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COMPLETE SYNCHRONIZATION, ANTI-SYNCHRONIZATION AND HYBRID SYNCHRONIZATION OF IDENTICAL 5D HYPERCHAOTIC SYSTEMS

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Abstract

This work examines the complete synchronization, anti-synchronization and hybrid synchronization based on nonlinear active control approach between two identical Lorenz 5D hyperchaotic systems. Stabilization of error dynamics for each phenomenon is realized by satisfying two analytical approaches, namely, Lyapunov second method and linear system theory. These two approaches have not been fully extended to a 5-D system and especially the Lorenz 5-D hyperchaotic system, thus, its consideration in this work. Controllers are designed by using the relevant variable of drive and response systems. Theoretical analysis and numerical simulations are shown to verify the results.

Keywords: Complete synchronization, anti-synchronization, hybrid synchronization, Lorenz 5-D hyperchaotic system, Lyapunov second method, active control

1.0 INTRODUCTION

The study of chaos in the last few decades had a tremendous impact on the foundations of science and engineering and one of the most recent exciting developments in this regard is the discovery of chaos synchronization, whose possibility was first reported by Fujisaka and Yamada [1] and later by Pecora and Carroll [2]. In their seminal paper, Pecora and Carroll [2] addressed the synchronization of chaotic system by using a drive-response configuration. The basic idea is to use the outputs of the drive system to control the response system so that the trajectories of the response's system can synchronize with those of the drive system.

The idea of chaos synchronization has been considered a major breakthrough in chaotic dynamics, due to its potential applications in modelling brain activities [3], chemical reactions [4]; and more importantly, in information processing and secure communication [5]. These foreseen applications have triggered the enormous research attention given to chaos synchronization for over two decades. In the last two decades, there has been considerable interest devoted to the synchronization of chaotic and hyperchaotic systems.

The phenomena of chaos synchronization have been studied from theoretical [6] and experimental [7] point of view. Varieties of synchronization schemes have been reported in literatures, which include complete synchronization [8]; lag synchronization [9]; generalized synchronization [10]; phase synchronization [11]; partial synchronization [12]; projective synchronization [13]; anti-synchronization [14,15]; inverse lag synchronization [16] and so on. Overtime, many approaches have been proposed for chaos synchronization such as backstepping design [17, 18], active control [19, 20], linear feedback [21] and so on.

Hyperchaotic system was first reported by Rossler in 1979 which is a four-dimensional system [22]. Since then, some other hyperchaotic systems have also been found [23-25]. A hyperchaotic system is generally defined as a chaotic system with more than one positive Lyapunov exponents. Thus, the dynamics of a hyperchaotic system are expanded in several directions simultaneously. In this light, hyperchaotic systems are particularly useful in secure communication as it is difficult for

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intruders to break into them.

Based on the Lorenz system and state feedback control, a new 5D hyperchaotic system was reported by Yang and Chen [26] and the dynamical properties of this system was studied. Vaidyanathan et al. [27] generated another new 5D hyperchaotic system based on the Lorenz system and perform chaos control and synchronization via adaptive control when the parameters for this system are unknown.

In this work,

- The equilibrium and stability analysis of the 5D hyperchaotic Lorenz system is discussed
- Theoretical analysis of
    - Complete synchronization of identical 5D hyperchaotic Lorenz system
    - Anti-synchronization of identical 5D hyperchaotic Lorenz system via active control
    - Hybrid synchronization of identical 5D hyperchaotic Lorenz system via active control
    - Numerical simulation of above

To the best of our knowledge, the above has not been applied to the novel 5D hyperchaotic Lorenz system with three positive Lyapunov exponents. The rest of this work is structured as follows; in section 2, a brief discussion of the description, equilibrium and stability is presented. Sections 3 and 4 deal with theoretical analysis and numerical simulations respectively. Finally, a brief concluding remark is given in section 5.

2.0 SYSTEM DESCRIPTION, EQILIBRIUM AND STABILITY

2.1 Systems Description

Vaidyanathan et al. [27] introduced a novel 5D hyperchaotic Lorenz system, which is describable by the following nonlinear differential equation

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_4 + x_5 \\
\dot{x}_2 &= cx_1 - x_1x_3 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + px_4 \\
\dot{x}_5 &= qx_1
\end{aligned} \tag{1}$$

where x_1, x_2, x_3, x_4, x_5 are the states of the system and a, b, c, p, q are constant, positive parameters of the system that control the dynamics of the system.

With $a = 10;\ b = 8/3;\ c = 28;\ p = 4/3;\ q = 5/2$; system (1) is hyperchaotic with phase space plot as shown in figure (1) and has three positive Lyapunov exponents, i.e. $LE_1 = 0.4195, LE_2 = 0.2430,\ LE_3 = 0.0145;$ other Lyapunov exponents are $LE_4 = 0$ and $LE_5 = -13.0405$.

