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RADIAL DISTANCE AND AZIMUTHAL ANGLE VARYING TENSOR FIELD EQUATION EXTERIOR TO HOMOGENEOUS SPHERICAL MASS DISTRIBUTION

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***Abstract*:**

Einstein's field equation is a set of equations in general relativity which describe the fundamental interaction of gravitation as a manifestation of curved space-time with the curvature being produced by mass-energy and momentum content of the space-time. In this research work, we formulate exterior solution to Einstein's geometrical gravitational field equation for a homogeneous spherical distribution of mass whose tensor field varies with radial distance and azimuthal angle (r,ϕ). From covariant and contravariant metric tensors, we obtain the coefficients of affine connection which has thirteen components only, we then construct Riemann-Christoffel tensor, Ricci tensor and Einstein tensor and seek a solution in the form of power series. The obtained result satisfies equivalence principle of Physics, yields a function f which to the order of $c^0, c^{-2}, \ldots$ satisfies Laplace's equation, contains Newton dynamical gravitational scalar potential, post Newtonian additional terms. This solution puts Einstein's geometrical theory of gravitation at the same footing with Newton dynamical theory. Thus the results obtain are applicable to homogeneous spherical distribution of mass rotating with uniform angular speed and accelerating about a fixed diameter.

Keywords: General Relativity, Radial Distance, Azimuthal angle, Einstein Tensor

1.0 Introduction

After Einstein's publication on geometrical theory of gravitation, the search for solution to its inherent geometrical field equations for various mass distributions began different approaches to the solutions of Einstein geometrical field equation (EGFE) have so far been proposed and studied, the first approach has been to seek a mapping under which the metric tensor assume a simple form. Such as vanishing of the off-diagonal components. This led to the famous analytical solution- the famous Schwarzchild's solution. The second assumes that the metric tensor has symmetries-assumed forms of the killing vectors. This led to the solution found by Weyl and Levi-Cavita. The third approach requires that the metric tensor leads to a particular type of classification of Weyl and Riemann- Christoffel tensors. This led to the fronted wave solutions. The fourth approach has been to seek Taylor series expansion of some initial value hyper surface solution, subject to consistent initial value data, this methods has not proven successful in generating solutions [1,2,3,4].

The first exact solution of Einstein's gravitational field was constructed by Schwarzschild in 1916, it is a metric tensor exterior to static spherically symmetric distributions of pressure or mass. Schwarzschild is the mathematically most simple and astrophysically most satisfactory solution of Einstein's geometrical gravitational field equation in the space exterior to a static homogenous distribution of mass within a spherical region [5,6].

In this article as an extension of Schwarzschild we formulate exterior solution to Einstein field equation exterior to a rotating homogeneous spherical distribution of mass about a fixed diameter whose tensor field varies with radial distance and azimuthal angle only.

Theoretical Analysis

Considering an astrophysical mass distribution within spherical geometry in which the tensor field varies with radial distance and azimuthal angle. The covariant metric tensors for this distribution of mass or pressure is given by [7,8].

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$$g_{00} = \left[1 + \frac{2f(r,\phi)}{c^2}\right] \quad (1)$$

$$g_{11} = -\left[1 + \frac{2f(r,\phi)}{c^2}\right]^{-1} \quad (2)$$

$$g_{22} = -r^2 \quad (3)$$

$$g_{33} = -r^2 \sin^2\theta \quad (4)$$

$$g_{\mu\nu} = 0; \quad (5)$$

Using Quotient theorem of tensor analysis, the Contravariant metric tensors for radial distance and azimuthal angle is trivially constructed, the Coeffiecient of affine connection were obtained using [1,8,10] $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\xi}\left(g_{\alpha\xi,\beta} + g_{\beta\xi,\alpha} - g_{\alpha\beta,\xi}\right)$

In general relativity for analogy to hold, we need an equation involving the second order partial derivative of the metric tensor component $g_{\mu\nu}$ and we want the equation to be invariant, so that it is independent of the coordinate system used.
The Riemann-Christoffel Tensor is constructed and Ricci tensor is obtained from it using [8,10]

$$R_{\mu\nu} = R^{0}{}_{\mu\nu 0} + R^{1}{}_{\mu\nu 1} + R^{2}{}_{\mu\nu 2} + R^{3}{}_{\mu\nu 3} \quad (6)$$

