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OSCILLATIONS OF A CLASS OF DELAY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH CONSTANT IMPULSIVE JUMPS

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Abstract

In this paper, we investigate the oscillation properties of a certain class of second order impulsive delay differential equations. Some sufficient conditions for oscillation of every bounded solution are obtained. Examples are provided to illustrate the main result.

Keywords: *Bounded solution, impulsive, delay differential equation, second-order, oscillation.*2010 Mathematics subject classification; 34K12, 34R12, 34K11

1.1 INTRODUCTION

A number of processes in natural evolution experience an instantaneous change of state at certain moments of time. This has been the main reason for the development of the theory of impulsive ordinary differential equations which has become a new branch of the theory of ordinary differential equations. The first paper on oscillation of impulsive delay differential equations was published in 1989 by Gopalsamy and Zhang [1]. Recently the oscillatory behavior of impulsive delay differential equations has attracted the attention of many researchers. For some contributions in this area, the reader is referred to [2-10],. Relatively, it is known that stochastic functional differential equations with state-dependent delay, which are relevant in mathematical models of real phenomena, are important area of practical application of differential equations with impulses[9], [11].

In this paper, we are concerned with the problem of oscillation of bounded solutions of a class of second order impulsive delay differential equations.

In ordinary differential equations, the solutions are continuously differentiable, sometimes at least once, whereas the impulsive differential equations generally possess non-continuous solutions. Since the continuity properties of the solutions play an important role in the analysis of the behavior, the techniques used to handle the solutions of impulsive differential equations are fundamentally different including the definitions of some of the basic terms. In this work, we examine some of these changes.

Let an evolution process evolve in a period of time J in an open set $\Omega\subset J\times R^n$ and let the function $f:\Omega\to R^n$ be a continuous mapping fulfilling local Lipchitzian condition in $y\in R^n$, $\forall\,(t,y)\in\Omega$.

Definition 1.1

Impulsive differential equations with fixed moments of impulsive effect are of the form

$$\begin{cases} y'(t)=f(t,y(t)),\forall\, t\in T\setminus S \\ \Delta y(t_k)=f(t_k,y(t_k)),\forall\, t_k\in S, \end{cases} \qquad (1.1)$$

where $(t,y)\in\Omega\subset R\times R^n$ and the real numerical sequence $S=\{t_k\}_{k=1}^{\infty}\subset J$ is increasing and has no finite accumulation point. In the case of unfixed moments of impulsive effects, the impulsive points may be time and state dependent(that is, $t_k:=t_k(t,y(t))$). When the function t_k depends on the state of the system (1.1) it is said to have impulses at variable times. This is reflected in the fact that different solutions will tend to undergo impulses at different times. However, if the functions t_k are all constants the system is said to have impulses at fixed times in which case all solutions undergo impulses actions at

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the same time and the question of existence of solution of the system (1.1) is non-trivial when impulses occur at variable times. Even the precise notion of what a solution is must be carefully stated. It is fairly clear that solutions should be piecewise continuous and in fact piecewise continuously differentiable (or piecewise absolutely differentiable when considering generalized types of solutions). A solution will undergo simple jump discontinuity when it intersects impulse hyper-surfaces. Even after focusing on a particular class of relations $t(s,y(s))=0$ given by impulse hyper-surfaces, impulsive differential equations still exhibit some unusual behaviour (Berezansky, [12]).

In this work, we shall concentrate only on impulsive differential equations with fixed moments. Be that as it may, to obtain or discuss the solution of an impulsive differential equation, we must take into cognizance certain peculiarities of the model. We assume that for $t\in T\setminus S$, the solution x of equation (1.1) is determined by the ordinary differential equation $x'(t)=f\left(t_k, y(t_k)\right)$. For $t\in S$, a change by jump of the solution x occurs so that $y\left(t_k^-\right)=y\left(t_k\right)$ and $y\left(t_k^+\right)=y\left(t_k\right)+\Delta y\left(t_k\right)=y\left(t_k\right)+f\left(t_k,y\left(t_k\right)\right)$. After the jump at the moment $t=t_k$, the solution x of the system (1.1) coincides with the solution y of the initial value problem (Bainov and Simeonov, [7]):

$$\begin{cases} x'(t)=f\left(t, x(t)\right),\ t_k< t\le t_{k+1} \\ x\left(t_k\right)=x\left(t_k^+\right), t=t_k\in S. \end{cases} \tag{1.2}$$

