Journal of the Nigerian Association of Mathematical Physics
Volume 51 (May, 2019 Issue), pp55 – 64
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NONLINEAR DYNAMICS OF A SOIL-LIVESTOCK INTERACTION WITH HUMAN EFFORT: A MATHEMATICAL MODEL

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Abstract

In this paper a nonlinear mathematical deterministic model is presented and analysed to study the contribution of anthropogenic excesses to soil degradation. The proposed model is analysed qualitatively using stability theory of differential equations. Using MATLAB, it was shown that the deployment of both effort and livestock activity on the soil human would exacerbate degradation with varying levels of intensity. It is also shown that a facilitating the decomposition rate of livestock droppings can sustainably reclaim the fertility of degraded topsoil.

1.0. Introduction

Agriculture is crucial to the development and livelihoods of the rural population in Africa [1]. Animal production (livestock) is globally considered to be done on a greater percentage of agricultural land [2]. In its report, FAO and others outlined the enormous contribution of animal production (livestock) sector to the global agricultural GDP where it is estimated to employ over a billion people apart from being the major source of livelihoods for a billion people, majority of which are the world's poor [2]. Livestock products are very rich source of protein and other essential micronutrients [3]. Ironically, however, an estimated 925 million of the world's population are feared to be seriously undernourished for lack of appropriate and sufficient food supply [4, 5]. Livestock by-products are the major raw materials for a range of essential household products and farm manure [3]. In developing countries for instance, draught animals are estimated to provide 80% of the power used for farming [6]. It is estimated that about 52 percent of draught power comes from animals. In India, selling cattle dung for fuel to urban centres can supply up to 60% of the income of the poor village family. Its prospects in developing countries are bright following its increasing dietary preference over staple food. The number of ruminant animals (like goats and sheep) produced per unit of agricultural area in developing countries is almost double that of developed countries. This has continued to mount significant pressure on scarce resources such as arable land and water are serious concerns [7]. A high livestock density exerts a considerable pressure on soil's physical, biological and chemical conditions [8]. Categories of soil degradation include decline in soil fertility, erosion, deterioration of the structural composition of the soil and changes in alkalinity and acidity [9]. Additionally, animal agriculture contributes to greenhouse gas emission in the form of carbon dioxide, methane and nitrous oxide [4, 5] with Africa feared to be worst hit [6]. From the foregoing, an understanding of the ecological and other aspects of animal agriculture is critical to ensure access to safe and healthy food and sustainable environment [12]. Factors influencing animal production and utilisation range from mechanical, like farm management and soil condition, among others [7 – 14]; to biological, which include health, temperature, reproduction and nutrition etc. [11 – 14]; and management and socio-economic factors [15 – 17]. There is need to simultaneously consider these factors if efficient management systems of animal traction are to be developed [18].

In this study, we investigate the effect of both human effort (labour), through rearing (especially herding) and the direct interaction of livestock, through grazing. Open grazing, notwithstanding its attendant threats, is a basic farming feature of developing countries [2]. It is common to see herdsmen circumventing the lengths and breadths of Nigeria with their livestock in search of pasture. While it is true that the droppings (dungs and urine) of these animals have a fertility enriching effect on the soil, the uncontrolled removal of the vegetative cover of the soil as a result of grazing gradually but consistently exposes the soil. These exposures have become very important agents of soil depletion. Vast portions of arable land mass have been rendered defective due to depletion.

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2. Model formulation

In this section, we present our proposed model for the phenomenal depletion of fertile topsoil through the direct and indirect activities of human beings and grazing livestock. We envisage a concentration of human beings that with a pool of deployable effort or labour, represented by P, the density of grazing livestock denoted by R and two categories of mutually existing landmasses, fertile or arable and degrade topsoil, denoted, respectively, by S_{A} and S_{D}. it is considered that the primary agent for degrading a fertile topsoil is the deployment of unsustainable land use practices at excessive levels. We are basically concerned with the depletion of fertile topsoil due to land use practices. Thus, in emphasising on the extent of human labour and livestock grazing on the soil, we concentrate on the aggregate impact of the depletive potentiality of overgrazing and excessive human effort. The following assumptions are made:

1. The basic interaction pattern of the model is governed by simple law of mass action
2. Topsoil depletion is caused primarily by action of both human beings and grazing livestock on the soil
3. The interaction between human effort (labour) and livestock and that between livestock and the topsoil is symbiotic in nature
4. Depending on the aggression of an activity on the soil, a soil type could be permanently rendered completely unsuitable for any form of vegetation purposes (for instance mining sites)
5. Human labour, livestock density and natural growth rate of fertile topsoil are all constant
6. The words soil and land are used interchangeably to mean the top layer of owned ground

2.1. Basic properties of the model

Following from the designations above, human labour, livestock density and aggregate landmass are accrued at constant rates, each denoted respectively as Π, Φ and Δ. The rates of decreases of each is assumed to be either directly proportional to their respective cumulative concentrations and their number densities, or their individual number densities. Following from the above, the dynamics of the is proposed to be governed by the following system of nonlinear differential equations,

