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AN ACCURATE FOUR-POINT HYBRID BLOCK METHOD FOR THE SOLUTION OF GENERAL THIRD ORDER DIFFERENTIAL EQUATIONS

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Abstract

An accurate four-point hybrid block method for direct solution of general third order initial value problem of ordinary differential equations is considered. In the development of the method, collocation and interpolation approaches are adopted using the combination of power series and exponential function approximate solution as the basis function. The developed order six method is applied to solve some third order ordinary differential equations for the purpose of comparison with existing methods of same order of accuracy. From the numerical results obtained, it is observed that the new hybrid block method is better than the previous numerical methods in literature in terms of accuracy.

Keywords**:** Hybrid method, Power series and Exponential function, Continuous block method, Differential Equations, Third order

1. Introduction

The use of Mathematics to understand the physical world has been in use for centuries, but the manner and degree to which it can be used has drastically changed in recent years due to the intervention of computer and its ability to perform incredibly complex and computationally-intensive tasks. These tasks are especially applicable in the study of thin film flow of a liquid in fluid dynamics, electromagnetic waves, gravity driven flow, rocket launch trajectory analysis, airflow over airplane bodies (aerodynamics), transport and disposition of chemicals through the body, immune-assay chemistry for developing new blood tests, seismic wave propagation in the earth (earthquakes), predict the evolution of crystals growing in an industrial crystallizer, underwater acoustic signal processing, eco-systems, psychology to mention but just a few. The modeling of the physical, biological and social problems give rise to different forms of differential equations (des) of the form

$$y^{(\mu)} = f\left(x, y, y', ..., y^{(\mu-1)}\right), \ y(a) = \psi_0, \ y'(a) = \psi_1, ..., y^{(\mu-1)}(a) = \psi_{\mu-1} \qquad (1)$$

where μ is the order of the problem, $y \in C^{\mu}[a,b]$ and f is a continuous function of x, y, and derivatives of y. In this research work, the case $\mu = 3$ in equation (1) is considered to have.

$$y''' = f\left(x, y, y', y''\right), \ y(a) = \psi_0, \ y'(a) = \psi_1, y''(a) = \psi_2. \qquad (2)$$

Many researchers have focused attention to this type of problem in equation (2). It is worthy to note that not all these differential equations resulting from the modeling of these processes can be solved analytically. Therefore, the need to adopt numerical solutions for obtaining an approximate solution of these third order ordinary differential equations is expedient. A number of articles have been written by scholars proposing numerical methods and approximate solutions to equation (2) [1, 2]. However, Scholars who proposed method to finding approximate solutions to (2) and implement their methods in Predictor-Corrector modes include [3-6]. While some that implemented their methods in block mode with restriction of the number of interpolation points to the order of the differential equations to be solved include [7-10].

It is noted that all these methods implemented in block modes are developed in such a way that, the numbers of interpolation points are problem dependent (the number of interpolation point is equal to the order of the differential equations to be solved). However, this barrier conditions have been circumvented in this research work where the freedom of numbers of interpolation point is guaranteed.

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Therefore, the aim of this work is to construct a continuous implicit two-step four-point hybrid block method for the solution of initial value problems of general third order ordinary differential equations. To achieve this, the combination of power series and exponential function approximate solution is interpolated at both grid and off-grid point except at the end point and collocation is done at all the grid points. The resulting systems of linear equations are incorporated to augment the procedure and the implementation will be by a simultaneous application of the method to provide approximations to the solution of (2) at block points of $x = x_n,\ x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}},\ x_{n+1},\ x_{n+\frac{4}{3}},\ x_{n+\frac{5}{3}},\ x_{n+2}$ on a partition of $[a, b]$.

2. Methodology

2.1 Development of Hybrid block Method

This section shows the development of 2-step 4-point hybrid block method for the solution of general third order initial value problems of ordinary differential equations. The collocation approach is by using the combination of power series and exponential function as the approximate solution to (2) of the form.

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j + a_{r+s} \sum_{j=0}^{r+s} \frac{\alpha^j x^j}{j!}. \qquad (3)$$

where a_j is an unknown parameters, $t \in [a, b]$, the solution interval, α is a free parameter while r and s are the number of interpolation and collocation points respectively.