Giving explicitly by equation 7-10

$$R_{00} = -\frac{1}{c^2}\left(1 + \frac{2f(r,\phi)}{c^2}\right)\frac{\partial^2 f(r,\phi)}{\partial r^2} + \frac{2}{c^4 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-1}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 - \frac{2}{c^2 r}\left(1 + \frac{2f(r,\phi)}{c^2}\right)\frac{\partial f(r,\phi)}{\partial r}$$
$$-\frac{1}{c^2 r^2 \sin^2\theta}\left(\frac{\partial^2 f(r,\phi)}{\partial \phi^2}\right) \quad (7)$$

$$R_{11} = \frac{1}{c^2}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-1}\frac{\partial^2 f(r,\phi)}{\partial r^2} + \frac{2}{c^2 r}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-1}\frac{\partial f(r,\phi)}{\partial r} - \frac{1}{c^2 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(r,\phi)}{\partial \phi^2}\right) + \quad (8)$$
$$\frac{2}{c^4 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-3}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2$$

$$R_{22} = \frac{2r}{c^2}\frac{\partial f(r,\phi)}{\partial r} + \frac{2f(r,\phi)}{c^2} \quad (9)$$

$$R_{33} = \frac{2}{c^4}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 + \frac{2r\sin^2\theta}{c^2}\left(\frac{\partial f(r,\phi)}{\partial r}\right) + \frac{2\sin^2\theta f(r,\phi)}{c^2} \quad (10)$$

From the Ricci tensor, we construct the Ricci scalar with

$$R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}$$

Giving explicitly as

$$R = -\frac{1}{c^2}\left(\frac{\partial^2 f(r,\phi)}{\partial r^2}\right) + \frac{2}{c^4 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 - \frac{2}{c^2 r}\left(\frac{\partial f(r,\phi)}{\partial r}\right)$$
$$-\frac{1}{c^2 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(r,\phi)}{\partial \phi^2}\right) - \frac{1}{c^2}\frac{\partial^2 f(r,\phi)}{\partial r^2} - \frac{2}{c^2 r}\frac{\partial f(r,\phi)}{\partial r} \quad (11)$$
$$+\frac{1}{c^2 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-1}\left(\frac{\partial^2 f(r,\phi)}{\partial \phi^2}\right) - \frac{2}{c^4 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2$$
$$-\frac{2}{c^2 r}\left(\frac{\partial f(r,\phi)}{\partial r} + \frac{f(r,\phi)}{r}\right) - \frac{2}{c^4 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 - \frac{2}{c^2 r^2}\left(\frac{\partial f(r,\phi)}{\partial r}\right) - \frac{2f(r,\phi)}{c^2 r}$$

This reduces significantly to

$$R = -\frac{2}{c^2}\left(\frac{\partial^2 f(r,\phi)}{\partial r^2}\right) - \frac{8}{c^2 r}\left(\frac{\partial f(r,\phi)}{\partial r}\right) - \frac{4f(r,\phi)}{c^2 r^2} - \frac{4}{c^4 r^2 \sin^2\theta}\left(1 + \frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 \quad (12)$$

The Ricci scalar has a meaning similar to Gaussian curvature.
The Einstein filed equation is given by [9,10]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \qquad (13)$$

Explicitly giving by equation 14-17

$$G_{00} = \frac{1}{c^2}\left(1+\frac{2f(r,\phi)}{c^2}\right)^{-1}\frac{\partial^2 f(r,\phi)}{\partial r^2} - \frac{4}{c^2 r}\left(1+\frac{2f(r,\phi)}{c^2}\right)^{-1}\frac{\partial f(r,\phi)}{\partial r} \qquad (14)$$

$$G_{11} = -\frac{r^2}{c^2}\frac{\partial^2 f(r,\phi)}{\partial r^2} - \frac{4r}{c^2}\frac{\partial f(r,\phi)}{\partial r} - \frac{2f(r,\phi)}{c^2} - \frac{1}{c^4\sin^2\theta}\left(1+\frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 = 0 \qquad (15)$$

$$G_{22} = -\frac{2r}{c^2}\frac{\partial f(r,\phi)}{\partial r} - \frac{r^2}{c^2}\frac{\partial^2 f(r,\phi)}{\partial r^2} - \frac{1}{c^4\sin^2\theta}\left(1+\frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 = 0 \qquad (16)$$

$$G_{33} = \frac{1}{c^4}\left(1+\frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f}{\partial \phi}\right)^2 - \frac{2r\sin^2\theta}{c^2}\frac{\partial f(r,\phi)}{\partial r} - \frac{r^2\sin^2\theta}{c^2}\frac{\partial^2 f(r,\phi)}{\partial r^2} = 0 \qquad (17)$$

It is noted that all the solutions to the other recurrence differential equation (14), (16) and (17) converges everywhere within the exterior field.
Considering the exterior field equation (16)
Hence, equation (16)could equivalently be written as

$$\nabla^2 f(r,\phi) + \frac{1}{c^2 r^2 \sin^2\theta}\left(1+\frac{2f(r,\phi)}{c^2}\right)^{-2}\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 = 0 \qquad (18)$$

and $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial f(r,\phi)}{\partial r}\right] = \nabla^2 f(r,\phi)$