$$\begin{aligned}
\frac{dP}{dt} &= \Pi - \lambda P - \alpha PR - \phi_1 PS_{\mathrm{A}} - \phi_2 PS_{\mathrm{D}}, \\
\frac{dR}{dt} &= \Phi - \rho PR - \psi R, \\
\frac{dS_{\mathrm{A}}}{dt} &= \Delta - \gamma(\upsilon + \delta_{\mathrm{C}})PS_{\mathrm{A}} + [\theta(\psi + \xi) - \mu]RS_{\mathrm{A}} - (\delta_{\mathrm{S}} + \delta_{\mathrm{P}})S_{\mathrm{A}}, \\
\frac{dS_{\mathrm{D}}}{dt} &= \delta_{\mathrm{S}} S_{\mathrm{A}} - \gamma_1(\upsilon + \delta_{\mathrm{C}})PS_{\mathrm{D}} + [\theta_1(\psi + \xi) - \mu]RS_{\mathrm{D}} - \delta_{\mathrm{P}} S_{\mathrm{D}}.
\end{aligned} \tag{1}$$

where Π, Φ and Δ the constant labour recruitment rate, livestock acquisition rate and natural growth rate coefficient of the densities of human effort, livestock and landmass respectively. α, ϕ_1 and ϕ_2 are the deployment rate coefficients of human labour due to the interaction between human beings and, respectively, livestock, fertile and degraded topsoil. Further, the constant λ models the natural diminutive rate of labour (due to either death, migration or urbanisation). The constants υ and δ_C are designated to model, respectively, fodder harvest (including deforestation) and infrastructure expansion rate. While the modification parameters γ and γ_1 account for the relative intensity of the combined effect of labour on S_{A} and S_{D} respectively. The symbiotic relationship between livestock and landmass is modelled by the parameters ψ, ξ and μ. ψ and ξ represent the death/carcass deposition and dung dropping rate coefficients on the soil with the parameters θ and θ_1 modifying their combined decomposition rate. μ on its part accounts for the cumulative derivable benefit of livestock from the soil which unfortunately is exposing the soil to degradation. Finally, the processes outlined above are assumed to result in soil vegetation cover thus leading to degradation. As a result, the parameter δ_{S} accounts for reclaimable fertile topsoil loss that adds to the cumulative density of degraded topsoil, while δ_{P} models the permanent depletion (due to excessive mining activities and other extreme land use practices) of the vegetation cover of both fertile and degraded topsoil.

3. Stability analysis

In this section, we use the stability theory of differential equations to analyse the complete model (1) under two basic phenomenal settings, namely:

Case I: $\upsilon = \mu = \phi_1 = \phi_2 = \delta_{\mathrm{P}} = \delta_{\mathrm{S}} = 0$ (a zero degradative anthropogenic activity on soil topsoil)

Case II: $\upsilon > 0, \mu > 0, \phi_1 > 0, \phi_2 > 0, \delta_{\mathrm{P}} > 0, \delta_{\mathrm{S}} > 0$ (a constant degradative anthropogenic activity on soil topsoil)

The first case corresponds to an absolute zero-grazing (pen) feeding regime, whereas the second corresponds to an absolute grazing-dependent feeding system

3.1. Case I. When $\upsilon = \mu = \phi_1 = \phi_2 = \delta_{\mathrm{P}} = \delta_{\mathrm{S}} = 0$ (a zero-depletion-case-scenario). In this case, the model has only on nonnegative equilibrium $E_0 = \left(P^*, R^*, S_{\mathrm{A}}^*, 0\right)$ in the $P - R - S_{\mathrm{A}} - S_{\mathrm{D}}$ space.

To study the local stability behaviour of the of E_0, we propose the following theorem

Theorem 1. *Let the following inequalities hold*

$$g_1\left(\psi + \rho P^*\right)\left[\gamma\delta_{\mathrm{C}} P^* - \theta(\psi + \xi)R^*\right] > g_2\left[\phi(\psi + \xi)S_{\mathrm{A}}^*\right]^2,$$

Where g_1 and g_2 are positive constants to be suitably chosen. Then E_0 is locally asymptotically stable.

Proof. To establish the prove of the above theorem, we begin by linearizing the model (1) about E_0 by taking the following transformations:

$$P = P^* + P_1,\ R = R^* + R_1,\ S_{\mathrm{A}} = S_{\mathrm{A}}^* + S_{\mathrm{A1}},\ S_{\mathrm{D}} = S_{\mathrm{D}}^* + S_{\mathrm{D1}},$$

where $P_1,\ R_1,\ S_{\mathrm{A1}},\ S_{\mathrm{D1}}$ are small perturbations about E_0. Then we consider the following positive definite function:

$$F = \frac{1}{2}\left(P_1^2 + g_1 R_1^2 + g_2 S_{\mathrm{A1}}^2\right).$$

Then the time derivative of F along the linear version of the model (1) is

$$\begin{aligned}\dot{F} = &-\left(\lambda + \alpha R^* + \phi_1 S_{\mathrm{A}}^*\right)P_1^2 - \alpha P^* P_1 R_1 - \phi_1 P^* P_1 S_{\mathrm{A1}} - \rho g_1 R^* P_1 R_1 - g_1\left(\psi + \rho P^*\right)R_1^2 \\ &- g_2\gamma\delta_{\mathrm{C}} S_{\mathrm{A}}^* P_1 S_{\mathrm{A1}} + g_2\theta(\psi + \xi)S_{\mathrm{A}}^* R_1 S_{\mathrm{A1}} - g_2\left[\gamma\delta_{\mathrm{C}} P^* - \theta(\psi + \xi)R^*\right]S_{\mathrm{A1}}^{*2}.\end{aligned}$$

