Journal of the Nigerian Association of Mathematical Physics
Volume 48 (*Sept. & Nov., 2018 Issue*), **pp155 –160**

GLOBAL STABILITY ANALYSIS FOR THE MATHEMATICAL MODEL OF DYNAMICS OF DIABETES MELLITUS AND ITS COMPLICATIONS.

[1]Aye P .O., [2]Akinwande N.I., [3]Kayode D.J. and [4]Adebileje A.T.

[1,3]Dept. of Mathematical sciences, Adekunle Ajasin University, Akungba Akoko, Ondo State
[2]Dept. of Mathematics and Statistics, Federal University of Technology, Minna
[4]Dept. of Medical Nanotechnology, Tehran University of Medical Science, Tehran, Iran

Abstract

This paper present global stability analysis for a dynamic model. We discuss dynamic model of diabetes mellitus and its complications which is autonomous linear system. The model considers the development of individual from a healthy stage to susceptible stage, then stage of diabetes without complications, to the stage of diabetes with complications and then to stage of treatment. We investigated the stability of the model using quadratic lyapunov function method. The results show that the linear system is globally asymptotically stable.

Keywords: Diabetes mellitus, complications, global stability, quadratic lyapunov function

1.0 Introduction

Diabetes prevalence is rising dramatically worldwide and is expected to rise from 366 million in 2011 to 552 million by 2030. More than 10% of world health care expenditure and about 14% of U.S. healthcare costs are attributable to diabetes. Mathematical models can be used effectively to provide insight about incidence and prevalence of diabetes and help understand factors affecting disease development risk. For example, problems involving many complex parameters are arranged in one system known as dynamic system. A dynamic system is a system of equations that is affected by changes in motion and time. Furthermore, this dynamic system equation is often transformed into a state-space equation. One important study in the dynamic system problem is to investigate the condition of the system, whether the system is a stable or unstable. A good dynamic system must be stable. These conditions are needed to reduce errors in the system due to disturbance so that the system can represent the real problem.

In this paper we present a dynamic model of diabetic mellitus and its complications following previous mathematical models on diabetes [1-7]. The model illustrates the development of the diabetic population from healthy stage until treatment stage due to complications. The main purposes is to investigate the global stability of linear system. Stability problem can be solved by Lyapunov method.

2.0 Research Method

The method used in this study is literature review and reference collection of theories that support the completion of this research. We collect references about the characteristics of diabetes, the causes of diabetes, diabetes complications, the treatments, dynamical model of diabetes. From the diabetics population, we establish a dynamic model and prove the stability of linear system. Stability of linear system is investigated using Lyapunov function[9,10]. In addition, we give numerical example so that we can understand the theorems used to investigate the stability[11,12].

3.0 Results and Discussion

3.1 Formulation of the Model

In this paper, the mathematical model was constructed on the population of diabetics. Let

Corresponding Author**:** Aye P.O., Email: ayepatricko@gmail.com, Tel: +2347030596765

$H(t)$, $E(t)$, $D(t)$, $C(t)$ and $T(t)$ respectively be the numbers of healthy people, susceptible and diabetics without complications, diabetics with complications and diabetics with complications that undergo treatment. We considered the models developed by some authors
[1 - 4].

$$\left.\begin{aligned}
\dot{H} &= \beta - \mu H - \tau H + \sigma S \\
\dot{S} &= \tau H - \mu S - \alpha S - \sigma S \\
\dot{D} &= \alpha S - \mu D - \lambda D + \omega T \\
\dot{C} &= \lambda D - \mu C - \nu C - \delta C - \gamma C \\
\dot{T} &= \gamma C - \mu T - \omega T
\end{aligned}\right\} \qquad (1)$$

where β is the birth rate, α is the incidence of diabetes, μ is natural mortality deaths. parameter $\tau, is\ the$ rate at which healthy individual become susceptible, σ is the rate at which susceptible individual become healthy, λ is the probability of a diabetic person developing a complication. Parameter γ, is the rate at which diabetic with complication are treated and cured, v is the rate at which diabetic patients with complication become severely disabled, while ω is the rate at which diabetic with complication that recovered from complications after treatment return to diabetic without complications, δ is the mortality rate due to complication

3.2 Global Stability for Autonomous Linear System

In this section we prove stability of the system by considering the Lyapunov properties of the linear system.