This simply means that, after the jump at $t=t_k$, a new function $y(t)$ takes over control from $x(t)$. The controlling impulsive differential equation is given by

$$\begin{cases} y'(t)=f(t,y(t)), \forall\, t\in T\setminus S \\ \Delta y(t)=f(t_k,y(t_k)), \forall\, t_k\in S, \\ y(t_0)=y_0, t_0\in T\setminus S, (t_0,y_0)\in \Omega \end{cases} \tag{1.3}$$

Definition 1.2

The corresponding second order impulsive differential equation is of the form

$$\begin{cases} y''(t)=f(t,y,y'),\ t\ne t_k \\ \Delta y'(t_k)=f_k(y,y'),\ t=t_k, \end{cases} \tag{1.4}$$

where $y'=\dfrac{dy}{dt}$, $y''=\dfrac{d^2y}{dt^2}$, $(t,y(t))\in\Omega$, $\Delta y^{(i)}(t_k)=y^{(i)}(t_k^+)-y^{(i)}(t_k^-)$, $i=0,1$ and $y(t_k^-)$, $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t=t_k$, respectively. For the sake of definiteness, we shall suppose that the functions $y(t)$ and $y'(t)$ are continuous from the left at the points t_k such that $y'(t_k^-)=y'(t_k)$, $y(t_k^-)=y(t_k)$.

For the description of the continuous change of such processes, ordinary differential equations are used, while the moments and the magnitude of the change by jumps are given by the jump conditions. Now, in the case of unfixed moments of impulse effects, the impulse points may be time and state dependent, that is $t_k=t_k(t,y(t))$. When the function t_k depends on the state of the system (1.1), then it is said to have impulses at variable times. This is reflected in the fact that different solutions will tend to undergo impulses at different times.

In this paper, we shall restrict ourselves to the investigation of conditions for bounded oscillatory solutions of impulsive differential equations for which the impulse effects take place at fixed moments of time $\{t_k\}$. Our equation under consideration is of the form

$$\begin{cases} \left[r(t)y'(t)\right]'=p(t)y\left(g(t)\right), t\ge t_0, t\notin S \\ \Delta\left[r(t_k)y'(t_k)\right]=p_k y\left(g(t_k)\right), t_k\ge t_0, \forall t_k\in S \end{cases} \tag{1.5}$$

where $t,t_k\ge 0$. Without further mention throughout this paper, we will assume that every solution $y(t)$ of equation (1.5) that is under consideration here, is continuous from the left and is nontrivial. That is, $y(t)$ is defined on some half-line $[T_y,\infty)$ and $\sup\left\{|y(t)|:\ t\ge T\right\}>0$ for all $T\ge T_y$. Such a solution is called a regular solution of equation (1.5).

We say that a real valued function $y(t)$ defined on an interval $[a,+\infty)$ fulfills some property *finally* if there exists $T\ge a$ such that $y(t)$ has this property on the interval $[T,+\infty)$.

Definition 1.3
The solution $y(t)$ of the impulsive differential equation (1.5) is said to be

i) finally positive (finally negative) if there exist $T\geq 0$ such that $y(t)$ is defined and is strictly positive(negative) for $t\geq T$ [13];
ii) non-oscillatory, if it is either finally positive or finally negative; and
iii) oscillatory, if it is neither finally positive nor finally negative [14], [15].

Remark 1.1: All functional inequalities considered in thispaper are assumed to hold finally, that is, they are satisfied for all t large enough.