Writing the above expression further as sum of quadratic forms, gives

$$\dot{F} = -b_{11}P_1^2 - b_{12}P_1R_1 - b_{22}R_1^2 - b_{11}P_1^2 - b_{13}P_1S_{\mathrm{A1}} - b_{33}S_{\mathrm{A1}}^2 - b_{22}R_1^2 + b_{23}R_1S_{\mathrm{A1}} - b_{33}S_{\mathrm{A1}}^2,$$

where

$$b_{11} = \frac{1}{2}\left(\lambda + \alpha R^* + \phi_1 S_{\mathrm{A}}^*\right),\ b_{12} = \alpha P^* + \rho g_1 R^*,\ b_{22} = g_1\left(\psi + \rho P^*\right),$$

$$b_{13} = \phi_1 P^* + \gamma g_2 \delta_{\mathrm{C}} S_{\mathrm{A}}^*,\ b_{23} = g_2\theta(\psi + \xi)S_{\mathrm{A}}^*,\ b_{33} = \frac{1}{2}g_2\left[\gamma\delta_{\mathrm{C}} P^* - \theta(\psi + \xi)R^*\right].$$

Sufficient conditions for negative definiteness of dF/dt are that the following conditions hold:

$$b_{12}^2 < b_{11}b_{22} \text{ and } b_{13}^2 < b_{11}b_{33}. \quad (2)$$

By choosing

$g_1 = 1$ and $g_2 < \dfrac{\left(\psi + \rho P^*\right)\left[\gamma\delta_{\mathrm{C}} P^* - \theta(\psi + \xi)R^*\right]}{\theta^2(\psi + \xi)^2 S_{\mathrm{A}}^2}$, then obviously $\dfrac{\left(\psi + \rho P^*\right)\left(\lambda + \alpha R^* + \phi_1 S_{\mathrm{A}}^*\right)}{\left(\alpha P^* + \rho R^*\right)^2} > 1.$

Thus, the positive equilibrium, E_0, LAS.

To verify the global (nonlinear) stability behaviour of E_0, we propose the following theorem

Thus, F is a Lyapunov function with respect to E_0, whose domain contains the region of attraction D, and the prove follows.

The implication of this result is that human effort, density of livestock and mass of fertile topsoil settle down at steady state under some specific parametric conditions.

3.2. Case II. Here, we analyse the system (1) under the conditions: $\upsilon > 0, \mu > 0, \phi_1 > 0, \phi_2 > 0,$ and $\delta_{\mathrm{P}} > 0, \delta_{\mathrm{S}} > 0$ (a depletion-driven-interactional case scenario). Here, also as previously, the model is characterised by only one nonnegative equilibrium denoted by $E_1 = \left(P^{**}, R^{**}, S_{\mathrm{A}}^{**}, S_{\mathrm{D}}^{**}\right)$, in the $P - R - S_{\mathrm{A}} - S_{\mathrm{D}}$ space. where $P^{**}, R^{**}, S_{\mathrm{A}}^{**}$ and $, S_{\mathrm{D}}^{**}$ are the positive solutions of the following algebraic equations

$$\begin{aligned}&\Pi - \lambda P^{**} - \alpha P^{**}R^{**} - \phi_1 P^{**}S_{\mathrm{A}}^{**} - \phi_2 P^{**}S_{\mathrm{D}}^{**} = 0, \\ &\Phi - \rho P^{**}R^{**} - \psi R^{**} = 0, \\ &\Delta - \gamma\left(\upsilon + \delta_{\mathrm{C}}\right)P^{**}S_{\mathrm{A}}^{**} + \left[\theta(\psi + \xi) - \mu\right]R^{**}S_{\mathrm{A}}^{**} - \left(\delta_{\mathrm{P}} + \delta_{\mathrm{S}}\right)S_{\mathrm{A}}^{**} = 0, \\ &\delta_{\mathrm{S}}S_{\mathrm{A}}^{**} - \gamma_1\left(\upsilon + \delta_{\mathrm{C}}\right)P^{**}S_{\mathrm{D}}^{**} + \left[\theta_1(\psi + \xi) - \mu\right]R^{**}S_{\mathrm{D}}^{**} - \delta_{\mathrm{P}}S_{\mathrm{D}}^{**} = 0,\end{aligned} \quad (3)$$

given, respectively, by

$$P^{**} = \frac{\Pi}{\lambda + \alpha f_1(P^{**}) + \phi_1 f_2(P^{**}) + \phi_2 f_3(P^{**})}, R^{**} = \frac{\Phi}{\psi + \rho P^{**}} = f_1(P^{**}),$$

$$S_{\mathrm{A}}^{**} = \frac{\Delta}{(\delta_{\mathrm{P}} + \delta_{\mathrm{S}}) + \gamma(\upsilon + \delta_{\mathrm{C}})P^{**} - [\theta(\psi + \xi) - \mu] f_1(P^{**})} = f_2(P^{**}),$$

$$S_{\mathrm{D}}^{**} = \frac{\delta_{\mathrm{S}} S_{\mathrm{A}}^{**}}{\delta_{\mathrm{P}} + \gamma_1(\upsilon + \delta_{\mathrm{C}})P^{**} - [\theta_1(\psi + \xi) - \mu] f_1(P^{**})} = f_3(P^{**}).$$

It can be observed that $P^{**} > 0$ provided that $R^{**} > 0,$ $S_{\mathrm{A}}^{**} > 0$ and $S_{\mathrm{D}}^{**} > 0.$ Additionally, it can be noted that in the absence of any form of shear stress on arable topsoil, $\delta_{\mathrm{S}} = 0,$ (due to a zero overgrazing regime and sustainable land management practices), soil degradation will be non-existent, $S_{\mathrm{D}}^{**} \to 0.$

Now let, $F(P) = \Pi - \frac{1}{\psi}\{\lambda\psi + \alpha\Phi + \psi[\phi_1 f_1(P^{**}) + \phi_2 f_2(P^{**})]\}P.$ (4)