Also, the slave system is described by the controlled hyper chaotic Lorenz system

$$\begin{aligned}
\dot{y}_1 &= a(y_2 - y_1) + y_4 + y_5 + u_1 \\
\dot{x}_2 &= cy_1 - y_1y_3 - y_2 + u_2 \\
\dot{y}_3 &= y_1y_2 - by_3 + u_3 \\
\dot{y}_4 &= -y_1y_3 + py_4 + u_4 \\
\dot{y}_5 &= qy_1 + u_5
\end{aligned} \tag{2}$$

where y_1, y_2, y_3, y_4, y_5 are the states and u_1, u_2, u_3, u_4, u_5 are active controllers to be designed.

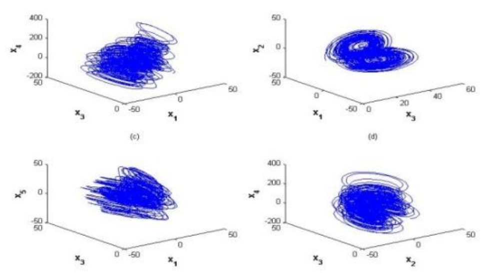


Figure 1: Phase Portrait of Hyperchaotic 5D Lorenz system in (a) $x_1 - x_3 - x_4$ plane; (b) $x_3 - x_1 - x_2$ plane; (c) $x_1 - x_3 - x_5$ plane; (d) $x_2 - x_3 - x_4$ plane.

The divergence of system (1) is in the form

$$\nabla V = \frac{d\dot{x}_1}{dx_1} + \frac{d\dot{x}_2}{dx_2} + \frac{d\dot{x}_3}{dx_3} + \frac{d\dot{x}_4}{dx_4} + \frac{d\dot{x}_5}{dx_5} \quad (3)$$

$= -(a+1+b-p+0) = -(10+1+8/3-4/3) = -37/3$ $\nabla V < 0,$ i.e. system (1) is dissipative.

2.2 Equilibrium and Stability Analysis

The equilibrium of the system (1) can be obtained by solving the following equations

$$\begin{aligned} & a(x_2 - x_1) + x_4 + x_5 = 0 \\ & cx_1 - x_1x_3 - x_2 = 0 \\ & x_1x_2 - bx_3 = 0 \\ & -x_1x_3 + px_4 = 0 \\ & qx_1 = 0 \end{aligned} \quad (4)$$

It is easy to find that system (4) has only one trivial equilibrium point $E_0(0,0,0,0,0)$.

The Jacobian matrix is given by

$$J = \begin{pmatrix} -a & a & 0 & 1 & 1 \\ c & -1 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & 0 \end{pmatrix} \quad (5)$$

To find the eigenvalues, the corresponding characteristic equation is:

$(b+\lambda)(p-\lambda)\left[(a+\lambda)(1+\lambda)\lambda - ac\lambda - (1+\lambda)q\right] = 0.$

The eigenvalues of the Jacobian matrix J are:

$\lambda_1 = 1.3333; \lambda_2 = 11.9057; \lambda_3 = -0.0092; \lambda_4 = -2.6667; \lambda_5 = -22.8966$

The eigenvalues are real and have opposite signs, this shows that the equilibrium of the system is a saddle point.

3.0 THEORETICAL ANALYSES

3.1 Complete Synchronization of Identical 5D Hyperchaotic Lorenz Systems via Active Control

The complete synchronization error is defined by

$e_i = y_i - \beta x_i, (i = 1,2,3,4,5)$.

Then the error dynamics is obtained as

$$\begin{aligned} \dot{e}_1 &= ae_2 - ae_1 + e_4 + e_5 + u_1 \\ \dot{e}_2 &= ce_1 - e_2 + x_1x_3 - y_1y_3 + u_2 \\ \dot{e}_3 &= -be_3 - x_1x_2 + y_1y_2 + u_3 \\ \dot{e}_4 &= pe_4 + x_1x_3 - y_1y_3 + u_4 \\ \dot{e}_5 &= qe_1 + u_5 \end{aligned} \quad (6)$$

We need to find the nonlinear active control law for $u_i, (i = 1,2,3,4,5)$, in such a manner that the error dynamics (6) is globally asymptotically stable.

