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ON THE INVERSE BURR DISTRIBUTION: ITS PROPERTIES AND APPLICATIONS

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Abstract

This paper presents a discussion on the mathematical properties of the inverse Burrdistribution. The application of the distribution was subjected to two lifetime data sets and some measures of goodness-of-fit such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Komolgorov-Smirnov (K-S) test statistics were considered to examine its flexibility in modeling lifetime data and the superiority over other well-known distributions. Results obtained from the two lifetime data sets, reveal that the inverse Burr distribution is an appropriate model in fitting the life time data sets and has superiority over other distribution considered.

Keywords: inverse Burrdistribution, Burr XII distribution; survival function; hazard rate; quantile function; moments; maximum likelihood.

1.0 Introduction

A family of twelve cumulative distribution functions was introduced in [1], for statistical modelling. In [2] it was suggested that the Burr III and XII have the simplest functional forms and therefore, are among the most desirable distributions in statistical modelling. This family of distributions suggested by [1] were obtained by differential equation of the form;

$$\frac{dy}{dx} = y(1-y)g(x,y), y = f(x)$$

where $g(x, y) > 0$, $0 \leq y \leq 1$, x is in the range over which the solution is being satisfied. By using different forms of $g\ (x, y)$, Burr obtained twelve distribution functions which are type I to type XII. The inverse Burrdistribution have received a wide range of applications in literature. Some of these applications are found in the works of [1, 3 – 9, 19]. This paper presents are view on the mathematical properties of the inverse Burrdistribution and its application to lifetime data set.

The remaining sections of this paper is organized as follows: In section 2, we introduce the density function, cumulative distribution function, the survival function, hazard rate function, the quantile function, the moments and related measures and moment generating function of the distribution. Section 3, contains an estimation procedure for the parameters of the distribution using maximum likelihood method. Application of the distribution to two real lifetime data sets is given in section 4 and finally in section 5, we give a concluding remark.

2.0 PDF and CDF of the Inverse BurrDistribution

The cumulative distribution of inverse Burrdistribution is given as:

$$F(x) = \left(1 + x^{-c}\right)^{-k}, \quad x > 0 \qquad (1)$$

where the parameters c > 0 and k > 0 are the shape parameters of the distribution.

The corresponding density function of the distribution can be obtain from the cdf in (1) as:

$$f(x) = ckx^{-(c+1)} \left(1 + x^{-c}\right)^{-(k+1)} \qquad (2)$$

The graphical presentation of the cdf and pdf functions of the inverse Burrdistribution for some fixed values of the parameters are shown in Figure 1 respectively as:

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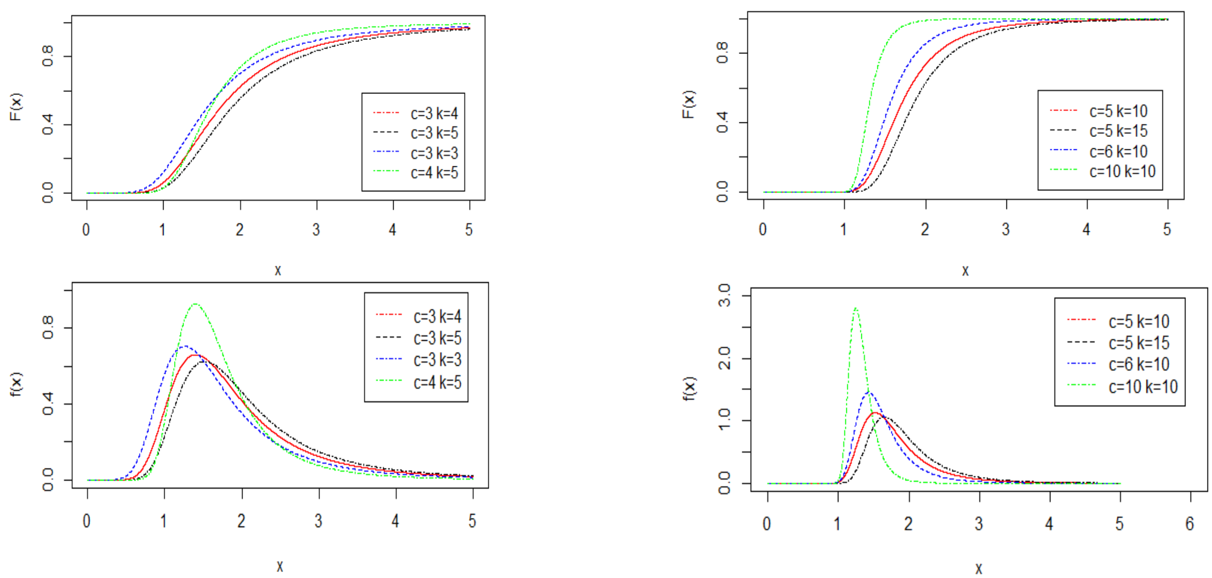