It easily to verified that $F(0) = \Pi > 0$ and $F\left(\frac{\Pi}{\lambda}\right) = -\frac{1}{\psi}\{\alpha\Phi + \psi[\phi_1 f_1(P^{**}) + \phi_2 f_2(P^{**})]\}P < 0.$ It can further be checked that $F'(P) < 0,$ provided that

$$\alpha\Phi + \psi\{\lambda + \phi_1[f_1(P^{**}) + Pf_1(P^{**})] + \phi_2[f_2(P^{**}) + Pf_2(P^{**})]\} > 0. \quad (5)$$

Thus, there exists a unique positive root (say P^{**}) in $0 < P^{**} < \Pi/\lambda$ provided the condition (5) holds. Therefore, using the value of $P^{**},$ the corresponding values of $R^{**}, S_{\mathrm{A}}^{**}$ and S_{D}^{**} can be computed from their respective defining expressions in (3) above.

To investigate the LAS of the $E_1,$ we have verified that each eigenvalue of the variational matrix V_1 corresponding to $E_1,$ given as

$$V^* = \begin{pmatrix} -\lambda - \alpha R^{**} - \phi_1 S_{\mathrm{A}}^{**} - \phi_2 S_{\mathrm{D}}^{**} & -\alpha P^{**} & -\phi_1 P^{**} & -\phi_2 P^{**} \\ -\rho R^{**} & -\psi - \rho P^{**} & 0 & 0 \\ -\gamma a_0 S_{\mathrm{A}}^{**} & a_1 S_{\mathrm{A}}^{**} & a_1 R^{**} - \gamma a_0 P^{**} - (\delta_{\mathrm{P}} + \delta_{\mathrm{S}}) & 0 \\ -\gamma a_0 S_{\mathrm{D}}^{**} & a_1 S_{\mathrm{D}}^{**} & \delta_{\mathrm{S}} & a_2 R^{**} - \gamma_1 a_0 P^{**} - \delta_{\mathrm{P}} \end{pmatrix},$$

where $a_0 = \upsilon + \delta_{\mathrm{C}},$ $a_1 = \theta(\psi + \xi) - \mu, a_2 = \theta_1(\psi + \xi) - \mu.$

has a negative value. Thus E_1 is LAS

To investigate the GAS of $E_1,$ we proceed as follows

Theorem 2: Let the following inequalities hold

$$\frac{9P^{**}R^{**}}{4\Phi\,\Pi}(\alpha P^{**} + \rho R^{**})^2 < 1,$$

$$\frac{16\Phi\,\Pi\Delta^2}{81 a_1^2 P^{**3} R^{**} S_{\mathrm{A}}^{**4}} > \left(\phi_1 + \phi_2 + \frac{4\gamma a_0 \Delta\Phi}{a_1^2 R^{**} S_{\mathrm{A}}^{**2}}\right), \quad (6)$$

then E_1 is locally asymptotically stable (LAS).

Pursuant to the investigation of the global stability behaviour of the $E_1,$ it is necessary that we state a lemma, which establishes the region of attraction for the system (1). The proof of the lemma is easy, and hence is omitted.

In the following theorem, the global stability behaviour of E_1 is studied

Theorem 3: Let the following inequalities hold,

$$\frac{P^* R^*}{\mathrm{F\,P}}(a P^* + r R^*)^2 < \frac{4}{9},$$

$$\frac{16\mathrm{F\,D}^2\mathrm{P}}{81 a_1^2 P^3 R^* S_{\mathrm{A}}^{*4}} > \left(f_1 + f_2 + g a_0 \frac{4\mathrm{F\,D}}{9 a_1^2 R^* S_{\mathrm{A}}^{*2}}\right)^2;$$

then E_1 is globally asymptotically stable (GAS).

Proof: Consider the following positive definite function about E_1

$$U = \frac{1}{2}\left\{ \left(P - P^{**}\right)^2 + c_1\left(R - R^{**}\right)^2 + c_2\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right)^2 + c_3\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right)^2 \right\}$$

where the $c_i = \{i = 1, 2, 3\}$ are positive constants to be suitably choosing

Thus, algebraically simplifying the time derivative of U along the solutions of the model (1) gives

$$\begin{aligned}\dot{U} &= \left(P - P^{**}\right)\left[-\left(\lambda + \alpha R^{**} + \phi_1 S_\mathrm{A}^{**} + \phi_2 S_\mathrm{D}^{**}\right)\right]P_1 - \alpha P^{**} R_1 - \phi_1 P^{**} S_{\mathrm{A}1} - \phi_2 P^{**} S_{\mathrm{D}1} \\ &+ c_1\left(R - R^{**}\right)\left[-\rho R^{**} P_1 - \left(\psi + \rho P^{**}\right)R_1\right] \\ &+ c_2\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right)\left\{-\gamma a_0 S_\mathrm{A}^{**} P_1 + a_1 S_\mathrm{A}^{**} R_1 - \left[\gamma a_0 P^{**} - a_1 R^{**} + \left(\delta_\mathrm{P} + \delta_\mathrm{S}\right)\right]S_{\mathrm{A}1}\right\} \\ &+ c_3\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right)\left\{-\gamma_1 a_0 S_\mathrm{D}^{**} P_1 + a_2 S_\mathrm{D}^{**} R_1 + \delta_\mathrm{S} S_{\mathrm{A}1} - \left(\gamma_1 a_0 P^{**} - a_2 R^{**} + \delta_\mathrm{P}\right)S_{\mathrm{A}1}\right\}.\end{aligned}$$