Combining equation (3) and (2) gives

$$f\left(x,\ y,\ y'\ y''\right) = \sum_{j=3}^{r+s-1} \left(j(j-1)(j-2) a_j x^{j-3} \right) + a_{r+s} \sum_{j=3}^{r+s-1} \frac{\alpha^j x^{j-3}}{(j-3)!}. \qquad (4)$$

Equations (3) was interpolated at all points, $x = x_{n+i}, i = 0, r, s, 1, u, v$ except the point of evaluation, while equation (4) was collocated at the three grid points, $x = x_{n+i},\ i = 0, 1, 2$, where $0 < r, s < 1$ and $1 < u, v < 2$.

The interpolated and collocated equations are respectively of the form

$$\sum_{j=0}^{c+i-1} a_j x_{n+i}^j = y_{n+j}, i = 0, r, s, 1, u, v \qquad (5)$$

$$\sum_{j=3}^{c+i-1} j(j-1)(j-2) a_j x^{j-3} + a_{c+i} \sum_{j=3}^{c+i} \frac{\alpha^j x^{j-3}}{(j-3)!} = f_{n+i}, \quad i = 0, 1, 2 \qquad (6)$$

where $f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i})$, and $y_{n+i} = y(x_{n+i})$.

The resulting system of nine equations from equations (5) and (6) are determined using Maple software to obtain the values of a_j's, $j = 0(1)8$. The real parameters a_j's are substituted back into (3) and after some manipulations yield, a continuous linear multistep method of the form

$$y(x) = \sum_{j=0}^{k-1} \zeta_j(x) y_{n+j} + \left\{ \tau_1(x) y_{n+r} + \tau_2(x) y_{n+s} + \tau_3(x) y_{n+u} + \tau_4(x) y_{n+v} \right\} + h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j}. \qquad (7)$$

Taking $k = 2$, the coefficients $\zeta_j(x)$ and $\beta_j(x)$ are expressed as function of $v = \dfrac{x - x_{n+1}}{h}$ as follows:

$$\zeta_0(v) = \frac{913}{5530} v^2 - \frac{1215}{632} v^4 + \frac{12879}{3160} v^6 - \frac{729}{553} v^8$$

$$\zeta_1(v) = 1 - \frac{28213}{2212} v^2 + \frac{11907}{316} v^4 - \frac{11421}{316} v^6 + \frac{22577}{2212} v^8$$

$$\tau_1(v) = \frac{61}{1092} v - \frac{239437}{172536} v^2 + \frac{124497}{8216} v^4 - \frac{243}{52} v^5 - \frac{204039}{8216} v^6 + \frac{243}{182} v^7 + \frac{447849}{57512} v^8$$

$$\tau_2(v) = -\frac{440}{273} v + \frac{172033}{21567} v^2 - \frac{149769}{4108} v^4 + \frac{243}{26} v^5 + \frac{197721}{4108} v^6 - \frac{243}{91} v^7 - \frac{209709}{14378} v^8$$

$$\tau_3(v) = \frac{440}{273} v + \frac{271939}{43134} v^2 - \frac{139563}{8216} v^4 - \frac{243}{26} v^5 + \frac{56295}{8216} v^6 + \frac{243}{91} v^7 - \frac{17739}{14378} v^8$$

$$\tau_4(v) = -\frac{61}{1092} v - \frac{262273}{862680} v^2 + \frac{20817}{8216} v^4 + \frac{243}{52} v^5 + \frac{78813}{81080} v^6 - \frac{243}{182} v^7 - \frac{49815}{57512} v^8$$

$$\beta_0(v)=\frac{4}{331695}v+\frac{41359}{104815620}v^2-\frac{77}{16432}v^4-\frac{1}{780}v^5+\frac{1327}{123240}v^6+\frac{1}{364}v^7-\frac{575}{115024}v^8$$

$$\beta_1(v)=-\frac{4778}{331695}v-\frac{423632}{26203905}v^2+\frac{1}{6}v^3+\frac{580}{3081}v^4-\frac{67}{195}v^5-\frac{6148}{15405}v^6+\frac{17}{182}v^7+\frac{928}{7189}v^8$$

$$\beta_2(v)=\frac{4}{331695}v+\frac{9769}{104815620}v^2-\frac{46}{49296}v^4-\frac{1}{780}v^5+\frac{157}{123240}v^6+\frac{1}{364}v^7+\frac{127}{115024}v^8 \tag{8}$$

