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LOCATION OF TRIANGULAR EQUILIBRIUM POINTS FOR THE BINARY LALANDE 21258 SYSTEM IN THE RESTRICTED PROBLEM OF THREE BODIES

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Abstract

This study examines the existence and location of the triangular equilibrium points under the framework of the circular restricted three-body problem where the primaries are triaxial rigid bodies and the infinitesimal body an oblate spheroid. The theory is applied to the binary Lalande 21258 system for eight different cases. Two triangular equilibrium points have been obtained in each case and it is seen from the numerical simulation that the triaxiality of the primaries have a significant effect on the location of the triangular equilibrium points while the oblateness of the infinitesimal body has an insignificant outcome on the positions of the triangular equilibrium points.

1.0 Introduction

The restricted three-body problem (R3BP) is the mathematical description of the motion of a massless (also called infinitesimal body) body moving under the gravitational influence of two point masses. The equilibrium points are a special kind of solution of R3BP. Five equilibrium points have been shown to exist under the framework of the circular R3BP.

Three out of the five equilibrium points called collinear equilibrium points and denoted by L_1, L_2 and L_3 lie along the line connecting the two point masses (known as primaries). The collinear equilibrium points are unstable especially for small masses i.e., for the mass ratio $\mu < 0.03852$ [1]. The remaining two equilibrium points which are dynamically stable form the apex of two triangles (equilateral or scalene depending on the distances between the primary and secondary bodies) which have the primaries as their vertices, respectively. These two points are called triangular equilibrium points (TEPs) and are denoted by L_4 and L_5 .

Many researchers have made modifications to the classical R3BP by taking into cognizance the shape of the primaries in their studies on the existence, location, stability and periodic orbits at and around the equilibrium points as the case may be, as well as including other perturbations like P-R Drag, radiation pressure, angular velocity, stoke's drag force, gravitational potential from belt et cetera. Some of such works can also be seen in [2-12].

In particular, establishing the existence of the equilibrium points is an important aspect of space dynamics. Recent research works concerning the investigation of the location of TEPs include those of Singh and Simeon [13], in which they investigated the motion of an infinitesimal body around the TEPs in the framework of the circular R3BP where the primaries are triaxial rigid bodies, radiating in nature and under the influence of P-R drag. They observed that the TEPs are not only seen to move towards the line joining the primaries, but are also unstable owing to the destabilizing influence of P-R drag.

Singh and Amuda [14] studied the existence and stability of a test particle around the equilibrium points in the photogravitational circular R3BP where the primaries are oblate spheroids under the influence of P-R drag and small perturbations given in the Coriolis and centrifugal forces. The theory was applied to eclipsing binary systems. The involved parameters influenced the location of the TEPs.

Furthermore, in the study of the location of equilibrium points in the elliptic R3BP with an oblate primary and a radiating secondary, Raman and Sharman 15] discovered that the triangular and collinear equilibrium points are different from those of the classical case.

Additionally, Zahra et al [16] investigated the existence of triangular equilibrium points when the primary is a triaxial rigid body and the secondary an oblate spheroidal body under the framework of the R3BP. They discovered that the locations of the TEPs are affected by the considered perturbations.

Also, in their study of the trajectory of the infinitesimal mass around L_4 of the TEPs in the R3BP, Singh et al [17] observed that oblateness and radiation pressure of the primaries have significant effect on the trajectory and stability of the infinitesimal mass around the TEPs.

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The motivation from this study stems from the fact that the shape of the infinitesimal body has to an extent always been taken to be spherical and not ellipsoidal. Singh and Haruna [18] examined the positions and stability of the equilibrium points with oblate primaries while the infinitesimal body is also an oblate spheroid together with small perturbations in the Coriolis and centrifugal forces.
In this paper, the locations of the triangular equilibrium points are obtained under the framework of the R3BP where the primaries are taken to be triaxial rigid bodies and the infinitesimal body is an oblate spheroid. The paper is organized in such a way that in section 2, the equations of motion are presented while in section 3 the coordinates of the TEPs are obtained. In section 4, numerical simulation is made using binary Lalande 21258 system and the conclusion is presented in the last section.

