
$k_1 = 0.21051708333333$
$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$
$k_2 = 0.1f\left(0.1 + \frac{0.1}{2}, 2.20517083333333 + \frac{0.21051708333333}{2}\right)$
$k_2 = 0.21604293750000$
$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$
$k_3 = 0.1f\left(0.1 + \frac{0.1}{2}, 2.20517083333333 + \frac{0.21604293750000}{2}\right)$
$k_3 = 0.21631923020833$
$k_4 = hf(x_1 + h, y_1 + k_3)$
$k_4 = 0.1f(0.1 + 0.1, 2.20517083333333 + 0.21631923020833)$
$k_4 = 0.22214900635417$
For $i = 2$ and $x_2 = x_0 + h = 0.1 + 0.1 = 0.2$
$k_1 = hf(x_2, y_2) = 0.1(2.4214025708509 - 0.2)$
$k_1 = 0.22214025708507$
$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$
$k_2 = 0.1f\left(0.2 + \frac{0.1}{2}, 2.4214025708509 + \frac{0.22214025708507}{2}\right)$
$k_2 = 0.22824726993932$
$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$

$k_3 = 0.1f\left(0.2 + \frac{0.1}{2}, 2.4214025708509 + \frac{0.22824726993932}{2}\right)$
$k_3 = 0.22855262058204$
$k_4 = hf(x_2 + h, y_2 + k_3)$
$k_4 = 0.1f(0.2 + 0.1, 2.4214025708509 + 0.22855262058204)$
$k_4 = 0.23499551914327$
Putting i, k_1, k_2, k_3 and k_4 in the equation below:
$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4$
$y_3 = 2.4214025708509 + \frac{1}{6}(0.22214025708507 + 2 * 0.22824726993932 + 2 * 0.22855262058204 + 0.23499551914327)$
$y_3 = 2.64985849706254$
For $i = 3$ and $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$,
$k_1 = hf(x_3, y_3) = 0.1(2.64985849706254 - 0.3)$
$k_1 = 0.23498584970625$
$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right)$
$k_2 = 0.1f\left(0.3 + \frac{0.1}{2}, 2.64985849706254 + \frac{0.23498584970625}{2}\right)$
$k_2 = 0.24173514219157$
$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right)$
$k_3 = 0.1f\left(0.3 + \frac{0.1}{2}, 2.64985849706254 + \frac{0.24173514219157}{2}\right)$
$k_3 = 0.24207260681583$
$k_4 = hf(x_3 + h, y_3 + k_3)$
$k_4 = 0.1f(0.3 + 0.1, 2.64985849706254 + 0.24207260681583)$
$k_4 = 0.24919311038784$
$y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4$
$y_4 = 2.64985849706254 + \frac{1}{6}(0.23498584970625 + 2 * 0.24173514219157 + 2 * 0.24207260681583 + 0.24919311038784)$
$y_4 = 2.89182424008069$
For $i = 4$ and $x_4 = x_3 + h = 0.3 + 0.1 = 0.4$
$k_1 = hf(x_4, y_4) = 0.1(2.89182424008069 - 0.4)$
$k_1 = 0.24918242400807$
$k_2 = hf\left(x_4 + \frac{h}{2}, y_4 + \frac{k_1}{2}\right)$
$k_2 = 0.1f\left(0.4 + \frac{0.1}{2}, 2.89182424008069 + \frac{0.24918242400807}{2}\right)$
$k_2 = 0.25664154520847$
$k_3 = hf\left(x_4 + \frac{h}{2}, y_4 + \frac{k_2}{2}\right)$
$k_3 = 0.1f\left(0.4 + \frac{0.1}{2}, 2.89182424008069 + \frac{0.25664154520847}{2}\right)$
$k_3 = 0.25701450126849$
$k_4 = hf(x_4 + h, y_4 + k_3)$
$k_4 = 0.1f(0.4 + 0.1, 2.89182424008069 + 0.25701450126849)$
$k_4 = 0.26488387413492$
Putting i, k_1, k_2, k_3 and k_4 in the equation below:
$y_5 = y_4 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4$

