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ONE STEP SECOND DERIVATIVE METHOD USING CHEBYSHEV COLLOCATION POINT FOR THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

One-step second derivative block method with three off-step points is developed using Chebyshev collocation points for the solution of stiff IVPs of ODEs. The stability properties of the new method are tested and found to be of order five and convergent, and test the accuracy of the method on some numerical examples, results obtained show that this method is better in terms of accuracy when compared with the methods of the same order and higher order methods.

Keywords**:** One step method, Second derivative method, Block method, Initial Value Problems, Chebyshev collocation point.
AMS Subject Classification: 65L04, 65L05

1.0 Introduction

Physical problems around us give rise to differential equations, such problems arise in fields of engineering and sciences such as chemical reactions and change of a magnitude. Solutions to these differential equations are usually obtained using analytical methods, but, some differential equations are very difficult to solve analytically except the approximate solution by the use of numerical methods [1]. This article considers solving Initial Value Problems (IVPs) of first order Ordinary Differential Equations (ODEs)

$$y' = f(x, y), \quad y(x_0) = y_0, \; a \le x \le b \qquad (1)$$

where $f \; :[a,b] \times \mathrm{R}^m \to \mathrm{R}^m, \; y_0 \in \mathrm{R}^m$ is continuous and differentiable. However, f is assumed to satisfy the existence and uniqueness theorem [2].

Applications in some areas such as chemical reactions, control systems, and electronic networks lead to systems of stiff differential equations, they are stiff problems because, the solution being sought is varying slowly, but there are nearby solutions that vary rapidly, for such reason, the numerical method must take small steps in order to obtain satisfactory results.

One step method for the solution of IVPs in the form of eq. (1) is of the form

$$y_{n+1} = y_n + h_n \phi(x_n, y_n; h_n) \qquad (2)$$

where $\phi(x, y; h)$ is the increment function and h_n is the step size in the subinterval $[x_n, x_{n+1}]$ [3]. Euler and RungeKutta method are some of the examples of one step method. Hybrid one step method for the solution of eq. (1) can be written in the form

$$\sum_{j=0}^{1} \alpha_j y_{n+j} + \alpha_k y_{n+k} = h\left[\sum_{j=0}^{1} \beta_j f_{n+j} + \beta_k f_{n+k}\right] \qquad (3)$$

where k is a rational number, if $f_{n+1} = 0$, then eq. (3) is called explicit otherwise it is called implicit hybrid one step method.

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Hybrid methods and second derivative methods were first designed to take care of the Dahlquist barrier theorems [4] whereby the conventional Linear Multistep Method (LMM) was modified by incorporating off-step points and second derivative term in the derivation process in order to increase the order of the method while preserving good stability properties. The second derivative terms provide more freedom to derive methods which are highly stable, convergent, with large regions of absolute stability and suitable for solving systems with large Lipschitz constants [5].
Milne in 1953 proposed block method in order to generate starting values for predictor-corrector LMM [6] and since then, several block methods has been developed by researchers such as [7 – 12] among others.
Implicit one step hybrid block method of order seven was developed by [1] for the solution of eq. (1) by considering polynomial approximate solution based on collocation and interpolation approach and thus, report that, this method performs better when compared with other methods established in literature.

Orthogonal polynomial with respect to the weight function $w(x) = x^2 - 1$ was considered by [13] to construct a basis function. Applying collocation and interpolation technique, this basis function is used to develop three classes of finite deference one step methods of order 3, 4 and 5 respectively by varying the off-step points for solving IVPs and reported that, these methods are accurate and convergent.
This article, therefore is concerned with the development of hybrid second derivative one step method using Chebyshev collocation point for the solution of stiff IVPs. The method combined the efficiency of single step methods and the high order of accuracy of LMMs.

2.0 Methodology
2.1 Mathematical Background

(i) Consider the approximate solution

$$y(x) = \sum_{n=0}^{k} \alpha_n x^n \tag{4}$$

where $x \in [a,b]$, $a_n \in \mathrm{R}$ are the unknown parameters to be determined.