2. Equations of Motion

The equations of motion in the barycentric, synodic and dimensionless coordinate system Oxyz of the massless body (see [2, 9, 18]) are represented by

$$\ddot{x} - 2n\dot{y} = \Omega_x, \qquad \ddot{y} + 2n\dot{x} = \Omega_y, \tag{1}$$

where the symbol Ω represent the pseudo force (potential function)

$$\Omega = \frac{n^2}{2}(x^2+y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)(2s_1-s_2)}{2r_1^3} + \frac{\mu(2s_1'-s_2')}{2r_2^3} - \frac{3(1-\mu)(s_1-s_2)y^2}{2r_1^5} - \frac{3\mu(s_1'-s_2')y^2}{2r_2^5} + \frac{(1-\mu)A_3}{2r_1^3} + \frac{\mu A_3}{2r_2^3}, \tag{2}$$

with

$$r_1 = \left[(x-\mu)^2 + y^2\right]^{\frac{1}{2}}, \; r_2 = \left[(x+1-\mu)^2 + y^2\right]^{\frac{1}{2}} \tag{3}$$

being the distances of the third body from the primary and secondary body, respectively. The perturbed, due to the triaxility of the primaries, mean motion n is given by the formula:

$$n = \sqrt{1 + \frac{3}{2}\left[(2s_1 - s_2) + (2s_1' - s_2')\right]}, \tag{4}$$

with $s_1 = (\alpha_{t_1}^2 - \alpha_{t_3}^2)/5R^2$, $s_2 = (\alpha_{t_2}^2 - \alpha_{t_3}^2)/5R^2$, $s_1' = (\alpha_{t_1}'^2 - \alpha_{t_3}'^2)/5R^2$, $s_2' = (\alpha_{t_2}'^2 - \alpha_{t_3}'^2)/5R^2$,
and $A_3 = (\alpha_{o_1}^2 - \alpha_{o_3}^2)/5R^2$ with $s_1, s_2, s_1', s_2', A_3 \ll 1$, where α_{t_1}, α_{t_2}, α_{t_3} are the semi-axes of the larger primary body, α_{t_1}', α_{t_2}', α_{t_3}' are the semi-axes of the smaller one, and $\alpha_{o_1}, \alpha_{o_2}, \alpha_{o_3}$ $(\alpha_{o_1} = \alpha_{o_2} > \alpha_{o_3})$ being the semi-axes of the infinitesimal body and R is the dimensional distance between the primaries.
The configuration of the rotating coordinate system for the restricted three-body problem have presented in Figure 1.

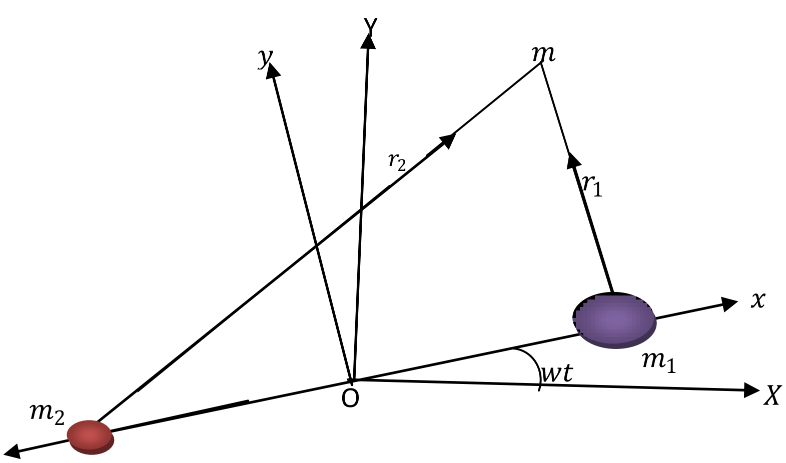

Figure 1: The configuration of the rotating coordinate system for the restricted three-body problem where m_1, m_2 and m are the triaxial primaries and an oblate infinitesimal body respectively.

3. Location of the Triangular points

The equilibrium points or Lagrangian points are obtained when the acceleration and velocity of the infinitesimal body are zero. In other words, the equilibrium points are obtained when the infinitesimal body is experiencing a state of rest. These equilibrium points are the solutions of the system $\Omega_x = \Omega_y = 0$. As a result, we have the following set of partial derivatives

$$\Omega_x = n^2 x + \frac{15(1-\mu)(x-\mu)(s_1-s_2)y^2}{2r_1^7} - \frac{3A_3(1-\mu)(x-\mu)}{2r_1^5} - \frac{3(1-\mu)(x-\mu)(2s_1-s_2)}{2r_1^5} - \frac{(1-\mu)(x-\mu)}{r_1^3} \tag{5}$$
$$+\frac{15\mu(x-\mu+1)(s_1'-s_2')y^2}{2r_2^7} - \frac{3A_3\mu(x-\mu+1)}{2r_2^5} - \frac{3\mu(x-\mu+1)(2s_1'-s_2')}{2r_2^5} - \frac{\mu(x+1-\mu)}{r_2^3},$$