Choosing the control functions $u = \{u_1, u_2, u_3, u_4, u_5\}^T$ as

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= y_1y_3 - x_1x_3 + v_2 \\ u_3 &= x_1x_2 - y_1y_2 + v_3 \\ u_4 &= y_1y_3 - x_1x_3 + v_4 \\ u_5 &= v_5 \end{aligned} \quad (7)$$

Substituting equation.(7) into (6) we have

$$\begin{aligned}\dot{e}_1 &= ae_2 - ae_1 + e_4 + e_5 + v_1\\ \dot{e}_2 &= ce_1 - e_2 + v_2\\ \dot{e}_3 &= -be_3 + v_3\\ \dot{e}_4 &= pe_4 + v_4\\ \dot{e}_5 &= qe_5 + v_5\end{aligned} \qquad (8)$$

Where $v = \{v_1, v_2, v_3, v_4, v_5\}$ are the linear control inputs chosen such that the system (6) becomes stable. Let us consider

$$v = [v_1, v_2, v_3, v_4, v_5]^T = A[e_1, e_2, e_3, e_4, e_5] \qquad (9)$$

Where A is a 5 x 5 constant matrix, In order to make above system stable, the matrix A should be selected in such a way that all of its eigenvalues are with negative real parts. Consider the following choice of matrix A as

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 & -1\\ -c & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -p-1 & 0\\ 0 & 0 & 0 & 0 & -q-1\end{pmatrix} \qquad (10)$$

$$\left[V_{1,}V_{2,}V_{3,}V_{4,}V_5\right] = \begin{pmatrix} 0 & -a & 0 & -1 & -1\\ -c & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -p-1 & 0\\ 0 & 0 & 0 & 0 & -q-1\end{pmatrix} \begin{pmatrix} e_1\\ e_2\\ e_3\\ e_4\\ e_5\end{pmatrix}$$

$$\begin{aligned}v_1 &= -ae_2 - e_4 - e_5\\ v_2 &= -ce_1\\ v_3 &= 0\\ v_4 &= -pe_4 - e_4\\ v_5 &= -qe_5 - e_5\end{aligned} \qquad (11)$$

With this choice, the error system becomes

$$\begin{aligned}\dot{e}_1 &= -ae_1\\ \dot{e}_2 &= -e_2\\ \dot{e}_3 &= -be_3\\ \dot{e}_4 &= -e_4\\ \dot{e}_5 &= -e_5\end{aligned} \qquad (12)$$

Substituting equation (11) into (7), hence, systems (1) and (2) will synchronize if the nonlinear active control is chosen as

$$\begin{aligned}u_1 &= -ae_2 - e_4 - e_5\\ u_2 &= -ce_1 + y_1y_3 - x_1x_3\\ u_3 &= x_1x_2 - y_1y_2\\ u_4 &= -pe_4 - e_4 + y_1y_3 - x_1x_3\\ u_5 &= -qe_1 - e_5\end{aligned} \qquad (13)$$

Proof. Based on the Lyapunov second method, we construct a positive definite Lyapunov candidate function as

$$V(e) = e^T Pe = \tfrac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2\right) \qquad (14)$$

Where $P = diag[1,1,1,1,1]$. The derivative of the Lyapunov function $V(e)$ with respect to time is

$$\begin{aligned}\dot{V}(e) &= e_1\dot{e}_2 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5\\ &= e_1(-ae_1) + e_2(-e_2) + e_3(-be_3)\\ &+ e_4(-e_4) + e_5(-e_5)\end{aligned} \qquad (15)$$

$$\dot{V}(e) = -ae_1^2 - e_2^2 - be_3^2 - e_4^2 - \tfrac{1}{2}e_5^2 = -e^T Qe$$

Where

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Every diagonal matrix with positive diagonal elements are positive definite; so $Q > 0$.

Therefore, $\dot{V}(e)$ is negative definite. And according to the Lyapunov asymptotical stability theory, the nonlinear active controller is achieved and the synchronization of the hyperchaotic systems is achieved. The proof is now complete.

3.2 Anti-Synchronization of Identical 5D Hyperchaotic Lorenz Systems via Active Control

Here, we discuss anti-synchronization behaviour of identical hyperchaotic Lorenz system evolving from different initial conditions. We define the error states for anti- synchronization as

$e_i = y_i + x_i, (i = 1, 2, 3, 4, 5)$

By applying the definition of error state, the error dynamics is obtained as

$$\begin{aligned} \dot{e}_1 &= ae_2 - ae_1 + e_4 + e_5 + u_1 \\ \dot{e}_2 &= ce_1 - e_2 - x_1x_3 - y_1y_3 + u_2 \\ \dot{e}_3 &= -be_3 + x_1x_2 + y_1y_2 + u_3 \\ \dot{e}_4 &= pe_4 - x_1x_3 - y_1y_3 + u_4 \\ \dot{e}_5 &= qe_1 + u_5 \end{aligned} \qquad (16)$$

We need to find the nonlinear active control law for $u_i, (i = 1, 2, 3, 4, 5)$ in such a manner that the error dynamics of (16) is globally asymptotically stable.