Evaluating (8) at $v=1$ gives the discrete method

$$y_{n+2}=\frac{256}{39}y_{n+\frac{5}{3}}-\frac{395}{39}y_{n+\frac{4}{3}}+\frac{395}{39}y_{n+\frac{2}{3}}-\frac{256}{39}y_{n+\frac{1}{3}}+y_n+\frac{h^3}{9477}\left(28f_{n+2}-1856f_{n+1}+28f_n\right). \tag{9}$$

To derive the new hybrid block method, additional equations are needed since equation (9) will not be sufficient if the solution at $x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}},\ x_{n+1},\ x_{n+\frac{4}{3}},\ x_{n+\frac{5}{3}},\ x_{n+2}$ are to be obtained simultaneously. The additional methods are obtained by evaluating the first and second derivatives of (7) at $x=x_n,\ x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}},\ x_{n+1},\ x_{n+\frac{4}{3}},\ x_{n+\frac{5}{3}},\ x_{n+2}$. While the remaining equations are obtained from the third derivative of (7) at $x=x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}},\ x_{n+\frac{4}{3}}$ to give the following equations

$$hy'_n=-\frac{72421}{11060}y_n+\frac{321560}{21567}y_{n+\frac{1}{3}}-\frac{340675}{21567}y_{n+\frac{2}{3}}+\frac{5480}{553}y_{n+1}-\frac{28715}{12324}y_{n+\frac{4}{3}}-\frac{15664}{107835}y_{n+\frac{5}{3}}+h^3\left[\frac{32353}{5240781}f_n+\frac{328504}{5240781}f_{n+1}+\frac{109}{248683}f_{n+2}\right]$$

$$hy'_{n+\frac{1}{3}}=-\frac{1506}{2765}y_n-\frac{111333}{28756}y_{n+\frac{1}{3}}+\frac{52852}{7189}y_{n+\frac{2}{3}}-\frac{2145}{553}y_{n+1}+\frac{6558}{7189}y_{n+\frac{4}{3}}+\frac{4477}{143780}y_{n+\frac{5}{3}}-h^3\left[\frac{6161}{8734635}f_n+\frac{181625}{8734635}f_{n+1}+\frac{311}{8734635}f_{n+2}\right]$$

$$hy'_{n+\frac{2}{3}}=\frac{873}{11060}y_n-\frac{528}{553}y_{n+\frac{1}{3}}-\frac{1168}{553}y_{n+\frac{2}{3}}-\frac{2088}{553}y_{n+1}-\frac{1749}{2212}y_{n+\frac{4}{3}}+\frac{8}{2765}y_{n+\frac{5}{3}}+h^3\left[\frac{32}{223965}f_n+\frac{2936}{223965}f_{n+1}+\frac{2}{223965}f_{n+2}\right]$$

$$hy'_{n+1}=\frac{61}{1092}y_{n+\frac{1}{3}}-\frac{440}{273}y_{n+\frac{2}{3}}+\frac{440}{273}y_{n+\frac{4}{3}}-\frac{61}{1092}y_{n+\frac{5}{3}}+h^3\left[\frac{4}{331695}f_n-\frac{4778}{331695}f_{n+1}+\frac{4}{331695}f_{n+2}\right]$$

$$hy'_{n+\frac{4}{3}}=-\frac{873}{11060}y_n+\frac{3704}{7187}y_{n+\frac{1}{3}}-\frac{9}{1027}y_{n+\frac{2}{3}}-\frac{2088}{553}y_{n+1}+\frac{83725}{28756}y_{n+\frac{4}{3}}+\frac{15696}{35945}y_{n+\frac{5}{3}}+h^3\left[-\frac{653}{2911545}f_n+\frac{83176}{2911545}f_{n+1}-\frac{263}{2911545}f_{n+2}\right]$$

$$hy'_{n+\frac{5}{3}}=\frac{1506}{2765}y_n-\frac{14815}{4108}y_{n+\frac{1}{3}}+\frac{33100}{7189}y_{n+\frac{2}{3}}-\frac{92510}{7189}y_{n+\frac{4}{3}}+\frac{152959}{20540}y_{n+\frac{5}{3}}+h^3\left[\frac{2749}{174627}f_n-\frac{222668}{1746927}f_{n+1}+\frac{1529}{1746927}f_{n+2}\right]$$