and

$$\Omega_y = n^2 y + \frac{15(1-\mu)(s_1-s_2)y^3}{2r_1^7} - \frac{3A_3(1-\mu)y}{2r_1^5} - \frac{3(1-\mu)y(s_1-s_2)}{r_1^5} - \frac{3(1-\mu)y(2s_1-s_2)}{2r_1^5} - \frac{(1-\mu)y}{r_1^3}$$
$$+\frac{15\mu(s_1'-s_2')y^3}{2r_2^7} - \frac{3A_3\mu y}{2r_2^5} - \frac{3\mu(s_1'-s_2')y}{r_2^5} - \frac{3\mu y(2s_1'-s_2')}{2r_2^5} - \frac{\mu y}{r_2^3},$$

or the last equation can also be written as

$$\Omega_y = \left[n^2 + \frac{15(1-\mu)(s_1-s_2)y^2}{2r_1^7} - \frac{3A_3(1-\mu)}{2r_1^5} - \frac{3(1-\mu)(s_1-s_2)}{r_1^5} - \frac{3(1-\mu)(2s_1-s_2)}{2r_1^5} - \frac{(1-\mu)}{r_1^3} \right. \tag{6}$$
$$\left. +\frac{15\mu(s_1'-s_2')y^2}{2r_2^7} - \frac{3A_3\mu}{2r_2^5} - \frac{3\mu(s_1'-s_2')}{r_2^5} - \frac{3\mu(2s_1'-s_2')}{2r_2^5} - \frac{\mu}{r_2^3}\right] y.$$

The triangular equilibrium points are obtained from the solutions of Equations (5) and (6) when $y \neq 0$. The absence of the triaxiality and oblateness parameters (that is, when $s_1 = s_2 = s_1' = s_2' = 0 = A_3$) in Equations (5) and (6) yield the solutions $r_1 = r_2 = 1$ and from Equation (4), we have $n = 1$.

Now, with the presence of the perturbation parameters, that is, for $s_1, s_2, s_1', s_2', A_3 \neq 0$ we assume that the solutions of Equations (5) and (6) are

$$r_1 = 1+\alpha, \quad r_2 = 1+\beta, \tag{7}$$

where $\alpha, \beta \ll 1$.

Next, we substitute the values of $r_i, i = 1,2$ from Equations (7) into Equations (3) and solving for x and y, we retain only linear terms in α and β to obtain

$$x = \beta - \alpha + \mu - \frac{1}{2}, \quad y = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(\alpha+\beta)\right]. \tag{8}$$

In order to find the values of the small quantities α and β, we make use of $r_{1,2}$, x and y from Equations (7) and (8) respectively and n^2 from Equation (4). These are then substituted into Equations (5) and (6) appropriately, such that higher order terms in s_1, s_2, s_1', s_2' and A_3 are neglected. Thus, we get

$$\alpha = \left[\frac{-11+11\mu}{8(1-\mu)}\right]s_1 + \left[\frac{11+11\mu}{8(1-\mu)}\right]s_2 + \left[\frac{-1}{1-\mu} + \frac{3\mu}{2(1-\mu)}\right]s_1' + \left[\frac{1}{2(1-\mu)} - \frac{\mu}{1-\mu}\right]s_2' \tag{9}$$
$$+\left[\frac{1}{2(1-\mu)} - \frac{\mu}{2(1-\mu)}\right]A_3,$$
$$\beta = \left[\frac{1}{2\mu} - \frac{3}{2}\right]s_1 + \left[\frac{-1}{2\mu} + 1\right]s_2 - \frac{11}{8}s_1' + \frac{11}{8}s_2' + \frac{1}{2}A_3.$$

As a result, the coordinates of the triangular equilibrium points (TEPs) are obtained after substituting Equations (9) into Equations (8)

$$x = \mu - \frac{1}{2} + \left[\frac{1}{2\mu} - \frac{1}{8}\right]s_1 + \left[\frac{-1}{2\mu} - \frac{3}{8}\right]s_2 + \left[\frac{-11}{8} + \frac{2-3\mu}{2(1-\mu)}\right]s_1' + \left[\frac{11}{8} + \frac{-1+2\mu}{2(1-\mu)}\right]s_2', \tag{10}$$

$$y=\pm\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{\left(\frac{1}{2\mu}-\frac{23}{8}\right)s_1+\left(\frac{-1}{2\mu}+\frac{19}{8}\right)s_2+\left(\frac{-11}{8}+\frac{2-3\mu}{2(1-\mu)}\right)s_1'\right.\right.$$
$$\left.\left.+\left(\frac{11}{8}+\frac{-1+2\mu}{2(1-\mu)}\right)s_2'+A_3\right\}\right].$$

It can be observed from Equations (10) that the coordinates of the triangular equilibrium points are affected by the triaxiality of the primaries as well as the oblateness of the infinitesimal body.