Choosing the control functions $u = \{u_1, u_2, u_3, u_4, u_5\}^T$ as

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= y_1y_3 + x_1x_3 + v_2 \\ u_3 &= -x_1x_2 - y_1y_2 + v_3 \\ u_4 &= y_1y_3 + x_1x_3 + v_4 \\ u_5 &= v_5 \end{aligned} \qquad (17)$$

Substituting equation (17) into (16)

$$\begin{aligned} \dot{e}_1 &= ae_2 - ae_1 + e_4 + e_5 + v_1 \\ \dot{e}_2 &= ce_1 - e_2 + v_2 \\ \dot{e}_3 &= -be_3 + v_3 \\ \dot{e}_4 &= pe_4 + v_4 \\ e_5 &= qe_5 + v_5 \end{aligned} \qquad (18)$$

Where $v = \{v_1, v_2, v_3, v_4, v_5\}$ are the linear control inputs chosen such that the system (16) becomes stable. Let us consider

$$v = [v_1, v_2, v_3, v_4, v_5]^T = A[e_1, e_2, e_3, e_4, e_5] \qquad (19)$$

Where A is a 5 x 5 constant matrix, In order to make above system stable, the matrix A should be selected in such a way that all of its eigenvalues are with negative real parts. Consider the following choice of matrix A as

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 & -1 \\ -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p-1 & 0 \\ 0 & 0 & 0 & 0 & -q-1 \end{pmatrix}$$

$$\left[V_1, V_2, V_3, V_4, V_5\right] = \begin{pmatrix} 0 & -a & 0 & -1 & -1 \\ -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p-1 & 0 \\ 0 & 0 & 0 & 0 & -q-1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}$$

$$\begin{aligned} v_1 &= -ae_2 - e_4 - e_5 \\ v_2 &= -ce_1 \\ v_3 &= 0 \\ v_4 &= -pe_4 - e_4 \\ v_5 &= -qe_1 - e_5 \end{aligned} \qquad (20)$$

Substituting equation (20) into (17), consequently, System (1) will anti- synchronize with the system (2) if nonlinear active control is chosen as

$$\begin{aligned} u_1 &= -ae_2 - e_4 - e_5 \\ u_2 &= -ce_1 + y_1y_3 + x_1x_3 \\ u_3 &= -x_1x_2 - y_1y_2 \\ u_4 &= -pe_4 - e_4 + y_1y_3 + x_1x_3 \\ u_5 &= -qe_1 - e_5 \end{aligned} \qquad (21)$$

Proof: Based on the Lyapunov second method, we construct a positive definite Lyapunov candidate function as

$$V(e) = e^T Pe = \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2) \qquad (22)$$

Where P = diag [1, 1, 1, 1, 1]. The derivative of the Lyapunov function $V(e)$ with respect to time is

$$\begin{aligned} \dot V(e) &= e_1\dot e_2 + e_2\dot e_2 + e_3\dot e_3 + e_4\dot e_4 + e_5\dot e_5 \\ &= e_1(-ae_1) + e_2(-e_2) + e_3(-be_3) + e_4(-e_4) + e_5(-e_5) \\ \dot V(e) &= -ae_1^2 - e_2^2 - be_3^2 - e_4^2 - \tfrac{1}{2}e_5^2 = -e^T Qe \end{aligned} \qquad (23)$$

Where

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Every diagonal matrix with positive diagonal elements are positive definite, so Q > 0.

Therefore, $\dot V(e)$ is negative definite. And according to the Lyapunov stability theory, and hence anti-synchronization of the hyperchaotic systems is achieved. The proof is now complete.

3.3 Hybrid Synchronization of Identical 5D Hyperchaotic Lorenz Systems via Active Control

Hybrid synchronization based on the nonlinear active control between identical 5D hyperchaotic Lorentz systems.
The hybrid synchronization error is defined by

$e_i = y_i - \beta x_i, (i = 1,2,3,4,5)$ and $\beta = \pm 1$ for hybrid-synchronization to take place.

$\beta = +1;$ if $i = odd$ and $\beta = -1;$ if $i = even$.

Then the error dynamics is obtained as

$$\begin{aligned} \dot e_1 &= ae_2 - ae_1 + e_4 + e_5 - 2ax_2 - 2x_4 + u_1 \\ \dot e_2 &= ce_1 - e_2 + 2cx_1 - x_1x_3 - y_1y_3 + u_2 \\ \dot e_3 &= -be_3 - x_1x_2 + y_1y_2 + u_3 \\ \dot e_4 &= pe_4 - x_1x_3 - y_1y_3 + u_4 \\ \dot e_5 &= qe_1 + u_5 \end{aligned} \qquad (24)$$

We need to find the nonlinear active control law for $u_i, (i=1,2,3,4,5),$ in such a manner that the error dynamics of (24) is globally asymptotically stable.