$$y_5 = 2.89182424008069 + \frac{1}{6}(0,.24918242400807 + 2 * 0.25664154520847 + 2 * 0.25701450126849 + 0.26488387413492)$$
$y_5 = 3.14872063859684$
For $i = 5$ and $x_5 = x_4 + h = 0.4 + 0.1 = 0.5$
$k_1 = hf(x_5, y_5) = 0.1(3.14872063859684 - 0.5)$
$k_1 = 0.26487206385968$
$$k_2 = hf\left(x_5 + \frac{h}{2}, y_5 + \frac{k_1}{2}\right)$$
$$k_2 = 0.1f\left(0.5 + \frac{0.1}{2}, 3.14872063859684 + \frac{0.26487206385968}{2}\right)$$
$k_2 = 0.27311566705267$
$$k_3 = hf\left(x_5 + \frac{h}{2}, y_5 + \frac{k_2}{2}\right)$$
$$k_3 = 0.1f\left(0.5 + \frac{0.1}{2}, 3.14872063859684 + \frac{0.27311566705267}{2}\right)$$
$k_3 = 0.27352784721232$
$k_4 = hf(x_5 + h, y_5 + k_3)$
$k_4 = 0.1f(0.5 + 0.1, +3.14872063859684 + 0.27352784721232)$
$k_4 = 0.28222484858092$
Putting$i$,k_1,k_2,k_3 and k_4 in the equation below:
$$y_6 = y_5 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4$$
$$y_6 = 3.14872063859684 + \frac{1}{6}(0.26487206385968 + 2 * 0.27311566705267 + 2 * 0.27352784721232 + 0.28222484858092)$$
$y_6 = 3.42211796209193$
For $i = 6$ and $x_6 = x_5 + h = 0.5 + 0.1 = 0.6$
$k_1 = hf(x_6, y_6) = 0.1(3.42211796209193 - 0.6)$

$k_1 = 0.28221179620919$
$$k_2 = hf\left(x_6 + \frac{h}{2}, y_6 + \frac{k_1}{2}\right)$$
$$k_2 = 0.1f\left(0.6 + \frac{0.1}{2}, 3.42211796209193 + \frac{0.28221179620919}{2}\right)$$
$k_2 = 0.29132238601965$
$$k_3 = hf\left(x_6 + \frac{h}{2}, y_6 + \frac{k_2}{2}\right)$$
$$k_3 = 0.1f\left(0.6 + \frac{0.1}{2}, 3.42211796209193 + \frac{0.29132238601965}{2}\right)$$
$k_3 = 0.29177791551018$
$k_4 = hf(x_6 + h, y_6 + k_3)$
$k_4 = 0.1f(0.6 + 0.1, +3.42211796209193 + 0.29177791551018)$
$k_4 = 0.30138958776021$
Putting i, k_1, k_2 , k_3 and k_4 in the equation below:
$$y_7 = y_6 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4$$
$$y_7 = 3.42211796209193 + \frac{1}{6}(0.28221179620919 + 2 * 0.29132238601965 + 2 * 0.29177791551018 + 0.30138958776021)$$
$y_7 = 3.71375162659678$
For $i = 7$ and $x_7 = x_6 + h = 0.6 + 0.1 = 0.7$
$k_1 = hf(x_7, y_7) = 0.1(3.71375162659678 - 0.7)$
$k_1 = 0.30137516265968$