A little simplification gives

$$\begin{aligned}\dot{U} &= -\left(\lambda + \alpha R^{**} + \phi_1 S_\mathrm{A}^{**} + \phi_2 S_\mathrm{D}^{**}\right)\left(P - P^{**}\right)^2 - \alpha P^{**}\left(P - P^{**}\right)\left(R - R^{**}\right) - \phi_1 P^{**}\left(P - P^{**}\right)\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right) \\ &- \phi_2 P^{**}\left(P - P^{**}\right)\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right) - c_1 \rho R^{**}\left(P - P^{**}\right)\left(R - R^{**}\right) - c_1\left(\psi + \rho P^{**}\right)\left(R - R^{**}\right)^2 \\ &- \gamma a_0 c_2 S_\mathrm{A}^{**}\left(P - P^{**}\right)\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right) + a_1 c_2 S_\mathrm{A}^{**}\left(R - R^{**}\right)\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right) \\ &- c_2\left[\gamma a_0 P^{**} - a_1 R^{**} + \left(\delta_\mathrm{P} + \delta_\mathrm{S}\right)\right]\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right)^2 - \gamma_1 a_0 c_3 S_\mathrm{D}^{**}\left(P - P^{**}\right)\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right) \\ &+ a_2 c_3 S_\mathrm{D}^{**}\left(R - R^{**}\right)\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right) + c_3 \delta_\mathrm{S}\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right)\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right) - c_3\left(\gamma_1 a_0 P^{**} - a_2 R^{**} + \delta_\mathrm{P}\right)\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right)^2.\end{aligned}$$

Further simplifying the above expression as the sum of quadratic forms gives

$$\begin{aligned}\dot{U} &= -\frac{1}{2}a_{11}\left(P - P^{**}\right)^2 + a_{12}\left(P - P^{**}\right)\left(R - R^{**}\right) - \frac{1}{2}a_{22}\left(R - R^{**}\right)^2 \\ &- \frac{1}{2}a_{11}\left(P - P^{**}\right)^2 + a_{13}\left(P - P^{**}\right)\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right) - \frac{1}{2}a_{33}\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right)^2 \\ &- \frac{1}{2}a_{22}\left(R - R^{**}\right)^2 + a_{23}\left(R - R^{**}\right)\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right) - \frac{1}{2}a_{33}\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right)^2 \\ &- \frac{1}{2}a_{11}\left(P - P^{**}\right)^2 + a_{14}\left(P - P^{**}\right)\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right) - \frac{1}{2}a_{44}\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right)^2 \\ &- \frac{1}{2}a_{22}\left(R - R^{**}\right)^2 + a_{24}\left(R - R^{**}\right)\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right) - \frac{1}{2}a_{44}\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right)^2 \\ &- \frac{1}{2}a_{33}\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right)^2 + a_{34}\left(S_\mathrm{A} - S_\mathrm{A}^{**}\right)\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right) - \frac{1}{2}a_{44}\left(S_\mathrm{D} - S_\mathrm{D}^{**}\right)^2.\end{aligned}$$

where

$$\begin{aligned}&a_{11} = \frac{2}{3}\left(\lambda + \alpha R^{**} + \phi_1 S_\mathrm{A}^{**} + \phi_2 S_\mathrm{D}^{**}\right),\ \ a_{12} = -\left(\alpha P^{**} + \rho c_1 R^{**}\right),\ \ a_{22} = \frac{2}{3}c_1\left(\psi + \rho P^{**}\right), \\ &a_{13} = -\left(\phi_1 + \phi_2 + \gamma a_0 c_2 S_\mathrm{A}^{**}\right)P^{**},\ \ a_{14} = -\gamma_1 a_0 c_3 S_\mathrm{D}^{**},\ \ a_{23} = a_1 c_2 S_\mathrm{A}^{**},\ \ a_{24} = a_2 c_3 S_\mathrm{D}^{**}, \\ &a_{33} = \frac{2}{3}c_2\left[\gamma a_0 P^{**} - a_1 R^{**} + \left(\delta_\mathrm{P} + \delta_\mathrm{S}\right)\right],\ \ a_{34} = c_3 \delta_\mathrm{S},\ \ a_{44} = \frac{2}{3}c_3\left(\gamma_1 a_0 P^{**} - a_2 R^{**} + \delta_\mathrm{P}\right).\end{aligned} \tag{7}$$

Thus, dU/dt would be sufficiently negative definite on the condition that the following inequalities hold

$$\begin{aligned}&a_{12}^2 < a_{11}a_{22}, \\ &a_{13}^2 < a_{11}a_{33}, \\ &a_{14}^2 < a_{11}a_{44}, \\ &a_{23}^2 < a_{22}a_{33}, \\ &a_{24}^2 < a_{22}a_{44}, \\ &a_{34}^2 < a_{33}a_{44}.\end{aligned} \tag{8}$$

That is

$$c_1 > \frac{9P^{**}R^{**}}{4\Phi\,\Pi}\left(\alpha P^{**}+\rho c_1 R^{**}\right)^2,\quad c_2 > \frac{9P^{**3}S_{\mathrm{A}}^{**}}{4\Delta\Pi}\left(\phi_1+\phi_2+\gamma a_0 c_2 S_{\mathrm{A}}^{**}\right)^2,\quad c_1\frac{4\Phi\,\Pi}{9a_1^2P^{**3}S_{\mathrm{A}}^{**}} > c_2,$$

$$\frac{4\Pi}{9P^{**}\left(\gamma_1 a_0 S_{\mathrm{D}}^{**}\right)^2}\left(\gamma_1 a_0 P^{**}-a_2R^{**}+\delta_{\mathrm{P}}\right) > c_3,\quad c_1\frac{4\Phi}{9R^{**}\left(a_2 S_{\mathrm{D}}^{**}\right)^2}\left(\gamma_1 a_0 P^{**}-a_2R^{**}+\delta_{\mathrm{P}}\right) > c_3,$$

$$c_2\frac{4\Delta}{9\delta_{\mathrm{S}}^2 S_{\mathrm{A}}^{**}}\left(\gamma_1 a_0 P^{**}-a_2R^{**}+\delta_{\mathrm{P}}\right) > c_3.$$