(ii) To generate the collocation points, we considered the Gauss Chebyshev Lobatto scheme to construct the Chebyshev grid points in the interval $[-1,1]$, given as

$$x_k = \cos\left(\frac{k\pi}{N}\right), k = 0,1,2,...,N. \tag{5}$$

Taken $N = 4,$ to get grid points $\left[-1 - \frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}\ 1\right]$ and applying linear transformation $\zeta = \frac{1+\sigma}{2}$, $\sigma \in [-1,1]$ to get $\zeta = 0, \frac{2-\sqrt{2}}{4}, \frac{1}{2}, \frac{2+\sqrt{2}}{4}, 1.$ Now, evaluate eq. (4) at point x_n, the first derivative of eq. (4) at points $[x_n, x_{n+\frac{2-\sqrt{2}}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{2+\sqrt{2}}{4}}, x_{n+1}]$ and the second derivative of eq. (4) at point $[x_{n+\frac{1}{2}}, x_{n+1}]$ to give the system

$$XA = U \tag{6}$$

where

$$A = [a_0,\ a_1, a_2,\ a_3\ ...,\ a_k]^T,\ k = r + s + \tau - 1.$$

$$U = \left[y_n,\ y_{n+1,}\ ...,\ y_{n+r},\ \ y'_n,\ ...,\ y'_{n+s}\ \ y''_n,\ ...,\ y''_{n+\tau}\right]^T.$$

(iii) Impose the following conditions

$$y(x_{n+j}) = y_{n+j}, \quad j = 0,1,...r$$
$$y'(x_{n+j}) = f_{n+j}, \quad j = 0,1,..,s$$
$$y''(x_{n+j}) = g_{n+j}, \quad j = 0,1,..,\tau$$

on equation (4), gives

$$X=\begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+r} & x_{n+r}^2 & \cdots & x_{n+r}^k \\ 0 & 1 & 2x_n & \cdots & kx_n^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{n+s} & \ldots & kx_{n+s}^{k-1} \\ 0 & 0 & 2 & \ldots & k(k-1)x_n^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 2 & \ldots & k(k-1)x_{n+\tau}^{k-2} \end{bmatrix}$$

(iv) Solve (6) for the unknown parameters using Cramer's rule and substitute the results into (4) to give

$$y_{n+t}=\sum_{j=0}^{r}\alpha_j(t)y_{n+j}+h\sum_{j=0}^{s}\beta_j(t)f_{n+j}+h^2\sum_{j=0}^{\tau}\sigma_j(t)g_{n+j} \tag{7}$$

where $\alpha_j(t),\ \beta_j(t),$ and $\sigma_j(t)$ are polynomial of degree $r+s+\tau-1$ and $t=\dfrac{x-x_n}{h}$.

(v) Evaluate (7) when $t=[\ x_{n+\frac{2-\sqrt{2}}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{2+\sqrt{2}}{4}}, x_{n+1}]$ to give

$$\begin{aligned} y_{n+\frac{2-\sqrt{2}}{4}} &= y_n+\left(\frac{17}{3360}\sqrt{2}+\frac{101}{2240}\right)hf_n+\left(\frac{43}{210}-\frac{13}{210}\sqrt{2}\right)hf_{n+\frac{2-\sqrt{2}}{4}} \\ &+\left(\frac{4}{21}-\frac{23}{168}\sqrt{2}\right)hf_{n+\frac{1}{2}}+\left(\frac{1}{10}-\frac{3}{28}\sqrt{2}\right)hf_{n+\frac{2+\sqrt{2}}{4}}+\left(\frac{57}{1120}\sqrt{2}-\frac{271}{6720}\right)hf_{n+1} \\ &+\left(\frac{11}{1680}\sqrt{2}+\frac{1}{2240}\right)h^2g_{n+\frac{1}{2}}+\left(\frac{1}{420}-\frac{11}{3360}\sqrt{2}\right)h^2g_{n+1} \end{aligned} \tag{8}$$

$$\begin{aligned} y_{n+\frac{1}{2}} &= y_n+\frac{67}{1680}hf_n+\left(\frac{5\sqrt{2}-37}{210\sqrt{2}-420}\right)hf_{n+\frac{2-\sqrt{2}}{4}}+\frac{4}{21}hf_{n+\frac{1}{2}} \\ &+\left(\frac{16}{105}-\frac{9}{140}\sqrt{2}\right)hf_{n+\frac{2+\sqrt{2}}{4}}-\frac{59}{1680}hf_{n+1}-\frac{43}{1680}h^2g_{n+\frac{1}{2}}+\frac{1}{420}h^2g_{n+1} \end{aligned} \tag{9}$$