2.1 STATEMENT OF THE PROBLEM

At this point, we recall that the problem under consideration is the second order linear impulsive differential equation with delay of the form

$$\begin{cases}\left[r(t)y'(t)\right]'=p(t)y(g(t)),\ t\geq t_0,\ t\notin S\\ \Delta\left[r(t_k)y'(t_k)\right]=p_k y(g(t_k)),\ t_k\geq t_0,\ \forall t_k\in S\end{cases}\tag{2.1}$$

where $\Delta\left(r(t_k)y'(t_k)\right)=r(t_k^+)y'(t_k^+)-r(t_k)y'(t_k)$.

We introduce the following conditions:

C2.1: $g(t)\in C([t_0,\infty),R)$, $g(t)$ is a non-decreasing function in R_+, $g(t)\geq t$ for $t\in R_+$ and $\lim_{t\to\infty} g(t)=+\infty;$

C2.2: $r\in PC^1([t_0,\infty),R_+)$ and $r(t)>0,\ r(t_k^+)>0,$ for
$t,t_k\in R_+$;

C2.3: $p\in PC([t_0,\infty),R_+)$ and $p_k\geq 0,\ k\in N;$

At the initial point $t_0\geq 0$ the following initial conditions are imposed on the solution of equation (2.1):

$y(t)=\phi(t)$ for $t\in E_{t_0}$, $y'(t_0^+)=y_0'$, where $E_{t_0}=\{t_0\}\cup\{g(t):g(t)<t_0,\ t>t_0\}$; $\phi\in C(E_{t_0},R)$. Our aim is to establish some sufficient conditions for every bounded solution of equation (2.1) to be oscillatory. Here, we demonstrate how well-known mathematical techniques and methods, after suitable modification, is extended in proving an oscillation theorem for impulsive delay differential equations.

3.1MAIN RESULTS

By proper impulse imposition, the following theorem extends Theorem 4.3.1 of the monograph by Ladde*et al* [16]. The salient techniques for the proof are obtained from studies by Bainov and Simeonov [14].

Theorem 3.1: Assume that

i) conditions C2.1—C2.3 hold
ii) $p(t)>0$ for $t\in R_+$;
iii) $r(t)$ is a non-decreasing function in R_+ and $\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{r(s)}=\infty$
iv)
$$\lim_{t\to\infty}\sup\frac{1}{r(t)}\left[\int_{g(t)}^{t}(u-g(t))p(u)du+\sum_{g(t)\leq t_k<t}(t_k-g(t_k))p_k\right]>1\tag{3.1}$$

Then every bounded solution $y(t)$ of equation (2.1) is oscillatory.

Proof: Assume by contradiction that $y(t)$ is a bounded finally positive solution of equation (2.1). That is, there exist constants $T_1>0$ and $L>0$ such that $0<y(t)\leq L$ for $t\geq T_1$.

Hence, $(r(t)y'(t))'\geq 0$, $\Delta(r(t_k)y'(t_k))\geq 0$ for $t\geq T_2\geq T_1$ and $\forall k: t_k\geq T_2\geq T_1$, where T_2 is sufficiently large. This means that $r(t)y'(t)$ is a non-decreasing function for $t\geq T_2$.

Here, we observe that there exist two possibilities:

a) If $r(t)y'(t)>c>0$ and $t\geq T_3\geq T_2$, then $y'(t)>\frac{c}{r(t)}$, $t\neq t_k$ and $\Delta y(t_k)=0$ for $t\geq T_3$. Integrating this and taking into account condition (iii) results in the fact that $y(t)$ is an unbounded function which contradicts our earlier assumption.

b) If $r(t)y'(t)\leq 0$ for $t\geq T_2$, then $y'(t)\leq 0$ for $t\geq T_2$. That is $y(t)$ is a non-increasing function for $y(t)\geq T_2$.