Figure 1: *Density and cumulative functions of the inverse Burrdistribution for some fixed values of the parameters*

2.1 Survival and Hazard Rate Function of the Inverse BurrDistribution

Let X be a continuous random variable with density function *f(x)* and cumulative distribution function *F(x)*. The survival (reliability) function of the inverse Burrdistribution is defined by: $S(t) = 1 - F(x)$

For the inverse Burrdistribution, the survival function is given by

$$S(t) = [1 - (1 + x^{-c})^{-k}] \qquad (3)$$

where $F(x)$ is the cumulative distribution function (cdf) of the inverse Burrdistribution.

Similarly, the hazard function $h(t) = \dfrac{f(x)}{S(t)}$ of the inverse Burrdistribution is.

$$h(t) = \frac{ck}{x(x^{c})[(1+x^{-c})^{k}-1]} \qquad (4)$$

The graph of the survival function and the hazard rate of the inverse Burrdistribution for different values of the parameters is given in Figure 2 respectively as:

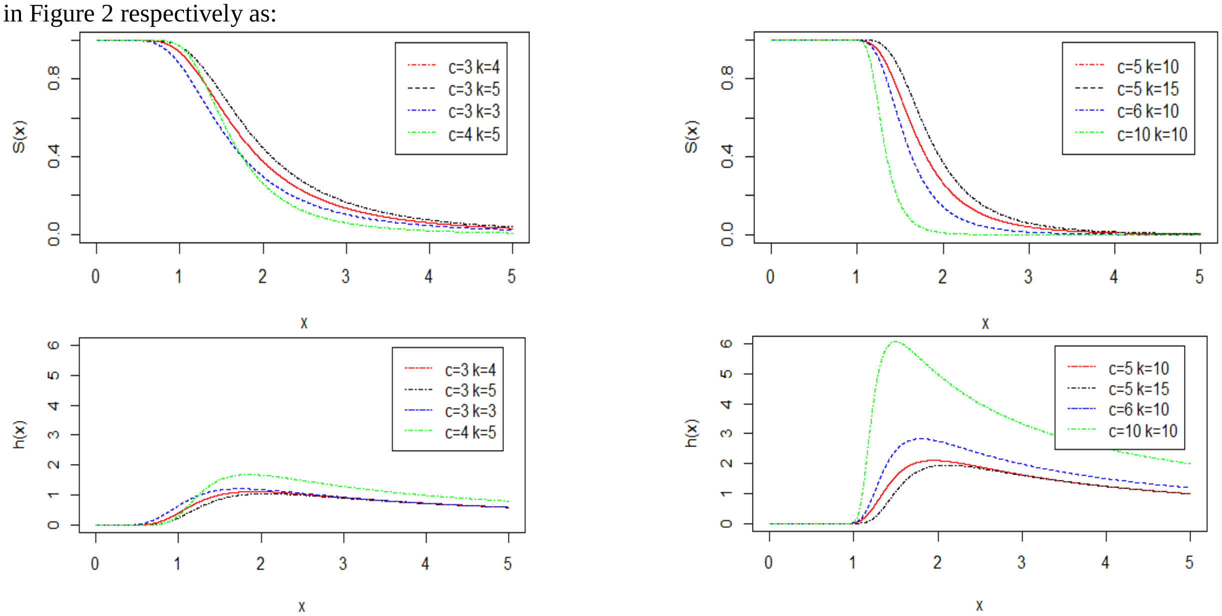

Figure 2*The survival function and hazard rate of the inverse Burr distribution with parameter c and k.*