Thus, by choosing

$$c_1 = 1,\ c_2 < \frac{4\Phi\,\Pi}{9a_1^2R^{**}S_{\mathrm{A}}^{**3}} \quad \text{and}$$

$$c_3 < \min\frac{4\left(\gamma_1 a_0 P^{**}-a_2R^{**}+\delta_{\mathrm{P}}\right)}{9}\left\{\frac{\Pi}{P^{**}\left(\gamma_1 a_0 S_{\mathrm{D}}^{**}\right)^2},\ \frac{4\Phi}{R^{**}\left(a_2 S_{\mathrm{D}}^{**}\right)^2},\ \frac{4\Phi\,\Delta^2}{a_1^2\delta_{\mathrm{S}}^2R^{**}S_{\mathrm{A}}^{**4}}\right\}. \qquad (9)$$

it can easily be checked that the inequalities in (8) reduce to

$$1 > \frac{9P^{**}R^{**}}{4\Phi\,\Pi}\left(\alpha P^{**}+\rho R^{**}\right) \quad \text{and} \quad \frac{4\Delta\Phi}{9a_1^2R^{**}S_{\mathrm{A}}^{**3}} > \frac{9P^{**3}S_{\mathrm{A}}^{**}}{4\Delta\Pi}\left(\phi_1+\phi_2+\frac{4\gamma a_0\Delta\Phi}{9a_1^2R^{**}S_{\mathrm{A}}^{**2}}\right)^2. \qquad (10)$$

Following from (10), we note that the conditions in (8) are satisfied. Thus, U is a Lyapunov function with respect to E_1, whose domain contains the region of attraction D, proving the theorem.

This shows that the density of human effort, density of livestock and depth of fertile topsoil settle down at steady state under certain parametric conditions. It is also noted that the depth of fertile topsoil decreases as anthropogenic activities increases on the surface of the soil.

4. Numerical Simulation

In this section we perform numerical analyses of the model (1). Pursuant to this, we choose the following values of parameters:

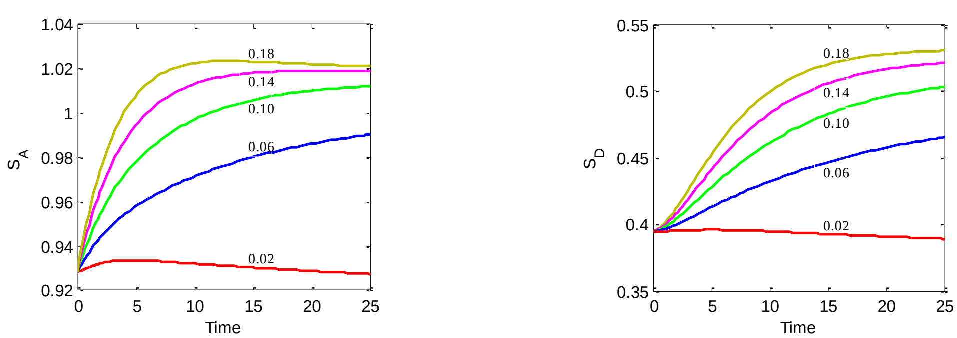
$$\begin{aligned}&\Pi = 0.7,\ \lambda = 0.6,\ \alpha = 0.43,\ \phi_1 = 0.35,\ \phi_2 = 0.05,\ \Phi = 0.127,\ \rho = 0.043,\ \psi = 0.02,\\ &\Delta = 0.55,\ \theta = 0.05,\ \theta_1 = 0.01,\ \xi = 0.01,\ \gamma = 0.05,\ \gamma_1 = 0.01,\ \delta_{\mathrm{C}} = 1.77,\ \mu = 0.025,\\ &\delta_{\mathrm{S}} = 0.17,\ \delta_{\mathrm{P}} = 0.35.\end{aligned} \qquad (11)$$

Using the parameter values above, our computer simulation shows that the positive equilibrium E_1 of the model (1) exists, and it is given by

$$P^{**} = 0.290408,\ R^{**} = 4.210277,\ S_{\mathrm{A}}^{**} = 0.917087,\ S_{\mathrm{D}}^{**} = 0.380116. \qquad (12)$$

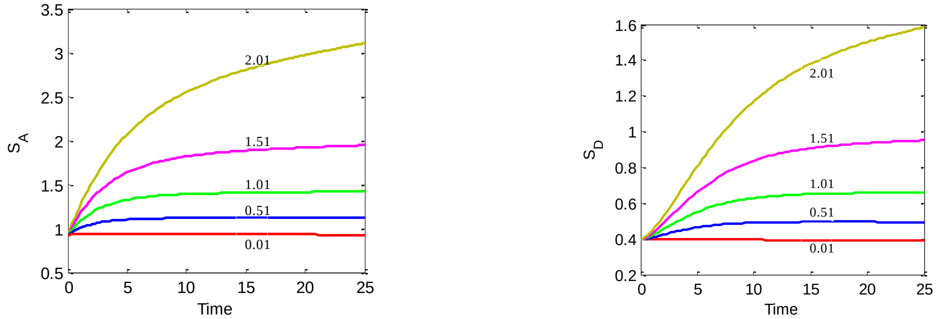
Furthermore, it can be checked that the values of the parameters give in (11) produce positive solutions for the expressions in (3). Thus, confirming the local asymptotic stability of E_1. Moreover, the conditions in (8) can equally be verified to be satisfied by the set of parameters given in (11) to establish the global asymptotic stability of E_1.