Theorem 1 (Lyapunov' s theorem for Local stability)[8]:
Let $\dot{x} = f(x), f(x^*) = 0$ where x^* is in the interior of $\Omega \subset \Re^n$. Assume that

V: $\Omega \to \Re$ is a C^1 function. If

(1) $V(x^*) = 0$

(2) $V(x) > 0$, for all $x \in \Omega, x \neq x^*$

(3) $V(x) \leq 0$ along all trajectories of the system in Ω

$\Rightarrow$ x^* is locally stable

Furthermore, if also

(4) $\overset{*}{V} < 0$ for all $x \in \Omega, x \neq x^*$

$\Rightarrow$ x^* is locally asymptotically stable

Lyapunov' s method requires one to choose a positive definite Lyapunov function (candidate) and then prove that its derivative is negative (semi) definite. Quadratic Lyapunov functions can be used to test stability of linear systems.

Theorem 2 (Quadratic form of Lyapunov function)[8]: Consider $\overset{*}{x} = A(x), x \in \Re^n$. The system (origin) is globally asymptotically stable if and only if there exists a positive definite matrix $P = P^T > 0$ such that $A^T P + PA$ is negative definite or $A^T P + PA < 0$. Equivalently if, for a given $Q = Q^T > 0$, it is possible to find a $P = P^T > 0$ such that

$$A^T P + PA = -Q \qquad (2)$$

then the system is globally asymptotically stable.

Theorem 3 [8]: For a given $Q = Q^T > 0$ there exists a unique $P = P^T > 0$ satisfying the Lyapunov equation $A^T P + PA = -Q$ so that the system is globally asymptotically stable

Proof:

$$Jacobian matrix = \begin{bmatrix} -(\mu+\tau) & \sigma & 0 & 0 & 0 \\ \tau & -(\mu+\alpha+\sigma) & 0 & 0 & 0 \\ 0 & \alpha & -(\mu+\lambda) & 0 & \omega \\ 0 & 0 & \lambda & -(\mu+\nu+\delta+\gamma) & 0 \\ 0 & 0 & 0 & \gamma & -(\mu+\omega) \end{bmatrix} \quad (3)$$

Let

$$A = \begin{bmatrix} -\eta_1 & \sigma & 0 & 0 & 0 \\ \tau & -\eta_2 & 0 & 0 & 0 \\ 0 & \alpha & -\eta_3 & 0 & \omega \\ 0 & 0 & \lambda & -\eta_4 & 0 \\ 0 & 0 & 0 & \gamma & -\eta_5 \end{bmatrix} \quad (4)$$

Where

$\eta_1 = \mu+\tau$, $\eta_2 = \mu+\alpha+\sigma$, $\eta_3 = \mu+\lambda$, $\eta_4 = \mu+\nu+\delta+\gamma, \eta_5 = \mu+\omega$

$$A^T = \begin{bmatrix} -\eta_1 & \tau & 0 & 0 & 0 \\ \sigma & -\eta_2 & \alpha & 0 & 0 \\ 0 & 0 & -\eta_3 & \lambda & 0 \\ 0 & 0 & 0 & -\eta_4 & \gamma \\ 0 & 0 & \omega & 0 & -\eta_5 \end{bmatrix} \quad (5)$$

We choose $Q = Q^T = I$ so from equation $A^T P + PA = -I$, we obtain

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\ P_{12} & P_{22} & P_{23} & P_{24} & P_{25} \\ P_{13} & P_{23} & P_{33} & P_{34} & P_{35} \\ P_{14} & P_{24} & P_{34} & P_{44} & P_{45} \\ P_{15} & P_{25} & P_{35} & P_{45} & P_{55} \end{bmatrix} \quad (6)$$

$$-Q = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (7)$$

Evaluating (2), we obtain equations (8) to (32)

$$2\tau P_{12} - 2\eta_1 P_{11} = -1 \quad (8)$$

$$\tau P_{22} - (\eta_1+\eta_2) P_{12} + \sigma P_{11} + \alpha P_{13} = 0 \quad (9)$$

$$\tau P_{23} - (\eta_1+\eta_3) P_{13} + \lambda P_{14} = 0 \quad (10)$$

$$\tau P_{24} - (\eta_1+\eta_4) P_{14} + \gamma P_{15} = 0 \quad (11)$$

$$\tau P_{25} - (\eta_1+\eta_5) P_{15} + \omega P_{13} = 0 \quad (12)$$

$$\sigma P_{11} - (\eta_1+\eta_2) P_{12} + \alpha P_{13} + \tau P_{22} = 0 \quad (13)$$