2.2 Quantile Function of the Inverse BurrDistribution

The quantilefunction of a random variable X is also known as the inverse cdf of X. It is defined as $Q_X(p) = F^{-1}(p)$. For the inverse Burr distribution, it is given by:

$$Q(p) = \left(p^{-1/k} - 1\right)^{-1/c} \quad (5)$$

The median of the inverse Burrdistribution is obtained from (5) by setting p = 0.5

2.3Moment of The Inverse BurrDistribution

The rth moment of a random variable *X*following a probability distribution provided it exist; is given by

$$\mu^r = \int_{-\infty}^{\infty} x^r \ f(x)dx \quad (6)$$

and thus, the rth moment about the origin of the inverse Burrdistribution is obtain from the relation

$$\mu^r = \int_0^{\infty} x^r \ ck \ x^{-(c+1)} \left(1 + x^{-c}\right)^{-(k+1)} dx.$$

With some substitution and the help of complete Beta function, we obtain

$$\mu^r = k \int_0^{\infty} \frac{t^{r/c}}{[1 + t]^{(t+1)}} \ dt$$

Thus the r^{th} moment of the inverse Burrdistribution about the origin is given as:

$$\mu^r = k\beta\left[1 + \frac{r}{c}, \ k - \frac{r}{c}\right] \quad (7)$$

where $\beta(x,y)$ is the complete Beta function.

3.0 Maximum Likelihood Estimate of Inverse Burr Distribution

Let $(x_1, x_2, ..., x_n)$ be random samples from the inverse Burr, then the likelihood function is defined as,

$$L = \sum_{i=1}^{n} In\, f(x_i) \quad (8)$$

where $f(x)$ is as defined in (2)

The log likelihood function is,

$$L = n\, ln\, c + n\, ln\, k - (c+1)\sum_{i=1}^{n} In\, x_i - (k+1)\sum_{i=1}^{n} In\, x_i\, (1 + x_i^{1-c}) \quad (9)$$

Differentiate (9) with respect to c and k to obtain

$$\frac{\partial L}{\partial c} = \frac{n}{c} - \sum_{m}^{n} \ln x_i - (k+1)\left[\frac{\left(1 + x_i^{-c}\right)}{u \ln x_i}\right] = 0 \quad (10)$$

$$\frac{\partial L}{\partial k} = \frac{n}{k} - \sum_{m}^{n} \ln\left(1 + x_i^{-c}\right) = 0 \quad (11)$$

The next is to solve for *k* and *c*. iteratively using any of the derivative with the R software.

4.0 Application of the Inverse Burr Distribution

In this paper, we fit the inverse Burr distribution to two real data sets (See Appendix) alongside with some well-known lifetime distributions and examine its fits by considering some measures of goodness-of-fitwhich includes; Akaike Information Criterion [AIC = *2k-2log(L)*], Bayesian Information Criterion [BIC =*klog(n)-2log(L)*] and Kolmogorov-Smirnov (*K-S*) test statistic.

Key notations:

IBD: Inverse Burr Distribution
Norm: Normal Distribution
Logis: Logistics Distribution
Gumbel: Gumbel Distribution
Inweibull: Inverse Weibull distribution
Invgamma: Inverse Gamma Distribution.