To see the effect of the various parameters on S_{A} and S_{D}, we perform the numerical simulation of the model (1) using MATLAB. From these figures we note the relative increases of the depth of fertile topsoil as



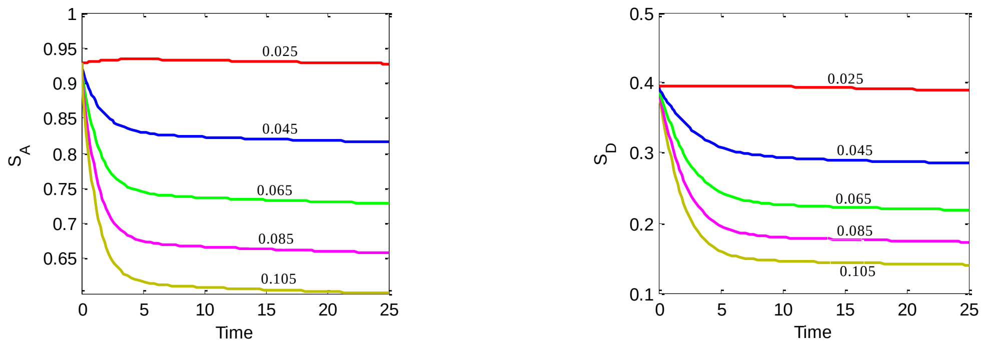
(a) (b)

Fig. 1. Plot of (a) S_{A} and (b) S_{D} against t for different values of ψ with other parametric values as in (11)



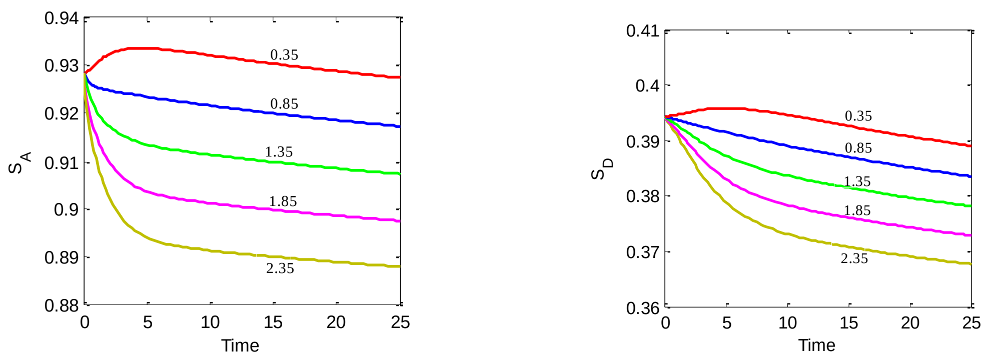
(a) (b)

Fig. 2. Plot of (a) S_A and (b) S_D against *t* for different values of ξ with other parametric values as in (11)



(a) (b)

Fig. 3. Plot of (a) S_A and (b) S_D against *t* for different values of υ with other parametric values as in (11)



(a) (b)

Fig. 4. Plot of (a) S_A and (b) S_D against *t* for different values of μ with other parametric values as in (11)

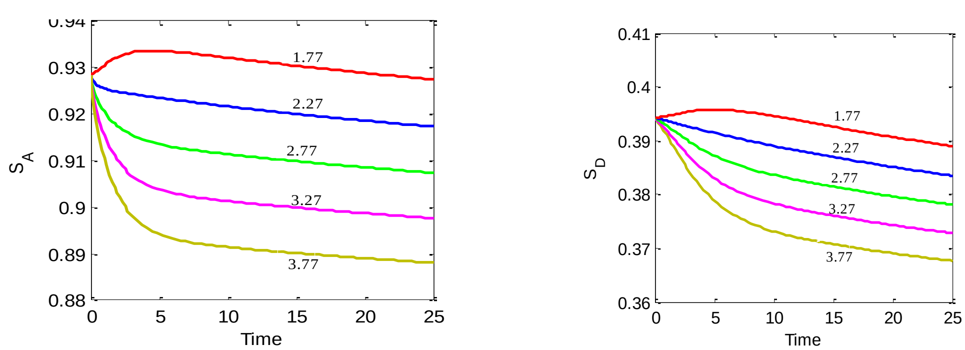


Fig. 5. Plot of (a) S_A and (b) S_D against *t* for different values of δ_C with other parametric values as in (11)

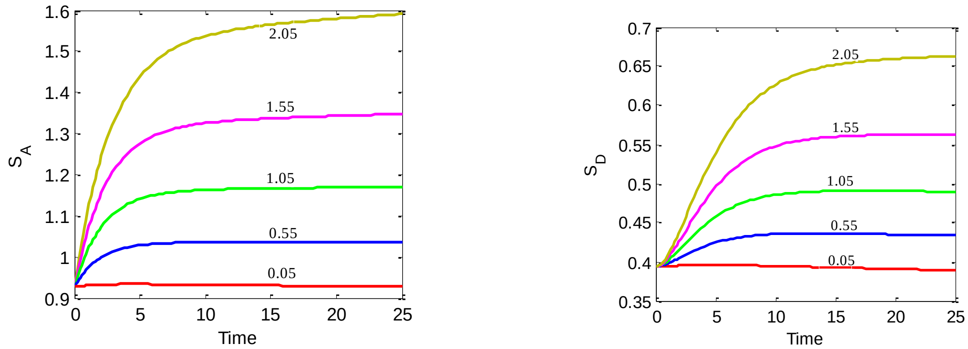


Fig. 6. Plot of S_P against t for different values of θ with other parametric values as in (11)