$k_2 = hf\left(x_7 + \frac{h}{2}, y_7 + \frac{k_1}{2}\right)$
$k_2 = 0.1f\left(0.7 + \frac{0.1}{2}, 3.71375162659678 + \frac{0.30137516265968}{2}\right)$
$k_2 = 0.31144392079266$
$k_3 = hf\left(x_7 + \frac{h}{2}, y_7 + \frac{k_2}{2}\right)$
$k_3 = 0.1f\left(0.7 + \frac{0.1}{2}, 0.30137516265968 + \frac{0.31144392079266}{2}\right)$
$k_3 = 0.31194735869931$
$k_4 = hf(x_7 + h, y_7 + k_3)$
$k_4 = 0.1f(0.7 + 0.1, +3.71375162659678 + 0.31194735869931)$
$k_4 = 0.32256989852961$
Putting$i$,k_1,k_2,k_3 and k_4 in the equation below:
$y_8 = y_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$y_8 = 3.71375162659678 + \frac{1}{6}(0.30137516265968 + 2 * 0.31144392079266 + 2 * 0.31194735869931 + 0.32256989852961)$
$y_8 = 4.02553956329231$
For $i = 8$ and $x_8 = x_7 + h = 0.7 + 0.1 = 0.8$
$k_1 = hf(x_8, y_8) = 0.1(4.02553956329231 - 0.8)$
$k_1 = 0.32255395632923$
$k_2 = hf\left(x_8 + \frac{h}{2}, y_8 + \frac{k_1}{2}\right)$
$k_2 = 0.1f\left(0.8 + \frac{0.1}{2}, 4.02553956329231 + \frac{0.32255395632923}{2}\right)$
$k_2 = 0.33368165414569$
$k_3 = hf\left(x_8 + \frac{h}{2}, y_8 + \frac{k_2}{2}\right)$
$k_3 = 0.1f\left(0.8 + \frac{0.1}{2}, 4.02553956329231 + \frac{0.33368165414569}{2}\right)$
$k_3 = 0.33423803903652$
$k_4 = hf(x_7 + h, y_7 + k_3)$
$k_4 = 0.1f(0.8 + 0.1, 4.02553956329231 + 0.33423803903652)$
$k_4 = 0.34597776023288$
Putting$i$,k_1,k_2,k_3 and k_4 in the equation below:
$y_9 = y_8 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$y_9 = 4.02553956329231 + \frac{1}{6}(0.32255395632923 + 2 * 0.33368165414569 + 2 * 0.33423803903652 + 0.34597776023288)$
$y_9 = 4.35960141378007$
For $i = 9$ and $x_9 = x_8 + h = 0.8 + 0.1 = 0.9$
$k_1 = hf(x_9, y_9) = 0.1(4.35960141378007 - 0.9)$
$k_1 = 0.34596014137801$
$k_2 = hf\left(x_9 + \frac{h}{2}, y_9 + \frac{k_1}{2}\right)$
$k_2 = 0.1f\left(0.9 + \frac{0.1}{2}, 4.35960141378007 + \frac{0.34596014137801}{2}\right)$
$k_2 = 0.35825814844691$
$k_3 = hf\left(x_9 + \frac{h}{2}, y_9 + \frac{k_2}{2}\right)$
$k_3 = 0.1f\left(0.9 + \frac{0.1}{2}, 4.35960141378007 + \frac{0.35825814844691}{2}\right)$

$k_3 = 0.35887304880035$
$k_4 = hf(x_9 + h, y_9 + k_3)$
$k_4 = 0.1f(0.9 + 0.1, 4.35960141378007 + 0.35887304880035)$
$k_4 = 0.34597776023288$
Putting i, k_1, k_2, k_3 and k_4 in the equation below:

$$y_{10} = y_9 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{10} = 4.35960141378007 + \frac{1}{6}(0.34596014137801 + 2 * 0.35825814844691 + 2 * 0.35887304880035 + 0.34597776023288)$$

$y_{10} = 4.71827974413516$

Table 8: Tabular Representation of Runge Kutta Method for the step size *h = 0.1*

Value of x	Approx value of y[y(x)]	Exact Solution of y[Y(x)]	Error Incurred[e_{RK4M}]
0.0	2.00000000	2.00000000	0.00000000
0.1	2.205170833333333	2.20517091807565	8.500000×10^{-8}
0.2	2.42140257085069	2.42140275816017	1.880000×10^{-7}
0.3	2.64985849706254	2.64985880757600	3.100000×10^{-7}
0.4	2.89182424008069	2.89182469764127	4.57000×10^{-7}
0.5	3.14872063859684	3.14872127070013	6.32000×10^{-7}
0.6	3.42211796209193	3.42211880039051	8.38000×10^{-7}
0.7	3.71375162659678	3.71375270747048	1.08000×10^{-6}
0.8	4.02553956329231	4.02554092849247	1.36500×10^{-6}
0.9	4.35960141378007	4.35960311115695	1.69000×10^{-6}
1.0	4.71827974413516	4.71828182845905	2.08400×10^{-6}

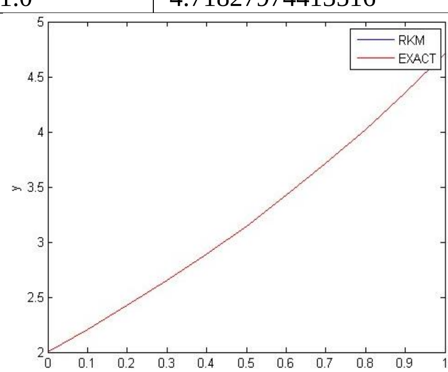

Figure 6: Graphical Illustration of Runge Kutta Solution and Exact Solution for *h = 0.1*
However, the Exact value of $y(x)$ is given as:
$y(x) = e^x + x + 1$
Then we generate the solution of the exact value to be:
$y(0.0)$ =2.00000000000000,
$y(0.1)$ =2.20517091807565,
$y(0.2)$ = 2.4214027581601,
$y(0.3)$ =2.64985880757600,
$y(0.4)$ =2.89182469764127,
$y(0.5)$ =3.14872127070013
$y(0.6)$ =3.42211880039051,