Table 1: Summary Statistics for Data set 1

Models	Estimates	*logL*	AIC	*BIC*	*K-S*	*P_value*
IBD	$c = 1.0331$ $k = 4.2084$	−426.7017	857.4035	863.1075	0.1015	0.1429
Norm	μ = 9.3584 σ = 10.4661	−482.2022	968.4044	974.1085	0.1907	0.0001817
Logis	$k = 7.5863$ $\lambda = 4.4810$	−456.6754	917.3508	923.0549	0.1578	0.003411
Gumbel	μ = 5.6427 $\beta = 5.4222$	−432.2489	868.4977	874.2018	0.1119	0.080093
Invweibull	$\alpha = 0.7521$ $\lambda = 0.1869$ $\beta = 3.2592$	−444.0144	894.0288	902.5849	0.1408	0.01248
Invgamma	$\alpha = 0.7147$ $\beta = 0.2865$ $\lambda = 1.7753$	−454.9451	915.8902	924.4463	0.1911	0.0001743

Table 2: Summary Statistics for Data set 2

Models	Estimates	*logL*	AIC	*BIC*	*K-S*	*P_value*
IBD	$c = 4.3292$ $k = 7.6461$	-15.4299	34.85985	36.85131	0.0909	0.9965
Norm	$\mu = 1.9001$ $\sigma = 0.6863$	-20.8498	45.6996	47.6911	0.2070	0.3525
Logis	$k = 1.7904$ $\lambda = 0.3389$	-19.2433	42.4867	44.4781	0.1429	0.8085
Gumbel	$\mu = 1.6189$ $\beta = 0.4387$	-16.3330	36.6661	38.6576	0.1341	0.8649
Invweibull	$\alpha = 4.0735$ $\lambda = -0.867$ $\beta = 1.5635$	-15.40872	36.8174	39.8046	0.1010	0.9854
Invgamma	$\alpha = 11.4809$ $\beta = -3.4567$ $\lambda = 1.7753$	-16.0464	38.0929	41.0800	0.1321	0.8762

$\boldsymbol{H_0}$: ***The distribution considered doesn't model the data set***
Vs
$\boldsymbol{H_1}$: ***The distribution considered does model the data set***

In hypothesis testing, we reject H_0 when the p_value is > 0.05.
From table 2, it is clearly seen that all distribution considered can model the data set 2, but the inverse Burr distribution has the superiority in modelling the data set. While in the data set 1, it is clearly seen that the normal, logistic, inverse Weibull and inverse gamma failed compared to the Gumbel and inverse Burr distribution which fit the data set.
The fit of the density and cumulative distribution of each distribution for the two real lifetime data sets are given in the Figures 3 and 4 respectively.