Choosing the control functions $u=\{u_1,u_2,u_3,u_4,u_5\}^T$ as

$$\begin{aligned}
u_1 &= 2ax_2 + 2x_4 + v_1 \\
u_2 &= y_1y_3 + x_1x_3 - 2cx_1 + v_2 \\
u_3 &= x_1x_2 - y_1y_2 + v_3 \\
u_4 &= y_1y_3 + x_1x_3 + v_4 \\
u_5 &= v_5
\end{aligned} \tag{25}$$

Substituting (25) into equation (24), one readily obtain

$$\begin{aligned}
\dot{e}_1 &= ae_2 - ae_1 + e_4 + e_5 + v_1 \\
\dot{e}_2 &= ce_1 - e_2 + v_2 \\
\dot{e}_3 &= -be_3 + v_3 \\
\dot{e}_4 &= pe_4 + v_4 \\
\dot{e}_5 &= qe_1 + v_5
\end{aligned} \tag{26}$$

Where $v=\{v_1,v_2,v_3,v_4,v_5\}$ are the linear control inputs chosen such that the system becomes stable. Let us consider

$$v=[v_1,v_2,v_3,v_4,v_5]^T = A[e_1,e_2,e_3,e_4,e_5] \tag{27}$$

where A is a 5 x 5 constant matrix. In order to make above system stable, the matrix A should be selected in such a way that all of its eigenvalues are with negative real parts. Consider the following choice of matrix A as

$$A=\begin{pmatrix} 0 & -1 & 0 & -1 & -1 \\ -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p-1 & 0 \\ 0 & 0 & 0 & 0 & -q-1 \end{pmatrix}$$

$$\left[V_{1,}V_{2,}V_{3,}V_{4,}V_5\right]=\begin{pmatrix} 0 & -a & 0 & -1 & -1 \\ -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p-1 & 0 \\ 0 & 0 & 0 & 0 & -q-1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix}$$

$$\begin{aligned}
v_1 &= -ae_2 - e_4 - e_5 \\
v_2 &= -ce_1 \\
v_3 &= 0 \\
v_4 &= -pe_4 - e_4 \\
v_5 &= -qe_1 - e_5
\end{aligned} \tag{28}$$

By substituting equation (28) into (25), system (1) will hybrid synchronize with system (2) if the nonlinear active control is chosen as

$$\begin{aligned}
u_1 &= -ae_2 - e_4 - e_5 + 2ax_2 + 2x_4 \\
u_2 &= -ce_1 + y_1y_3 + x_1x_3 - 2cx_1 \\
u_3 &= x_1x_2 - y_1y_2 \\
u_4 &= -pe_4 - e_4 + y_1y_3 + x_1x_3 \\
u_5 &= -qe_1 - e_5
\end{aligned} \tag{29}$$

Proof: Based on the Lyapunov second method, we construct a positive definite Lyapunov candidate function as

$$V(e) = e^T Pe = \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2) \tag{30}$$

Where $P = diag[1,1,1,1,1]$. The derivative of the Lyapunov function $V(e)$ with respect to time is

$$\begin{aligned}\dot{V}(e) &= e_1\dot{e}_2 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 \\ &= e_1(-ae_1) + e_2(-e_2) + e_3(-be_3) \\ &+ e_4(-e_4) + e_5(-e_5)\end{aligned} \quad (31)$$

$$\dot{V}(e) = -ae_1^2 - e_2^2 - be_3^2 - e_4^2 - \tfrac{1}{2}e_5^2 = -e^T Qe$$

Where

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Every diagonal matrix with positive diagonal elements are positive definite, so $Q > 0$.

Therefore, $\dot{V}(e)$ is negative definite; and according to the Lyapunov asymptotical stability theory, the nonlinear active controller is achieved and the synchronization of the hyperchaotic systems is achieved. The proof is now complete.

4.0 NUMERICAL SIMULATIONS

The numerical simulation is performed in order to establish the effectiveness and feasibility of the designed controller. The fourth-order Runge-Kutta scheme embedded in MATLAB software package was used with initial conditions (x_1, x_2, x_3, x_4, x_5) = (5, 10, 3, 12, 9) for the master system and $(y_1, y_2, y_3, y_4, y_5)$ = (0.4, 0.7, 0.6, 0.9, 1) for the slave system. The parameters values are taken as $a = 10;\ b = 8/3;\ c = 28;\ p = 4/3;\ q = 5/2.$

4.1. Complete Synchronization of Identical 5D Hyperchaotic Lorenz Systems

The complete synchronization of systems (1) and (2) is achieved using the controllers (13). The result obtained shows the state variable moves chaotically with time when the controller is de-activated. This shows that these systems depend sensitively on initial condition in the absence of controller, which is a major character of a chaotic system. It can be seen that when controllers are activated the system becomes asymptotically stable. Figures (2) and (3) show that in the absence of controllers the system does not synchronize but synchronization occurs when controllers are activated. Also from figure (4) the variable errors do not synchronize with time in the absence of controllers but when the controllers are introduced at $t \geq 5.75$ the variable errors synchronize with time. This is confirmed by the synchronization quality *e*, given by $e = \sqrt{e_1^2 + e_2^2 + e_3^2}$ as shown in figure (5).