$y(0.7)$ =3.71375270747048,
$y(0.8)$ =4.02554092849247 $y(0.9) = 4.35960311115695$, $y(1.0) = 4.71828182845905$

COMPARATIVE RESULT OF EXAMPLE 2
Table 9: The Comparative Result of Picard method, Modified Euler Method and Runge Kutta Method for the step size *h = 0.1*

Value of x	$[y(x)_{PM}]$	$[y(x)_{MEM}]$	$[y(x)RK4M]$	Exact Solution of y[Y(x)]
0.0	2.00000000	2.00000000	2.00000000	2,00000000
0.1	2.1952000000	2.205000000000	2.205170833333	2.205170918076
0.2	2.4186666667	2.421025000000	2.421402570850	2.421402758160
0.3	2.6491625000	2.649232625000	2.649858497063	2.649858807576
0.4	2.8916480001	2.890902050625	2.891824240081	2.891824697641
0.5	3.1486762157	3.147446765941	3.148720638597	3.148721270700
0.6	3.4221072460	3.420428676364	3.422117962092	3.422118800391
0.7	3.7137497278	3.711573687383	3.713751626597	3.713752707470
0.8	4.0255341567	4.022788924558	4.025539563292	4.025540928492
0.9	4.3596012103	4.356181761636	4.359601413780	4.359603111157
1.0	4.7182757778	4.714080846608	4.718279744135	4.718281828459

ERROR INCURRED IN EXAMPLE 2
Table 10: Error Incurred from Picard's Method, Modified Euler Method and Runge Kutta Method

Value of x	$[Y(x) - y(x) = e_{PM}]$	$[Y(x) - y(x) = e_{MEM}]$	$[Y(x) - y(x) = e_{RK4M}]$
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	9.970918×10^{-3}	1.70918×10^{-4}	8.500000×10^{-8}
0.2	2.736092×10^{-3}	3.77758×10^{-4}	1.880000×10^{-7}
0.3	6.963070×10^{-4}	6.26182×10^{-4}	3.100000×10^{-7}
0.4	1.766970×10^{-4}	9.22647×10^{-4}	4.57000×10^{-7}
0.5	4.505500×10^{-5}	1.274505×10^{-3}	6.32000×10^{-7}
0.6	1.155540×10^{-5}	1.690124×10^{-3}	8.38000×10^{-7}
0.7	2.98000×10^{-6}	2.179020×10^{-3}	1.08000×10^{-6}
0.8	6.77200×10^{-6}	2.752004×10^{-3}	1.36500×10^{-6}
0.9	1.90100×10^{-6}	3.421350×10^{-3}	1.69000×10^{-6}
1.0	6.05100×10^{-6}	4.200982×10^{-3}	2.08400×10^{-6}

DISCUSSION OF RESULT

The Picard's, Modified Euler and Runge Kutta methods have been derived in this work, computer programs have also been written for the implementation of these methods. The three methods where tested respectively on two numerical examples which are linear initial value problems of first order ordinary differential equation.

The result obtain by this three methods were tabulated as seen in table 1 to 10. From table 1 to 10, we observe that the result obtain by Picard's, Modified Euler and Runge Kutta Methods compare favorably with the exact solution. Also the Runge Kutta method converge faster to the exact solution than Picard's and Modified Euler methods, this could be seen from table 5 and 10 as the error incurred by Runge Kutta method is less when compared to Picard's and Modified Euler methods.

This makes Runge Kutta method more accurate and effective in solving initial value problems of first order ordinary differential equations, In addition each numerical example is graphically represented.

CONCLUSION

Solution has been obtained for the problem described in the given example and justified to conclude that the numerical solutions obtained by the three methods are in good agreement with the exact solutions. Although the Runge Kutta method is found to be generally more accurate for the solution of First order ordinary differential equation with initial value problems. However, the Picard's and Modified Euler Methods are found to be less accurate. Thus by the aim and objectives of this thesis, we successfully conclude that the Runge Kutta method is more accurate, efficiency and reliable for solving problems.

RECOMMENDATION:

Numerical Solution of first order ordinary differential equation using Picard's, Modified Euler and Runge Kutta Methods has been studied in this work, result shows that Runge Kutta Method give better accuracy than the other two Methods. We therefore recommend Runge Kutta for the solution of first order ordinary differential equation

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