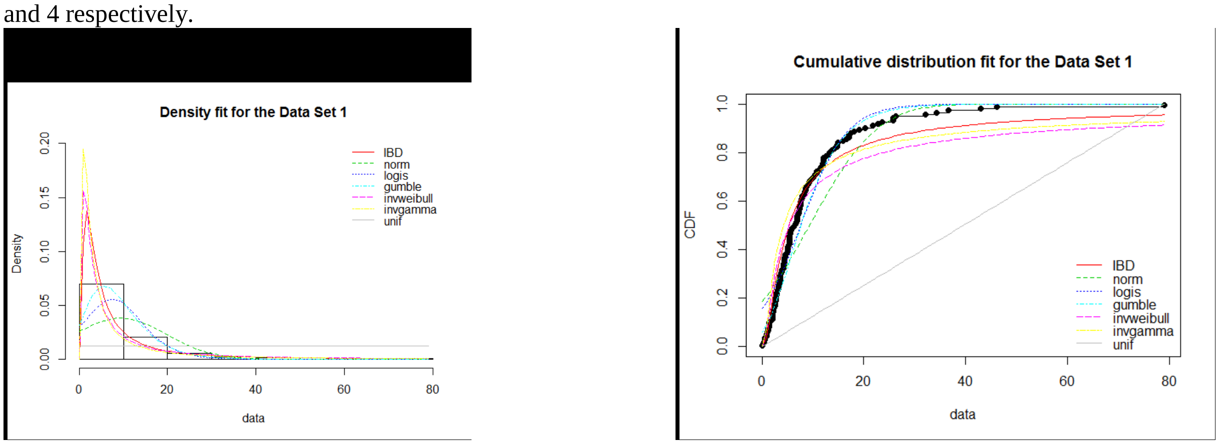

Figure 3: *Density and Cumulative Distribution fit for the Data Set 1*

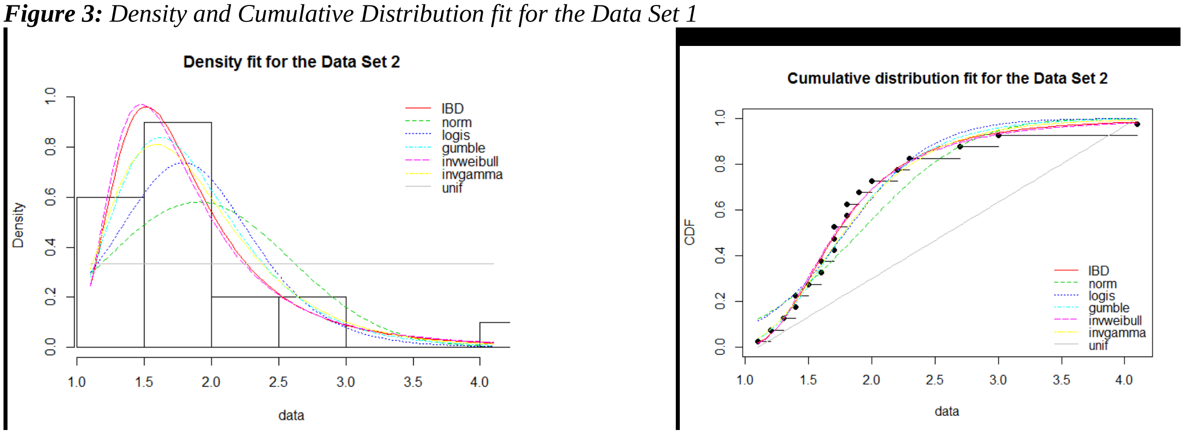

Figure 4: *Density and Cumulative Distribution fit for the Data Set 2*

4.1 P-P Plot

The probability-probability (p-p) plot is a graph of the empirical cdf values plotted against the theoretical cdf values. It is used to determine how well a specific distribution fits to the observed data. This plot will be approximately linear if the specified theoretical distribution is the correct model. The plot show how well data fit a particular distribution.

The probability-probability plot (p-p plot) of each distribution for the two real lifetime data sets are given in the Figures 5 and 6 respectively.