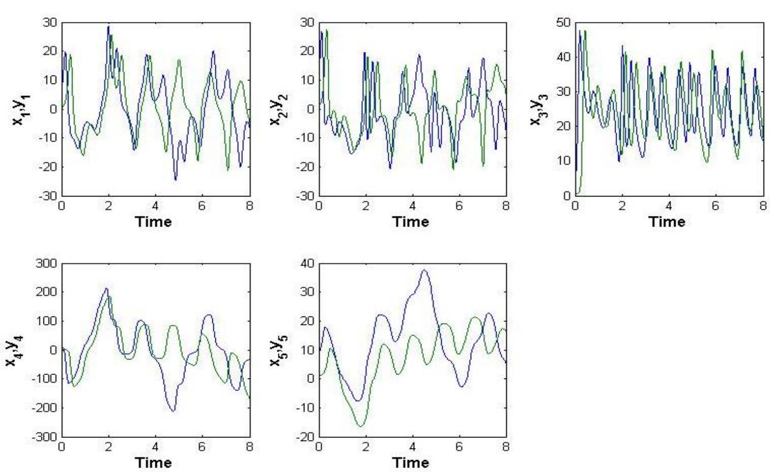


Figure 2: Time series of the master (system (1)) and the slave (system (2)) in the absence of the controllers.

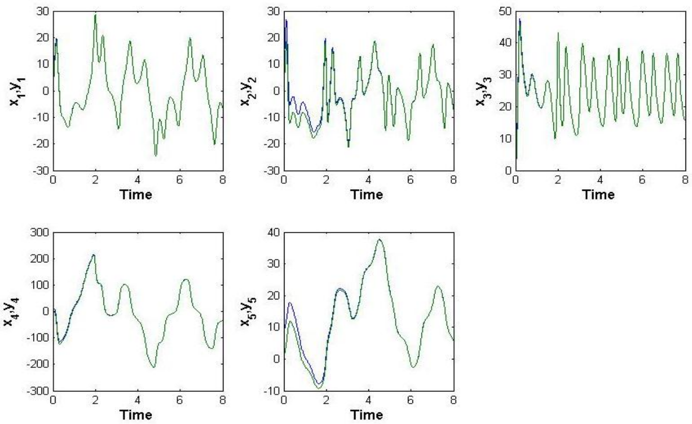


Figure 3: Time series of the master (system (1)) and the slave (system (2)) when the controllers are activated.

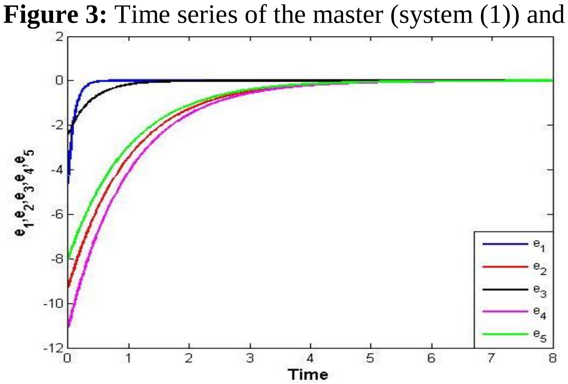


Figure 4: Error dynamics between systems (1) and (2) with the controllers deactivated for $0 < t = 5.75$ and activated for $t \geq 5.75$.

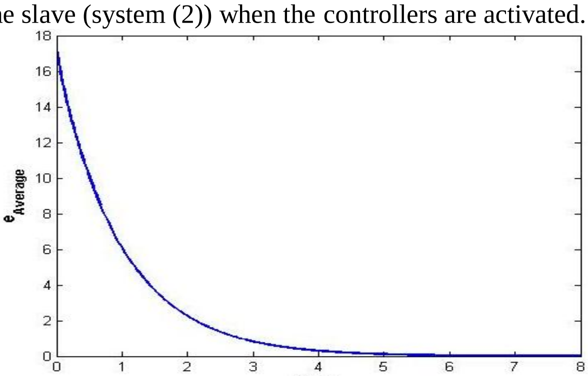


Figure 5: Synchronization quality between systems (1) and (2).