$$2\sigma P_{12} - 2\eta_2 P_{22} + 2\alpha P_{23} = -1 \quad (14)$$

$$\sigma P_{13} - (\eta_2+\eta_3) P_{23} + \alpha P_{33} + \lambda P_{24} = 0 \quad (15)$$

$$\sigma P_{14} - (\eta_2 + \eta_4) P_{24} + \alpha P_{34} + \gamma P_{25} = 0 \quad (16)$$

$$\sigma P_{15} - (\eta_2 + \eta_5) P_{25} + \alpha P_{35} + \omega P_{23} = 0 \quad (17)$$

$$\lambda P_{14} - (\eta_1 + \eta_3) P_{13} + \tau P_{23} = 0 \quad (18)$$

$$\lambda P_{24} - (\eta_2 + \eta_3) P_{23} + \sigma P_{13} + \alpha P_{33} = 0 \quad (19)$$

$$2\lambda P_{34} - 2\eta_3 P_{33} = -1 \quad (20)$$

$$\lambda P_{44} - (\eta_3 + \eta_4) P_{34} + \gamma P_{35} = 0 \quad (21)$$

$$\lambda P_{45} - (\eta_3 + \eta_5) P_{35} + \omega P_{33} = 0 \quad (22)$$

$$\gamma P_{15} - (\eta_1 + \eta_4) P_{14} + \tau P_{24} = 0 \quad (23)$$

$$\gamma P_{25} - (\eta_2 + \eta_4) P_{24} + \sigma P_{14} + \alpha P_{34} = 0 \quad (24)$$

$$\gamma P_{35} - (\eta_3 + \eta_4) P_{34} + \lambda P_{44} = 0 \quad (25)$$

$$2\gamma P_{45} - 2\eta_4 P_{44} = -1 \quad (26)$$

$$\gamma P_{55} - (\eta_4 + \eta_5) P_{45} + \omega P_{34} = 0 \quad (27)$$

$$\omega P_{13} - (\eta_1 + \eta_5) P_{15} + \tau P_{25} = 0 \quad (28)$$

$$\omega P_{23} - (\eta_2 + \eta_5) P_{25} + \sigma P_{15} + \alpha P_{35} = 0 \quad (29)$$

$$\omega P_{33} - (\eta_3 + \eta_5) P_{35} + \lambda P_{45} = 0 \quad (30)$$

$$\omega P_{34} - (\eta_4 + \eta_5) P_{45} + \gamma P_{55} = 0 \quad (31)$$

$$2\omega P_{35} - 2\eta_5 P_{55} = -1 \quad (32)$$

Value of the parameters

$\alpha = 6000000$, $\lambda = 0.66$, $\tau = 0.2$, $\gamma = 0.08$, $\omega = 0.08$, $\sigma = 0.08$, $\nu = 0.05$

$\delta = 0.05$, $\beta = 0.01623$, $\mu = 0.02$

Substituting the value of parameters above into (8) to (32), and solving the system of equations, we obtained the following values of $P's$.

$$P = \begin{bmatrix} 4.19363 & 2.11299 & 2.11299 & 1.80122 & 2.16915 \\ 2.11299 & 3.56446 & 3.56446 & 2.9149 & 2.62544 \\ 2.11299 & 3.56446 & 3.56448 & 2.9149 & 2.62544 \\ 1.80122 & 2.9149 & 2.9149 & 3.56829 & 2.67073 \\ 2.16915 & 2.62544 & 2.62544 & 2.67073 & 7.10035 \end{bmatrix}$$