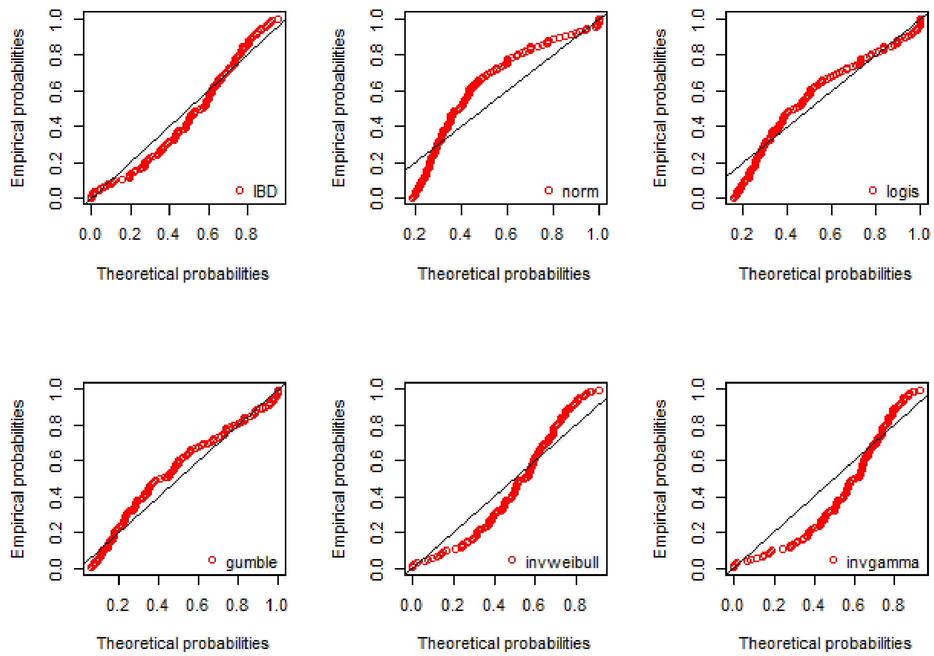

Figure 5: *p-p plot for the Data Set 1*

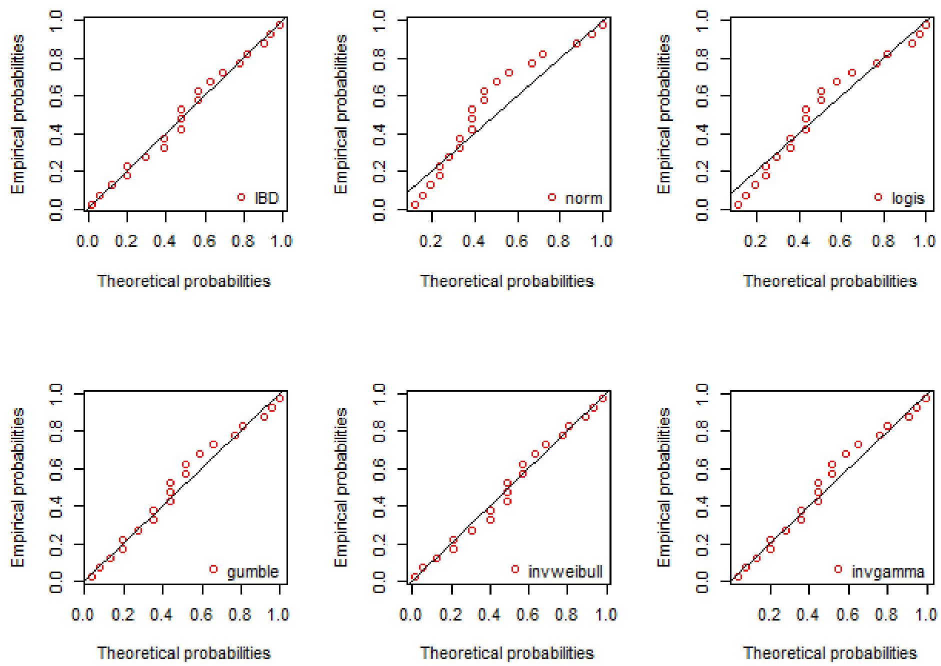

Figure 6: *p-p plot for the Data Set 2*

4.2 Discussion of Results

The superiority of a model over another can be characterized by the model having the least value in terms of *-2logL, AIC, BIC and K-S* Statistic. Table 1 and 2 shows that the IBD having the least values in terms of *-2logL, AIC, BIC and K-S Statistic,* demonstrates superiority over the uniform distribution (Burr I), the logistic distribution (Burr II), the Gumbel, inverse Gamma and inverse

Weibull distributions in modeling the lifetime data sets under study. This claim was further supported by visual inspection of the density, cumulative distribution fit and the p-p plot of the distributions for the two real lifetime data sets displayed in Figures 3-6 respectively.

5.0 Concluding Remark

In this paper, we have studied the properties of the inverse Burr distribution and its application to lifetime data. This distribution which have found wide range of applications in income, wage and wealth, in general is very effective and flexible in modelling lifetime data. Some goodness-of-fit measures which include the Akaike information criterion (AIC), the Bayesian information criterion (BIC) and Kolmogorov-Smirnov test were used to access its performance. The density, cumulative distribution and P-P plots of the distributions werefurther used to support the claim for the data set considered. The results showed that the inverse Burr distribution is flexible and displayed superiority over the other distributions used in fitting the real life data set considered.

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Appendix

Data Set 1: This data set represent the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). The data set is presented below

0.08, 2.09, 3.48,4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.8, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.3, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64 , 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28 , 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Data Set 2: Relief times (in minutes) of 20 patients receiving an analgesic.

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0