4.2. Anti-Synchronization of Identical 5D Hyperchaotic Lorenz Systems

The anti-synchronization of systems (1) and (2) is realised using the controllers (21). The results in figures (6) and (7) show that the state variable moves chaotically with time when the controllers are switched off but when controllers are activated the system becomes asymptotically stable. This also shows that these systems depend sensitively on initial condition in the absence of controller. We depict in figure (8) the plot of error dynamics as a function of time. Notably, when the controllers are de-activated, the error dynamics do not synchronize but when the contr0llers are switched on at $t \geq 5.75$ the variable errors synchronize with time. This is also confirmed by the synchronization quality in figure (9).

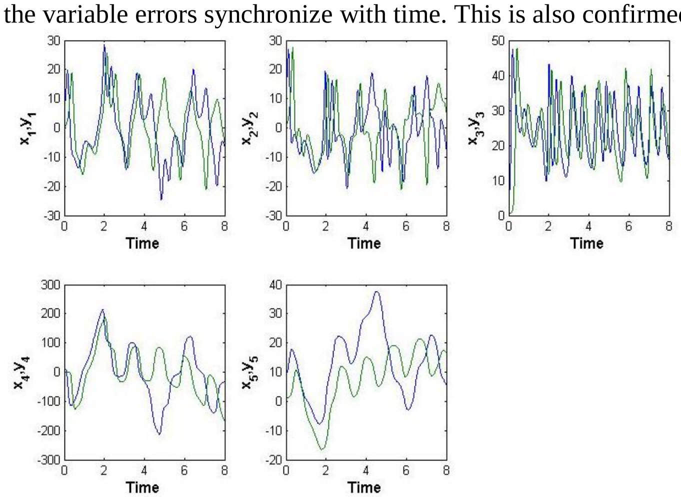


Figure 6: Time series of the master (system (1)) and the slave (system (2)) in the absence of the controllers.

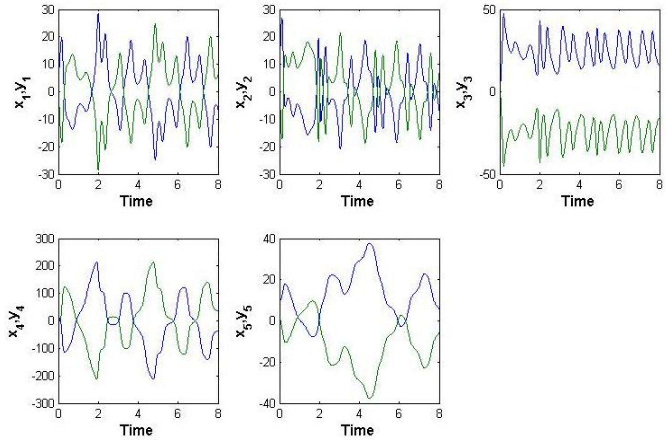
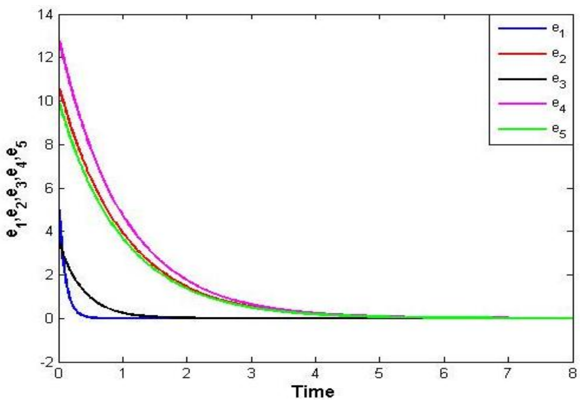


Figure 7: Time series of the master (system (1)) and the slave (system (2)) when the controllers are activated.



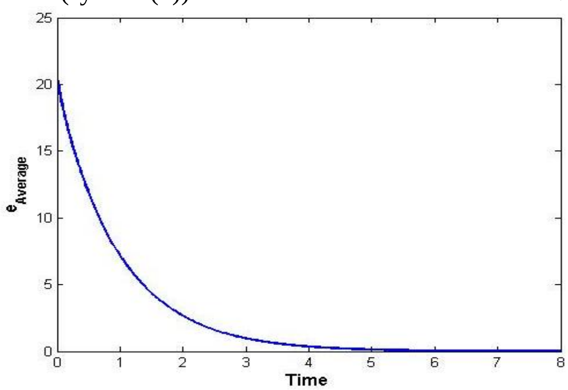


Figure 8: Error dynamics between systems (1) and (2) with the controllers deactivated for $0 < t = 5.75$ and activated for $t \geq 5.75$.

Figure 9: Synchronization quality between systems (1) and (2).