$$hy'_{n+2}=\frac{72421}{11060}y_n-\frac{307952}{7189}y_{n+\frac{1}{3}}+\frac{493522}{7189}y_{n+\frac{2}{3}}-\frac{5480}{553}y_{n+1}-\frac{1452853}{28756}y_{n+\frac{4}{3}}+\frac{1009048}{35945}y_{n+\frac{5}{3}}+h^3\left[\frac{24322}{1247805}f_n-\frac{1521944}{1247805}f_{n+1}+\frac{222904}{8734635}f_{n+2}\right] \tag{10}$$

and

$$h^2 y''_n = \frac{284317}{11060} y_n - \frac{1995172}{21567} y_{n+\frac{1}{3}} + \frac{5643145}{43134} y_{n+\frac{2}{3}} - \frac{47276}{553} y_{n+1} + \frac{1725631}{86268} y_{n+\frac{4}{3}} + \frac{157688}{107835} y_{n+\frac{5}{3}} - h^3\left[\frac{5350811}{52407810} f_n + \frac{14897524}{26203905} f_{n+1} + \frac{75281}{52407810} f_{n+2}\right]$$

$$h^2 y''_{n+\frac{1}{3}} = \frac{21423}{2765} y_n - \frac{302935}{28756} y_{n+\frac{1}{3}} - \frac{40196}{7189} y_{n+\frac{2}{3}} + \frac{12723}{1106} y_{n+1} - \frac{20369}{7189} y_{n+\frac{4}{3}} - \frac{41971}{143780} y_{n+\frac{5}{3}} + h^3\left[\frac{129823}{17469270} f_n + \frac{822062}{8734635} f_{n+1} + \frac{4633}{17469270} f_{n+2}\right]$$

$$h^2 y''_{n+\frac{2}{3}} = -\frac{9123}{11060} y_n - \frac{86696}{7189} y_{n+\frac{1}{3}} - \frac{322253}{14378} y_{n+\frac{2}{3}} + \frac{6708}{553} y_{n+1} - \frac{32651}{28756} y_{n+\frac{4}{3}} + \frac{6596}{35945} y_{n+\frac{5}{3}} - h^3\left[\frac{24107}{17469270} f_n + \frac{123988}{8734635} f_{n+1} + \frac{1877}{17469270} f_{n+2}\right]$$

$$h^2 y''_{n+1} = \frac{913}{2765} y_n - \frac{239437}{86268} y_{n+\frac{1}{3}} + \frac{344066}{21567} y_{n+\frac{2}{3}} + \frac{271939}{21567} y_{n+\frac{4}{3}} - \frac{262213}{43} y_{n+\frac{5}{3}} + h^3\left[\frac{41359}{52407810} f_n - \frac{847264}{26203905} f_{n+1} + \frac{9769}{52407810} f_{n+2}\right]$$

$$h^2 y''_{n+\frac{4}{3}} = -\frac{9123}{11060} y_n + \frac{40244}{7189} y_{n+\frac{1}{3}} - \frac{136445}{14378} y_{n+\frac{2}{3}} + \frac{6708}{553} y_{n+1} - \frac{404267}{28756} y_{n+\frac{4}{3}} + \frac{238856}{35945} y_{n+\frac{5}{3}} + h^3\left[-\frac{40697}{17469270} f_n + \frac{1535012}{8734635} f_{n+1} - \frac{18467}{17469270} f_{n+2}\right]$$

$$h^2 y''_{n+\frac{5}{3}} = \frac{21423}{2765} y_n - \frac{1470871}{28756} y_{n+\frac{1}{3}} + \frac{543770}{7189} y_{n+\frac{2}{3}} + \frac{12723}{1106} y_{n+1} - \frac{604337}{7189} y_{n+\frac{4}{3}} + \frac{5797709}{143780} y_{n+\frac{5}{3}} + h^3\left[\frac{395263}{17469270} f_n - \frac{14075758}{8734635} f_{n+1} + \frac{270073}{17469270} f_{n+2}\right]$$

$$h^2 y''_{n+2} = \frac{284317}{11060} y_n - \frac{3607720}{21567} y_{n+\frac{1}{3}} + \frac{12093337}{43134} y_{n+\frac{2}{3}} - \frac{47276}{553} y_{n+1} - \frac{11174753}{86268} y_{n+\frac{4}{3}} + \frac{8220248}{107835} y_{n+\frac{5}{3}} + h^3\left[\frac{4055719}{52407810} f_n - \frac{117025564}{26203905} f_{n+1} + \frac{9331249}{52407810} f_{n+2}\right] \quad (11)$$