$$\begin{aligned} y_{n+\frac{2+\sqrt{2}}{4}} &= y_n+\left(\frac{101}{2240}-\frac{17}{3360}\sqrt{2}\right)hf_n-\left(\frac{1}{70\left(\sqrt{2}-2\right)}\left(8\sqrt{2}-1\right)\right)hf_{n+\frac{2-\sqrt{2}}{4}} \\ &+\left(\frac{23}{168}\sqrt{2}+\frac{4}{21}\right)hf_{n+\frac{1}{2}}+\left(\frac{13}{210}\sqrt{2}+\frac{43}{210}\right)hf_{n+\frac{2+\sqrt{2}}{4}}+\left(-\frac{57}{1120}\sqrt{2}-\frac{271}{6720}\right)hf_{n+1} \\ &+\left(\frac{1}{2240}-\frac{11}{1680}\sqrt{2}\right)h^2g_{n+\frac{1}{2}}+\left(\frac{11}{3360}\sqrt{2}+\frac{1}{420}\right)h^2g_{n+1} \end{aligned} \tag{10}$$

$$\begin{aligned} y_{n+1} &= y_n+\frac{4}{105}hf_n+\left(\frac{40\sqrt{2}-72}{105\sqrt{2}-210}\right)hf_{n+\frac{2-\sqrt{2}}{4}}+\frac{8}{21}hf_{n+\frac{1}{2}} \\ &+\left(\frac{4}{105}\sqrt{2}+\frac{32}{105}\right)hf_{n+\frac{2+\sqrt{2}}{4}}-\frac{1}{35}hf_{n+1}-\frac{1}{105}h^2g_{n+\frac{1}{2}}+\frac{1}{210}h^2g_{n+1} \end{aligned} \tag{11}$$

(vi) writing equations (8) - (11) in block we get

$$A^{(1)}Y_{m+1}=A^{(0)}Y_m+hB^{(0)}F_m+hB^{(1)}F_{m+1}+h^2\gamma^{(1)}G_{m+1} \tag{12}$$

Where

$$Y_{m+1} = \left[y_{n+\frac{2-\sqrt{2}}{4}} \; y_{n+\frac{1}{2}} \; y_{n+\frac{2+\sqrt{2}}{4}} \; y_{n+1} \right]^T, \; Y_m = \left[y_{n-1} \; y_{n-2} \; y_{n-3} \; y_n \right]^T,$$

$$F_{m+1} = \left[f_{n+\frac{2-\sqrt{2}}{4}} \; f_{n+\frac{1}{2}} \; f_{n+\frac{2-\sqrt{2}}{4}} \; f_{n+1} \right]^T, \; F_m = \left[f_{n-1} \; f_{n-2} \; f_{n-3} \; f_n \right]^T,$$

$$G_{m+1} = \left[g_{n-1} \; g_{n-2} \; g_{n+\frac{1}{2}} \; g_{n+1} \right]^T,$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \left(\frac{17}{3360}\sqrt{2} + \frac{101}{2240}\right) \\ 0 & 0 & 0 & \frac{67}{1680} \\ 0 & 0 & 0 & \left(\frac{101}{2240} - \frac{17}{3360}\sqrt{2}\right) \\ 0 & 0 & 0 & \frac{4}{105} \end{bmatrix},$$

$$B^{(1)} = \begin{bmatrix} \left(\frac{43}{210} - \frac{13}{210}\sqrt{2}\right) & \left(\frac{4}{21} - \frac{23}{168}\sqrt{2}\right) & \left(\frac{1}{10} - \frac{3}{28}\sqrt{2}\right) & \left(\frac{57}{1120}\sqrt{2} - \frac{271}{6720}\right) \\ \left(\frac{5\sqrt{2}-37}{210\sqrt{2}-420}\right) & \frac{4}{21} & \left(\frac{16}{105} - \frac{9}{140}\sqrt{2}\right) & -\frac{59}{1680} \\ -\left(\frac{1}{70\left(\sqrt{2}-2\right)}\left(8\sqrt{2}-1\right)\right) & \left(\frac{23}{168}\sqrt{2} + \frac{4}{21}\right) & \left(\frac{13}{210}\sqrt{2} + \frac{43}{210}\right) & \left(-\frac{57}{1120}\sqrt{2} - \frac{271}{6720}\right) \\ \left(\frac{40\sqrt{2}-72}{105\sqrt{2}-210}\right) & \frac{8}{21} & \left(\frac{4}{105}\sqrt{2} + \frac{32}{105}\right) & -\frac{1}{35} \end{bmatrix},$$

$$\gamma^{(1)} = \begin{bmatrix} 0 & 0 & \left(\frac{11}{1680}\sqrt{2} + \frac{1}{2240}\right) & \left(\frac{1}{420} - \frac{11}{3360}\sqrt{2}\right) \\ 0 & 0 & -\frac{43}{1680} & \frac{1}{420} \\ 0 & 0 & \left(\frac{1}{2240} - \frac{11}{1680}\sqrt{2}\right) & \left(\frac{11}{3360}\sqrt{2} + \frac{1}{420}\right) \\ 0 & 0 & -\frac{1}{105} & \frac{1}{210} \end{bmatrix}.$$