Integrating equation (2.1) from s to t, we have

$$r(t)y'(t)-r(s)y'(s)=\int_s^t p(u)y(g(u))du+\sum_{s\leq t_k\leq t} p_k y(g(t_k)). \tag{3.3}$$

Again, integrating equation (3.3) in s from $g(t)$ to t, we obtain

$$r(t)y(t)(t-g(t))=\int_{g(t)}^t r(s)y'(s)ds+\int_{g(t)}^t\left(\int_s^t p(u)y(g(u))du+\sum_{s\leq t_k<t} p_k y(g(t_k))\right)ds \tag{3.4}$$

We now change the order of integration in equation (3.4), rearrange and obtain

$$0\geq r(t)y(t)-r(g(t))y(g(t))-y(g(t))(r(t)-r(g(t)))+\int_{g(t)}^t (u-g(t))p(u)y(g(u))du+$$

$$+\sum_{g(t)\leq t_k<t}(t_k-g(t))p_k y(g(t_k))$$

$$\geq r(t)(y(t)-y(g(t)))+\int_{g(t)}^t (u-g(t))p(u)y(g(u))du+\sum_{g(t)\leq t_k\leq t}(t_k-g(t))p_k y(g(t_k))$$

or

$$0\geq y(t)-y(g(t))+\frac{1}{r(t)}\left[\int_{g(t)}^t (u-g(t))p(u)y(g(u))du+\sum_{g(t)\leq t_k\leq t}(t_k-g(t))p_k y(g(t_k))\right]. \tag{3.5}$$

Dividing inequality (3.5) by $y(g(t))$ and using the monotonicity of $y(t)$, we get

$$0\geq\frac{y(t)}{y(g(t))}+\left[\frac{1}{r(t)}\left\{\int_{g(t)}^t (u-g(t))p(u)du+\sum_{g(t)\leq t_k<t}(t_k-g(t))p_k\right\}-1\right]$$

which is a contradiction to equation (3.1). This, therefore, completes the proof.

Corollary 3.1: Let $\tau\geq 0$, $p(t)\geq 0$ be continuous, $p_k\geq 0$ and $\tau^2 p(t)\geq 2$ for $t\geq 0$, then bounded solutions of the equation

$$\begin{cases} y''(t)-p(t)y(t-\tau)=0, & t\notin S \\ \Delta y'(t_k)-p_k y(t_k-\tau)=0, & \forall\, t_k\in S \end{cases}$$

are oscillatory.

Corollary 3.2: Let $\ell>0$, $p(t)\geq 0$ be piece-wise continuous, $p_k\geq 0$ and

$$\begin{cases} p(t)\geq\frac{2\ell^2}{((1-\ell)t)^2} \\ p_k\geq\frac{2\ell^2}{((1-\ell)t_k)^2} \end{cases}$$

for large t the bounded solutions of

$$\begin{cases} y''(t)-p(t)y(\frac{t}{\ell})=0, \; t\notin S \\ \Delta y'(t_k)-p_k y(\frac{t_k}{\ell})=0, \; \forall\, t_k\in S \end{cases}$$

are oscillatory.

Example 3.1: The equation

$$\begin{cases} \left(\frac{1}{t}y'\right)'-4ty\left(t-\pi\right)=0, t\geq 2, \; t\notin S \\ \Delta\left(\frac{1}{t_k}y'\right)-4t_k y\left(t_k-\pi\right)=0, t_k\geq 2, \; \forall\, t_k\in S \end{cases}$$

satisfies the conditions of Theorem 3.1. Therefore, all bounded solutions are oscillatory. In particular, $y(t)=\cos t^2$ is one such solution.

Example 3.2: The equation

$$\begin{cases} y''(t)-y(t-\tau)=0, \; t\notin S \\ \Delta y'(t_k)-y(t_k-\tau)=0, \; \forall\, t_k\in S, \end{cases}$$

where $0\leq\tau\leq 2e^{-1}$, does not satisfy the conditions of Theorem 3.1 as expected. This equation has a bounded non-oscillatory solution. $y(t)=e^{\lambda t}$ is one such solution.

Remark 3.1: If we do not require that $\int^{\infty}\frac{ds}{r(s)}=\infty$, but $r(t)$ is non-decreasing and condition (3.1) is satisfied, then the conclusion of Theorem 3.1 remains valid.