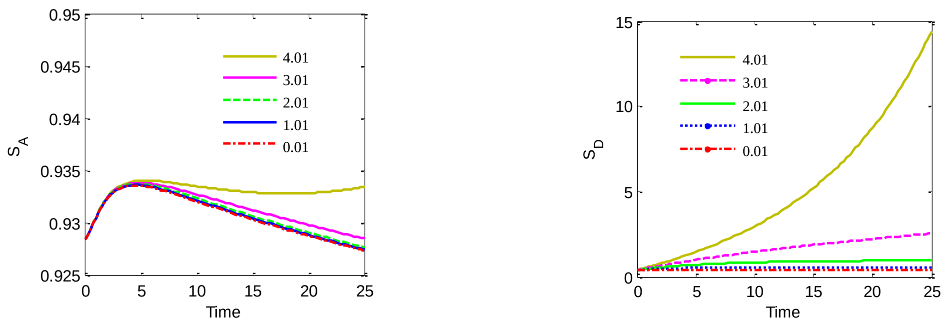


Fig. 7. Plot of (a) S_A and (b) S_D against t for different values of θ_1 with other parametric values as in (11)

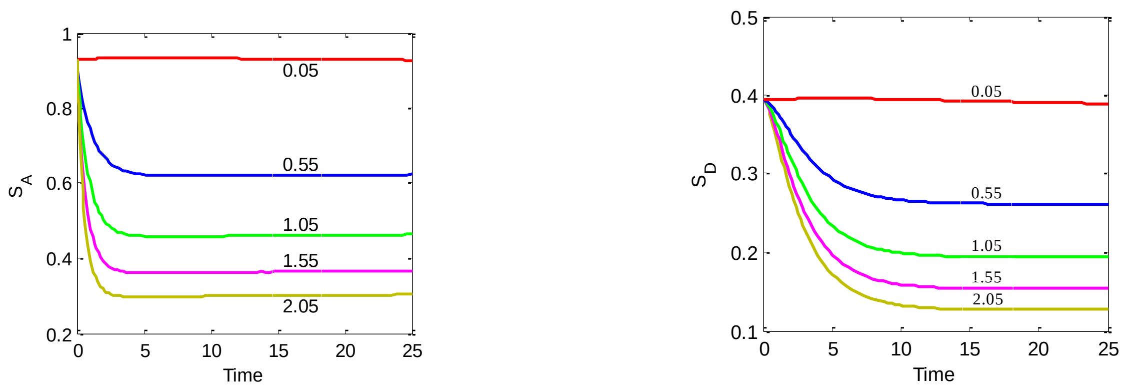


Fig. 8. Plot of (a) S_A and (b) S_D against t for different values of γ with other parametric values as in (11)

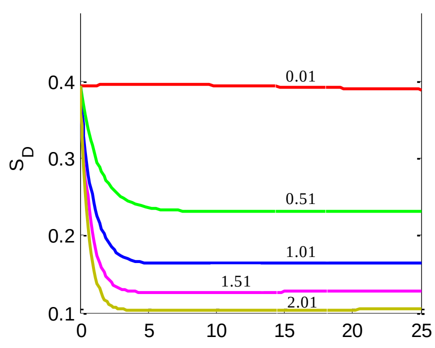


Fig. 9. Plot of S_D against t for different values of γ_1 with other parametric values as in (11)

It is observed from Fig. 1, Fig. 2, Fig. 1, Fig. 6 and Fig. 7 that the topsoil fertilities of S_{A} and S_{D} show varying levels of improvement as the numerical values of the parameters $\psi, \xi, \theta\ \psi,$ and θ_1 continue to increase. It is particularly depicted that enhancing the decomposition or decaying rate coefficient of both livestock carcass and droppings (dung or excrement) has the most significant effect on increasing the fertility of topsoil. Fig.7b specifically indicates the initial difficulty of reclaiming depletion, even with increasing decomposition, however, the substantial impact of facilitating and sustaining an increasing level of decomposition suggests the enormous potentials of organic reclamation of degraded topsoil. This presents a promising possibility in view of the comparative higher contribution of dungs than carcasses (compare Fig. 1 and Fig. 2). On the other hand, the depletive tendencies of excessive efforts ($\mu, \delta_{\text{C}}, \gamma$ and γ_1) at unsustainable levels is shown by Fig. 3, Fig. 4, Fig. 5, Fig. 8 and Fig. 9. The deployment of excessive human effort on depleted soil in addition to removal of vegetation and infrastructural expansion is found to exacerbate it further (see Fig. 9). The intensity in the removal of the vegetation cover of topsoil is found to closely trail in the exacerbation of depletion (Fig. 3). Fig. 4 and Fig. 5 show the depletive tendencies of overgrazing and infrastructural expansion.

5. Conclusion

In this paper, a nonlinear mathematical model has been proposed and analysed to study the effect of excessively deploying various land use practices on topsoil and the consequential effect on the fertility topsoil. It has been shown, both qualitatively and by numerical simulation, that the deployment of unsustainable land use practices has very deleterious consequences on the depth of fertility of topsoil. While a sustainable land use practices has the enormous potential for reclaiming depleted topsoil.

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