4.3. Hybrid Synchronization of Identical 5D Hyperchaotic Lorenz Systems

The hybrid synchronization of systems (1) and (2) is achieved with the controllers defined in (29). The results in figures (10) and (11) show that the state variable moves chaotically with time when the controllers are de-activated but when controllers are switched on (see figure 11) the system becomes asymptotically stable. Also from figure (12) the variable errors do not synchronize with time in the absence of controllers but when the controllers are introduced at $t \geq 5.75$ the variable errors become globally asymptotically stable. This is further confirmed by the synchronization quality displayed in figure (13).

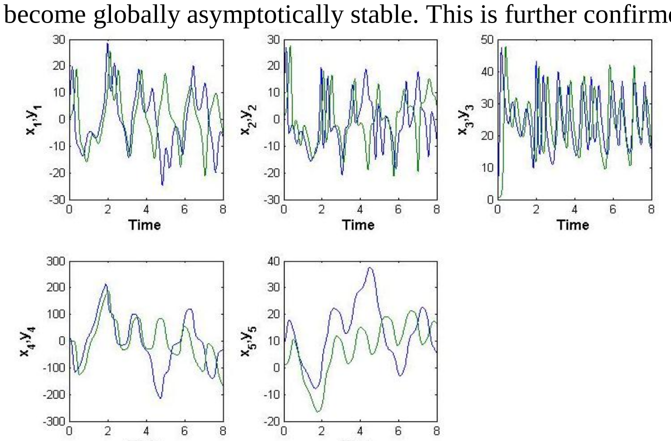


Figure 10: Time series of the master (system (1)) and the slave (system (2)) in the absence of the controllers.

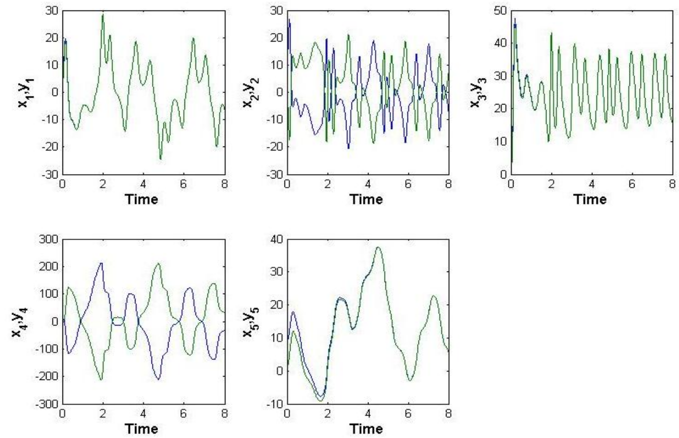


Figure 11: Time series of the master (system (1)) and the slave (system (2)) when the controllers are activated.

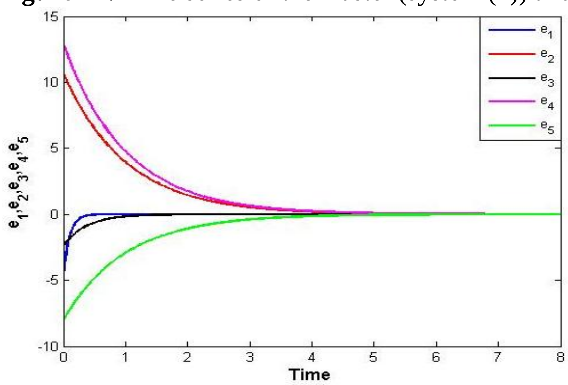


Figure 12: Error dynamics between systems (1) and (2) with the controllers deactivated for $0 < t = 5.75$ and activated for $t \geq 5.75$.

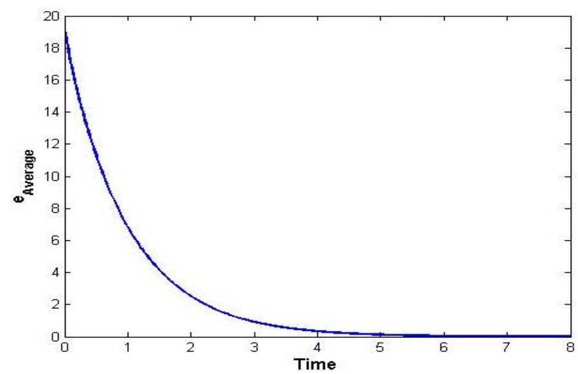


Figure 13: Synchronization quality between systems (1) and (2).

5.0 CONCLUSION

This paper presents the synchronization, anti-synchronization and hybrid synchronization scheme between two identical 5D hyperchaotic systems. The synchronization, anti-synchronization and hybrid synchronization are achieved via active control techniques. The Lyapunov second method and linear system theory utilized have not been fully extended to a 5-D system and especially the Lorenz 5-D hyperchaotic system, thus, its consideration in this work. The compensator controller and the active control law are derived in such a way that the time derivative of a Lyapunov function becomes strictly negative. Numerical simulations are shown to illustrate and validate the synchronization schemes presented in this study.

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