REFERENCES

[1] Gopalsamy, K. and Zhang,B. G., On delay differential equations with impulses, *Journal of Mathematical Analysis and Applications*, 139, pp. 110–122.

[2] Abasiekwere, U. A., Esuabana, I. M., Isaac, I. O., Lipcsey, Z,(2018). Classification of Non-Oscillatory Solutions of Nonlinear Neutral Delay Impulsive Differential Equations, *Global Journal of Science Frontier Research: Mathematics and Decision Sciences (USA)*, Volume 18, Issue 1, 49-63, doi:10.17406/GJSFR.

[3] Abasiekwere, U. A., Esuabana, I. M.(2017). Oscillation Theorem for Second Order Neutral Delay Differential Equations with Impulses, *International Journal of Mathematics Trends and Technology*, Vol. 52(5), 330-333, doi:10.14445/22315373/IJMTT- V52P548.

[4] Abasiekwere, U. A., Esuabana, I. M., Isaac, I. O., Lipcsey, Z.(2018). Existence Theorem For Linear Neutral Impulsive Differential Equations of the Second Order, *Communications and Applied Analysis, USA*, Vol. 22, No. 2, 135-147, doi:10.12732/caa.v22i2.1.

[5] Abasiekwere, U. A., Esuabana, I. M., Isaac, I. O., Lipcsey, Z.(2018). Oscillations of Second order impulsive Differential Equations with Advanced Arguments, *Global Journal of Science Frontier Research: Mathematics and Decision Sciences (USA)*, Volume 18, Issue 1, 25-32, doi:10.17406/GJSFR.

[6] Abasiekwere, U. A., Esuabana, I. M.(2018). Asymptotic behaviour of nonoscillating solutions of neutral delay differential equations of the second order with impulses, *Journal of Mathematical and Computational Science*, Vol 8, No 5, 620-629, doi:10.28919/jmcs/3765.

[7] Abasiekwere, U. A. and Moffat,I. U.(2017). Oscillation Theorems for Linear Neutral Impulsive Differential Equations of the Second Order with Variable Coefficients and Constant Retarded Arguments, *Applied Mathematics*, Vol. 7 No. 3, pp. 39-43. doi: 10.5923/j.am.20170703.01.

[8] Guo Liang Huang and JianHuaShen (2003).Oscillation of a second-order linear ODE with impulses, *J. Nat. Sci. Hunan Norm. Univ*., 26(4):8–10.

[9] Igobi, DodiK. And Ndiyo, Etop (2018). Results on Existence and Uniqueness of Solution of Impulsive Retarded Integro-Differential System, *Nonlinear Analysis and Differential Equation*, vol. 6, No. 1, 15 - 24.

[10] Jurang Yan (2004). Oscillation properties of a second-order impulsive delay differential equation, *Comput. Math. Appl*., 47(2-3):253–258.

[11] Chaolong Zhang and Xiaojian Hu (1989). Oscillations of higher order nonlinear ordinary differential equations with impulses, *J. Zhongkai Univ.*, 18(1):45–51.

[12] Berezansky,L. and Braverman, E. (1996) Oscillation of a linear delay impulsive differential equation, *Comm. Appl. Nonlinear Anal*., 3(1):61–77.

[13] Isaac, I. O. andLipcsey, Z. and Ibok,U. J. (2011).Nonoscillatory and Oscillatory Criteria for First Order Nonlinear Neutral Impulsive Differential Equations, *Journal of Mathematics Research*, Vol. 3 Issue 2, 52-65.

[14] Bainov, D. D. and SimeonovP. S. (1998).*Oscillation Theory of Impulsive Differential Equations*, International Publications Orlando, Florida.

[15] Isaac, I. O. andLipcsey, Z. (2010). Oscillations of Scalar Neutral Impulsive Differential Equations of the First Order with variable Coefficients, *Dynamic Systems and Applications*, 19, 45-62.

[16] Ladde, G. S., Lakshmikantham V. and Zhang, B. G., (1987)*Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York.