2.2 Stability Properties

Order and error constant of the Method

Theorem 1[14]

Let the linear operator L be defined by

$$L\{y(t):\ h\} = A^{(1)}X_{m+1} - A^{(0)}X_m - B^{(0)}hF_m - B^{(1)}hF_{m+1} - \gamma^{(1)}h^2G_{m+1} \tag{13}$$

be continuously differential within the interval $\left[t_0, t_f\right]$. Writing the numerical method in Taylor expansion about point t to obtain

$$l\{y(t):\ h\} = C_0 y(t) + C_1 hy^{/}(t) + C_2 y^{//}(t) + \ldots + C_p h^p y^{(p)}(t) + \ldots$$

where

$$C_p = \frac{1}{P!}\left[\sum_{j=1}^{r} j^p \theta_j - \frac{1}{(p-1)!}\sum_{j=1}^{r} j^{p-1}\gamma_j\right]$$

The method has order p, if

$$L[y(t), h)] = O\left(h^{p+1}\right), C_0 = C_1 = \ldots = C_p = 0, C_{p+1} \neq 0$$

hence C_{p+1} is called the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(t)$ is called the Local Truncation Error (LTE).

Proof: Evaluating each row of eq. (12) in a Taylor series about x_n gives

$$L[y(x); h] = y_{n+t} - \sum_{j=0}^{r}\alpha_j(t)y_{n+j} - h\sum_{j=0}^{s}\beta_j(t)f_{n+j} - h^2\sum_{j=0}^{\tau}\sigma_j(t)g_{n+j} \tag{14}$$

$$h^{p+n} \neq 0$$

$$p + n = 6$$

where n is the order of the differential equation. Therefore, the order of the new method is $p=[5,5,5,5]^T$ with error constant

$$\begin{bmatrix} \left(-\frac{101027}{313528320}\sqrt{2}+\frac{64103}{195955200}\right)h^6, \left(-\frac{8857}{58060800}\sqrt{2}-\frac{80137}{1567641600}\right)h^6, \\ \left(-\frac{2123}{43545600}\sqrt{2}-\frac{108131}{522547200}\right)h^6, \left(-\frac{42803}{97977600}\sqrt{2}+\frac{33571}{97977600}\right)h^6 \end{bmatrix}^T \quad (15)$$

Consistency of the Method

Theorem 2[15]

The block method (12) is consistent, if it has order $p \geq 1.$

Proof: The order of the method (12) is $p=[5,5,5,5]^T > 1$, therefore, it is consistent.

Zero Stability of the Block Method

Theorem 3[15]

The block method is said to be zero stable if the roots $z_s, s=1,2,...,n$ of the first characteristics polynomial $p(z)$ defined by

$$\rho(z)=\det\left[zA^{(1)}-A^{(0)}\right] \quad (16)$$

satisfies $|z_s| \leq 1$ and every root with $|z_s| \leq 1$ has multiplicity not exceeding the order of the differential equation.

Proof: The roots $z_s, s=1,2,3,...n$ of the first characteristics polynomial $\rho(z)$ defined by

$\rho(\lambda)=\det\left[A^{(1)}\lambda-A^{(0)}\right]=0$ is

$\lambda^4-\lambda^3=\lambda^3(\lambda-1)=0,$

solving for $\lambda,$ we have $\lambda=[0,0,0,1]^T$. Hence the block method (12) is zero stable.

Convergence

Theorem 4[11]

The necessary and sufficient condition for a numerical method to be convergent are that it must be consistent and zero stable.

Proof: Since the method (12) is consistent and zero-stable, therefore, it is convergent.