4. Numerical simulation

We apply the theory of this consideration (the triaxiallity of the primaries as well as the oblateness of the infinitesimal body) to the binary Lalande 21258 system. Lalande 21258 A is taken to be the primary, while Lalande 21258 B is the secondary body and both are assumed to be triaxial rigid bodies. The photogravitational effect of the binary stem has been neglected in this investigation. The infinitesimal body is assumed to be an exoplanet moving in circular orbits under the gravitational influence of the binary Lalande 21258 system.

Lalande 21258 is also known as Gliese 412. The system is made of two stars forming a binary system. It is a nearby binary red dwarf star, sharing a common motion in the constellation Ursa Major.

In Table 1, we give a few properties of the binary system which are needed for this numerical investigation. The second column gives the visual luminosity of the stars while the third column furnishes the mass of each star with respect to the mass of the Sun represented as M_S (andys.wikia.com/wiki/Lalande21258_System, https://en.wikipedia.org/wiki/Gliese_412).

Table 1. Properties of Lalande 21258 system

Star	Visual Luminosity	Mass	Unit
Lalande 21258 A	0.00637	0.48	M_S
Lalande 21258 B	0.0000344	0.1	M_S

By using the data presented in Table 1, the mass parameter of the binary system is $\mu = 0.1724$. In Table 2, the coordinates of the triangular equilibrium points are presented for eight different cases which have been enumerated in the following manner:

Case 1. The classical case ($s_1 = s_2 = s_1' = s_2' = 0 = A_3$);

Case 2. The Triaxiality of the primary only ($s_1' = s_2' = 0 = A_3$);

Case 3. The Triaxiality of the secondary only ($s_1 = s_2 = 0 = A_3$);

Case 4. Oblateness of the infinitesimal body ($s_1 = s_2 = s_1' = s_2' = 0$);

Case 5. Triaxiality of the primaries ($A_3 = 0$);

Case 6. Triaxiality of the primary as well as oblateness of the infinitesimal body ($s_1' = s_2' = 0$);

Case 7. Triaxiality of the secondary as well as oblateness of the infinitesimal body ($s_1 = s_2 = 0$);

Case 8. Present problem ($s_1, s_2, s_1', s_2', A_3 \neq 0$).

Table 2. The coordinates of the triangular equilibrium points for the eight cases using Equations (10)

Case	s_1	s_2	s_1'	s_2'	A_3	$L_{4,5}$
1.	0	0	0	0	0	-0.32760000 ±0.86602540
2.	0.008	0.006	0	0	0	-0.32504954 ±0.86432280
3.	0	0	0.006	0.004	0	-0.32655831 ±0.86027586
4.	0	0	0	0	0.005	-0.32760000 ±0.86602540
5.	0.008	0.006	0.006	0.004	0	-0.32400785 ±0.85857286
6.	0.008	0.006	0	0	0.005	-0.32504954 ±0.86432240
7.	0	0	0.006	0.004	0.005	-0.32655831 ±0.86027546
8.	0.008	0.006	0.006	0.004	0.005	-0.32400785 ±0.85857286

From the results in Table 2, it can be seen that case 1 which is the classical case (sphericity of the primaries) coincides with that of case 4 in which the infinitesimal body is oblate. As such, the two cases provide the same effect on the TEPs. As such, the oblateness of the infinitesimal body has no significant effect on the TEPs. Also, cases 5 and 8 coincide. That is to say that their effects on the TEPs are similar. This also shows that the oblateness of the infinitesimal body has an insignificant outcome on the TEPs.

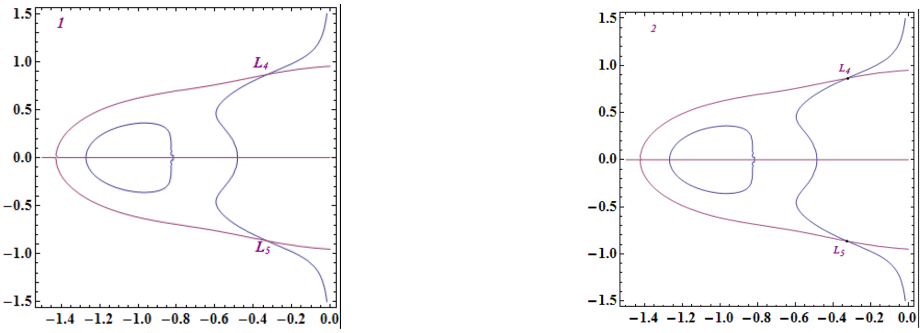

Figure 2: The positions of the triangular equilibrium points corresponding to the cases 1and 2