To the weak fieldlimit order of c^0, equation (18) reduces to

$$\nabla^2 f(r,\phi) = 0 \qquad (19)$$

Equation (19) agrees with the priori analogy of the General relativity, reduces to the Laplaces equation, and contains a scalar potential of two functions, Thus the auxiliary form of equation (19) is giving by

$$m^2 + \frac{2}{r}m = 0 \qquad (20)$$

$$f(r,\phi) = -\frac{2}{r} \qquad (21)$$

$f(r,\phi)$ is an arbitrary function that satisfies Laplaces equation with the dependency on two scalar functions

But According to Newton dynamical theory, Newton gravitational scalar potential exterior to a distribution of mass or pressure is given by

$$f(r) = -\frac{GM_0}{r} \qquad (22)$$

G- is the Universal Gravitational Constant

M_0 is the total mass of the spherical body

Deduction from Schwarzschild metric and Newton's theory of gravitation our obtain Scalar potential will be given by

$$f(r,\phi) = -\frac{k}{r} \qquad (23)$$

Where $k = 2GM_0$

Remarkably and interestingly we obtain an arbitrary function which is a function of radial and azimuthal equal to Newton's dynamical scalar potential, thus all the application of Newtons scalar potential could be used with our obtained scalar potentialwith much wider application such as the study of coupling effect of electromagnetism , weak field approximation and possibly quantum in optics with the existence of single operator for the fields..

Consider reducing equation (18) to the order of c^{-2}, it reduces to a which reduces to a second order PDE giving by

$$\nabla^2 f(r,\phi) + \frac{1}{c^2 r^2 \sin^2\theta}\left(1 - \frac{4f(r,\phi)}{c^2} + ...\right)\left(\frac{\partial f(r,\phi)}{\partial \phi}\right)^2 = 0 \qquad (24)$$

We now seek a series solution in the form

$$f(r,\phi) = \sum_{i=0}^{\infty} R_n(r)\exp in\omega\left(\phi - \frac{r}{c}\right) \qquad (25)$$

Where R_n is a functions of r only

Equating Coefficients of: exp(0) gives

$$R_0^{11} + \frac{2}{r}R_0^1 \qquad (26)$$

Equating Coefficients of: $\exp i\omega\left(\phi - \frac{r}{c}\right)$:

$$R_1^{11}(r) + 2\left(\frac{1}{r} - \frac{iw}{c}\right)R_1^1 + iw\left(\frac{iw}{c^2} - \frac{1}{c} + 1\right)R_1 \qquad (27)$$

This is our exact solution for R_1

This solution of equation (26) to the order of c^0, it reduces to

$$f(r,\phi) \approx -\frac{k}{r}(r)\exp i\omega\left(\phi - \frac{r}{c}\right) \qquad (28)$$

Equating Coefficients of: $\exp 2i\omega\left(\phi - \frac{r}{c}\right)$

$$R_2^{11}(r) + 2\left(\frac{1}{r} - \frac{2w}{c}\right)R_2^1 + 2w\left(\frac{2w}{c^2} - \frac{1}{c} + 1\right)R_2 \qquad (29)$$

Thus solutions to the series obtained in (28) and (29) contains the Newton gravitational scalar potential as well as post Newton additional terms which is convergent solution similar to result obtained by[9].

Since the gravitational scalar potential is additive, we can thus sum the 16 independent scalar geometrical field to obtain a resultant gravitational scalar potential due to the distribution of mass or pressure[7].

$$f = \left(t, r, \theta, \phi; f_1 + f_2 + f_3 + \ldots\ldots. f_{16}\right) \quad (30)$$

CONCLUSION

In this article we obtained thirteen affine connection unlike that of Schwarzschild which has only nine affine connections, thus we can remark that all the applications of Schwarzschild metric in the study of gravitational phenomenon could be apply with much precession using our own results. Global positioning system GPS is one of the most significant of the application in everyday life, this system uses stable atomic clock in satellite and on the ground to provide worldwide positioning and time determination [9,12].and gravitational phenomenon in the solar system such as navigation of aircraft

Thus, the results obtained are also applicable to all 2-D physical systems rotating or oscillating about a fixed and central point or with phenomenon originating from a central point [11].

Our obtained result is different from [5] in the sense that [5] is for a hypothetical symmetry which does not have physical significance, while [9] is for a non-static system constantly varying with time.

The door for research is open for the solution to all the other scalar potentials to obtain completely the resultant independent gravitational potential due to the distribution of mass or pressure along the 16 geometrical fields giving by (30).

It is observed that the solution to (27) and (29) converges everywhere within the exterior field, similarly all the solution of the 16 geometrical fields will converge within the exterior field.

Instructively, we realize that our solution to the obtain scalar potential (23) and (25) has a link to the pure Newtonian Dynamical Scalar Potential, this agrees to our assumption that Newton's Dynamical field Equation is a limiting case of Einstein Geometrical field equation, thus all the solution to Newton Dynamical field Equation and parameters could be obtain by geometrical quantities.

We have thus obtained completely the solution to Einstein's Geometrical Field Equations exterior to a homogeneous spherical distribution of mass whose tensor field varies with radial distance and azimuthal angle.

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