Region of Absolute Stability

This is achieved by substituting the test equation

$$y'=\lambda y \quad (17)$$

in the block formula to give

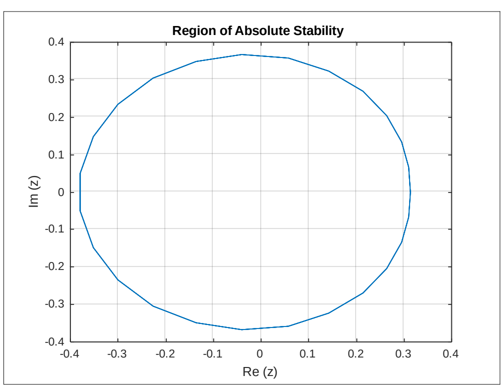

Figure 1: Region of absolute stability of the new method

3.0 Numerical Experiment

The following notations were used in this research:

Table 1: Notations

x	Step size
3BBDF	3 point Block Backward Differentiation Formula
3BEBDF	3 point Block Extended Backward Differentiation Formula
OMOOP	One-Step Method with One Off-Step Point
OMTOP	One-Step Method with Two Off-Step Point
OMTHOP	One-Step Method with Three Off-Step Point

We considered the following problems to test the efficiency of the new method:

Problem 1: [13]

$$y' = -y,\ y(0) = 1,\ h = 0.1,\ 0 \le x \le 1$$

with exact solution

$$y(x) = e^{-x}.$$

Table 2: Error for problem 1

x	OMOOP	OMTOP	OMTHOP	New Method
0.1	1.2574666e-8	4.4204169e-7	1.00267323e-6	1.2633456e-8
0.2	2.2756058e-8	7.999513e-7	1.81451150e-6	2.343354e-8
0.3	3.0885799e-8	1.0857388e-6	2.46275549e-6	3.1534333e-8
0.4	3.726217e-8	1.3098892e-6	2.97118945e-6	4.1621227e-8
0.5	4.2145258e-8	1.48154559e-6	3.36055237e-6	4.5142258e-8
0.6	4.5761528e-8	1.608669e-6	3.64890222e-6	4.9011128e-9
0.7	4.8307866e-8	1.698189e-6	3.85193834e-6	4.9993066e-9
0.8	4.995516e-8	1.756088e-6	3.98328687e-6	5.495227e-9
0.9	5.0851416e-8	4.420417e-7	4.05475314e-6	5.7866414e-9
1.0	5.1124784e-8	7.999513e-7	4.07654481e-6	5.9102785e-9

Problem 2: [13]

$$y' = 1 + y^2,\ y(0) = 0,\ h = 0.01,\ 0 \le x \le 1$$

with exact solution

$$y(x) = \tan x$$

Table 3: Error for problem 2

x	OMOOP	OMTOP	OMTHOP	New Method
0.01	5.558e-13	3.4572e-12	6.7362e-12	3.1472e-13
0.02	1.1116e-12	1.43294e-11	3.01544e-11	1.25249e-13
0.03	1.6673e-12	3.26397e-11	7.03047e-11	3.26997e-13
0.04	2.2234e-12	5.84296e-11	1.272766e-10	3.89299e-13
0.05	2.7792e-12	9.17568e-11	2.011988e-10	4.17968e-13
0.06	3.3357e-12	1.326973e-10	2.922383e-10	4.456932e-13
0.07	3.8923e-12	1.813437e-10	4.006017e-10	4.623473e-13
0.08	4.4494e-12	2.378066e-10	5.265376e-10	5.172165e-13
0.09	5.0065e-12	3.022165e-10	6.703345e-10	5.423361e-13
0.1	5.564e-12	3.74722e-10	8.32325e-10	5.84666e-13

Problem 3: [16]

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 95 \\ -1 & -97 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; y_1(0)=1, \; y_2(0)=1$$

$h = 0.01, 0 \le x \le 10$

with exact solution

$$y_1(x) = \frac{1}{47}\left(95e^{-2x} - 48e^{-96x}\right)$$

$$y_2(x) = \frac{1}{47}\left(48e^{-96x} - e^{-2x}\right)$$

The eigen values of the system is $\lambda = -2$ and $\lambda = -96.$

Table 4: Error for problem 3

x	3BEBDF	3BBDF	New Method
0.02	3.06562e-002	1.14619e[illegible]136	3.78600e-002
0.03	3.93978e-002	6.89917e-002	5.65676e-003
0.04	8.39129e-003	1.05286e-002	8.59142e-003
0.05	9.02719e-004	1.07915e-003	9.53313e-004
0.06	9.09318e-002	1.08172e-004	9.98889e-004

Problem 4: [16]

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -100 & 9.901 \\ 0.1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; y_1(0)=1, \; y_2(0)=10$$