The minor of matrix P are

$$\Delta_1 = P_{11} = 4.19363 > 0$$

$$\Delta_2 = \begin{bmatrix} 4.19363 & 2.11299 \\ 2.11299 & 3.56446 \end{bmatrix} = 10.48329965 > 0$$

$$\Delta_3 = \begin{bmatrix} 4.19363 & 2.11299 & 2.11299 \\ 2.11299 & 3.56446 & 3.56446 \\ 2.11299 & 3.56446 & 3.56448 \end{bmatrix} = 0.0002 > 0$$

$$\Delta_4 = \begin{bmatrix} 4.19363 & 2.11299 & 2.11299 & 1.80122 \\ 2.11299 & 3.56446 & 3.56446 & 2.9149 \\ 2.11299 & 3.56446 & 3.56448 & 2.9149 \\ 1.80122 & 2.9149 & 2.9149 & 3.56829 \end{bmatrix} = 0.0002480291329 > 0$$

$$\Delta_5 = \begin{bmatrix} 4.19363 & 2.11299 & 2.11299 & 1.80122 & 2.16915 \\ 2.11299 & 3.56446 & 3.56446 & 2.9149 & 2.62544 \\ 2.11299 & 3.56446 & 3.56448 & 2.9149 & 2.62544 \\ 1.80122 & 2.9149 & 2.9149 & 3.56829 & 2.67073 \\ 2.16915 & 2.62544 & 2.62544 & 2.67073 & 7.10035 \end{bmatrix} = \det(p) = 0.001195431495 > 0$$

Because the minor $\Delta_K > 0, k = 1, \cdots, 5$, it implies that P is positive definite matrix. The system is globally asymptotically stable.

4. Conclusion

Based on the description and explanation of the results and discussion, it can be concluded that the problem of autonomous linear system stability can be solved by using Lyapunov method, by determining quadratic forms of Lyapunov function in system (1). By choosing identity matrix, we can find a positive definite matrix, so that resulting system (1) is globally asymptotically stable.

References

[1] Permatasari, A. H., Tjahjana, R. H., Udjiani, T. (2018). Global stability for linear system and controlability for nonlinear system in the dynamics model of diabetics population. Journal of physics: Conf. series 1025 (2018)012086 doi:10. 1088/1742 1025/1/012086.

[2] Purnami, W., Rifqi, C. A., Dewi, R. S. S.(2018). A mathematical model for the epidimiology of diabetes mellitus with lifestyle and genetic factors. Journal of physics: Conf. series 1028 (2018)012110 doi:10. 1088/1742 - 6596/1028/1/012110.

[3] Adamu, I. I., Momoh, A. A., Tahir, A. (2016). Stability Analysis of the Mathematical Model for the Dynamics of Diabetic Population under the Combine Effect of Birth Rate and Treatment. Int. Journal of Science and Tech Vol. 5, No.1 ,pp 26 - 35.

[4] Akinsola, V. O & Oluyo, T. O. (2014). Mathematical Model of the Complications and Control Diabetes Mellitus. International Journal of Mathematics and Computer Applications Research (UMCAR) ISSN(P): 2249-6955; ISSN(E): 2249-8060 Vol. 4, Issue 5, Oct 2014, 1-14.

[5] Adewale, S.O., R. O. Ayeni, R.O., Ajala, O. A. (2007). A new Generalized Mathematical model for the study of Diabetes Mellitus. Research Journal of Applied Sciences 2(5): 629-632.

[6] Bouteyab, A., Twizell, E.H., Achouayb, K., & Chetouani, A. (2004). A Mathematical Model for the burden of diabetes and its complications, *Biomedical Engineering online*, Vol. 3, No 20., pp 1- 8.

[7] Sandhya, Deepak, K & Prerna (2011). An Ordinary Differential Equation Model of Diabetic Population in New Delhi. Indian Journal of Mathematics and Mathematical Sciences Vol.7, No. 1, (June 2011): 45 - 50.

[8] Hedrick, J.K and Girard, A (2005). *Control of Nonlinear Dynamic Systems: Theory and Applications*, Berkeley: University of California.

[9] Afuwape, A.U., Omeike, M.O (2010). Stability and Bounded-ness of Solutions of a Kind of Third Order Delay Differential Equations. Computational & Applied Mathematics, Vol. 29, N. 3 pp 329-342.

[10] Bainov, D.D (1997). Second Method of Lyapunov and Exis-tence of Periodic Solutions of Linear Impulsive Differential Difference Equations. Divulgaciones Mathematicas. V.S, No. ½ (1997), 29-36.

[11] Griggs, W. M., King, C. K., Shorten, R. N., Mason, O and Wulff, K. (2010). Quadratic Lyapunov functions for systems with state-dependent switching *Linear Algebra and its Applications*, 433**,** 52–63.

[12] Ignat'ev, A. O (2011). On the existence of a lyapunov function as a quadratic form for impulsive systems of linear differential equations *Ukrainian Mathematical Journal,* **62,** 1680-1689.