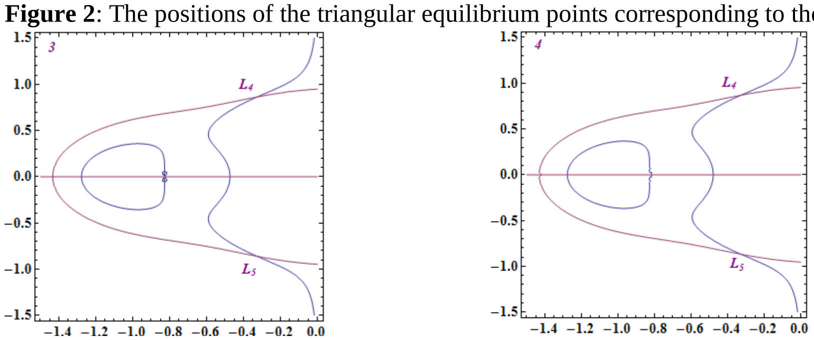

Figure 3: The positions of the triangular equilibrium points corresponding to the cases 3 and 4

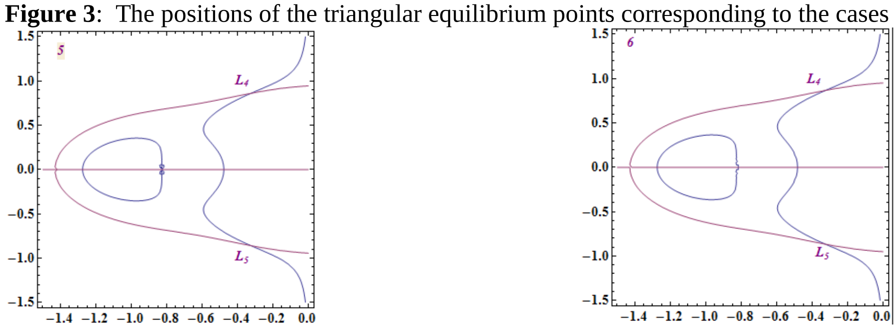

Figure 4: The positions of the triangular equilibrium points corresponding to the cases 5 and 6

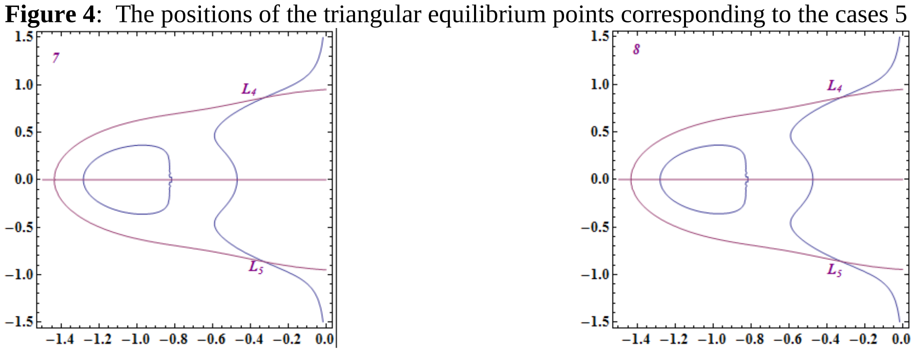

Figure 5: The positions of the triangular equilibrium points corresponding to the cases 7and 8

In figures 2 to 5, the positions of the TEPs are graphically shown. The points have been labelled L_4 and L_5 respectively.

5. Discussion and conclusion

The triangular equilibrium points have been located under the circular motion of the oblate infinitesimal body in the vicinity of the ellipsoidal (triaxial rigid) primaries. The total number of triangular equilibrium points for each of the eight cases considered remained two. It is found that the triaxiality of the primaries have significant effect on the TEPs while the oblateness of the infinitesimal body has an insignificant outcome on the location of the TEPs as seen for the binary Lalande 21258 system mass ratio. The result obtained agrees with those of Singh and Simeon [13] when the primaries are non-luminous and with the absence of P-R Drag. When the primaries are triaxial rigid bodies together with the sphericity of the infinitesimal body and in the absence of the Coriolis and centrifugal forces our result coincides with those of Singh [19]. In the case where $\alpha_{t_1} = \alpha_{t_2} > \alpha_{t_3}$ and

$\alpha'_{t_1} = \alpha'_{t_2} > \alpha'_{t_3}$ and the absence of the Coriolis and centrifugal forces, our result is agreement with Singh and Haruna [18].

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