$h = 0.1, 0 \le x \le 10$

with exact solution

$$y_1(x) = e^{-0.99x}$$

$$y_2(x) = 10e^{-0.99x}$$

The eigen values of the system is $\lambda = -0.99$ and $\lambda = -100.01.$

Table 5: Error for problem 4

x	3BEBDF	3BBDF	New Method
0.02	8.35359e - 002	1.78041 + 124	6.12353e - 003
0.03	9.10218e - 003	1.08962e - 002	7.43201e - 004
0.04	9.18073e - 004	1.09230e - 003	9.08032e - 004
0.05	9.18862e - 005	1.09257e - 004	9.58862e - 006
0.06	9.18939e - 006	1.09259e - 005	9.81944e - 007

4.0 CONCLUSION

Hybrid one-step second derivative block method is developed using Chebyshev collocation points for the solution of stiff IVPs. Three off-step points are considered to drive hybrid linear multistep method with implementation in block form. The new method developed is of order five and was verified to be convergent. The new method was also found to be stable within the region of absolute stability. A code was written using MATLAB 8.5 for the implementation of the new method. The accuracy of the method was tested on some numerical examples and compared the results produced by the new method with other methods established in literature. It can be seen from Table 2 to Table 5 that, the result obtained by the new method is better than the one obtained by other methods of the same order.

This paper show that, applying Chybeshev collocation points for the development of one-step numerical methods with off-step points yield methods that are better in terms of accuracy with better stability properties. The new method is suitable for solving problems with large Lipschitz constants.

References

[1] Areo, E. A. and Adeniyi, R. B., Block Implicit One-Step Method for the Numerical Integration of Initial Value Problems in Ordinary Differential Equations. *International Journal of Mathematics and Statistics Studies*, 2(3), (2014), 4-13.

[2] Lambert J.D.,*Numerical methods for ordinary differential system: the initial value problems*. Chichester, New York: John Wiley & Sons, (1991).

[3] Fatunla, S.O.,*Numerical Methods for Initial Value Problems in Ordinary Differential Equations*. New York: Academic Press Inc, (1988).

[4] Dahlquist, G., Convergence and stability in the numerical integration of Ordinary Differential Equation. *Math. Scand.* 4, (1956), 33-50.

[5] Yakubu, D. G., Bakari, A. I. and Markus, S., Two-step second-derivative high-order methods with two off-step points for solution of stiff systems. *AfrikaMatematika*,(2015), 1081-1093

[6] Lambert, J. D.,*Computational methods in ordinary differential equations*. London, UK: John Wiley & Sons, (1973).

[7] Cash, J. R., Block Runge-Kutta methods for the numerical integration of initial value problems in ordinary differential equations. Part I. The non-stiff case. *Mathematics of Computation*, 40(161), (1983), 175-191.

[8] Chu, M. T., and Hamilton, H., Parallel solution of ODEs by multi-block methods. *SIAM J. Sci. Stat. Comput.*, 8(1), (1987), 342-353.

[9] Fatunla, S.O., Block methods for second order ODEs. *International Journal of Computer Mathematics*, 41, (1991), 55-63.

[10] Muka, K.O., and Ikhile, M.N.O., Second derivative parallel one block two points stabilized Simpson's method. *Journal of interdisciplinary Mathematics*, 13, (2010), 186-191.

[11] Adesanya, A. O., Odekunle, M. R. and Adeyeye, A. O., Continuous block hybrid predictor corrector method for the solution of $y'' = f(x; y; y')$. *International Journal of Mathematics and Soft Computing*, 2(2), (2012), 35-42.

[12] Anake, T. A., Awoyemi, D. O. and Adesanya, A. O., One-step implicit hybrid block method for the direct solution of general second order ordinary differential equations. *IAENG International Journal of Applied Mathematics*, 42(4), (2012).

[13] Adeyefa, E. O. and Fadaka, O., A step block algorithm for solving first order initial value problems in ordinary differential equations. *FUOYE Journal of Engineering and Technology*, 2(1), (2017), 110-113.

[14] Yakubu, D.G. and Markus, S., The Efficiency of Second Derivative Multistep Methods for the Numerical Integration of Stiff Systems. *Journal of the Nigeria Mathematics Society*, 35, (2016), 107-127.

[15] Awoyemi, D. O., Adebile, E. A., Adesanya, A. O. and Anake, T. A., Modified block method for the direct solution of second order ordinary differential equations. *Intern. Journal of Applied Maths and Comp.*, 3(3), (2011), 181-188.

[16] Musa, H., Suleiman, M. B. &Senu, N., Fully Implicit 3-Point Block Extended Backward Differentiation Formula for Stiff Initial Value Problems. *Applied Mathematical Sciences*, 6(85), (2012), 4211-4228.