$$h^3 f_{n+\frac{1}{3}} = \left[-\frac{4410}{79} y_n + \frac{232890}{1027} y_{n+\frac{1}{3}} - \frac{353460}{1027} y_{n+\frac{2}{3}} + \frac{18180}{79} y_{n+1} + \frac{57210}{1027} y_{n+\frac{4}{3}} - \frac{1230}{1027} y_{n+\frac{5}{3}}\right] + \frac{h^3}{83187}\left[-575 f_n + 98035 f_{n+1} + 127 f_{n+2}\right]$$

$$h^3 f_{n+\frac{2}{3}} = \left[-\frac{72}{79} y_n - \frac{50640}{1027} y_{n+\frac{1}{3}} + \frac{157440}{1027} y_{n+\frac{2}{3}} - \frac{12240}{79} y_{n+1} + \frac{53160}{1027} y_{n+\frac{4}{3}} + \frac{96}{1027} y_{n+\frac{5}{3}}\right] + \frac{h^3}{83187}\left[-406 f_n + 79792 f_{n+1} - 55 f_{n+2}\right]$$

$$h^3 f_{n+\frac{4}{3}} = \frac{1}{79}\left[72 y_n - 480 y_{n+\frac{1}{3}} - 3360 y_{n+\frac{2}{3}} + 12240 y_{n+1} - 12840 y_{n+\frac{4}{3}} + 4368 y_{n+\frac{5}{3}}\right] + \frac{h^3}{6399}\left[13 f_n - 7280 f_{n+1} - 14 f_{n+2}\right] \quad (12)$$

3. The Block method

The general block formula proposed in [11] is adopted with modification for the implementation of the new method in the normalized form.

$$h^{\lambda}\sum_{j=1}^{q} a_{ij} y_{n+j}^{\lambda} = h^{\lambda}\sum_{j=0}^{q} e_{ij} y_{n}^{\lambda} + h^{\mu-\lambda}\left[\sum_{j=1}^{q} d_{ij} f_{n} + \sum_{j=1}^{q} b_{ij} f_{n+j}\right],\ i = 0, 1, \cdots, q \qquad (13)$$

where λ is the power of the derivative of the continuous method, μ is the order of the problem to be solved and $q = r + s.$

In vector notation, (13) can be written as

$$h^{\lambda}\bar{a}Y_m = h^{\lambda}\bar{e}y_m + h^{\mu-\lambda}\left[\bar{d}f(y_m) + \bar{b}F(Y_m)\right] \qquad (14)$$

The matrices $\bar{a} = (a_{ij}),\ \bar{b} = (b_{ij}),\ \bar{e} = (e_{ij}),\ \bar{d} = (d_{ij})$ are constant coefficients matrices and $Y_m = (y_{n+v_i},\ y_{n+j},\ \cdot\ \cdot\ \cdot\ ,\ y'_{n+v_i},\ y'_{n+j},\ \cdot\ \cdot\ \cdot\ , y''_{n+v_i},\ y''_{n+j})^T$, $y_m = (y_n - (r-1),\ y_n - (r-2),\ \cdot\ \cdot\ \cdot\ , y_n)^T$, $\bar{F}(Y_m) = (f_{n+v_i},\ f_{n+j})^T$ and $F(y_m) = (f_{n-j},\ \cdot\ \cdot\ \cdot\ , f_n),\ i = 1,\ \cdot\ \cdot\ \cdot\ , q$. The normalized version of (13) is given as

$$\bar{A}Y_m = h^{\lambda}\bar{E}y_m + h^{\mu-\lambda}\left[\bar{D}F(y_m) + \bar{B}F(Y_m)\right] \qquad (15)$$

and our block method is in the form

$$A^0 Y_m = A^i y_m + B_i F_m \qquad (16)$$

Combining and solving equations (9), (10), (11) and (12) using matrix inversion to obtain the coefficients of (16) of the new block method as

$A^0 = I_{18\times18}$ Identity matrix,

$$Y_m = \left[\ y_{n+\frac{1}{3}}\ \cdot\ \cdot\ \cdot\ y_{n+2}\ y'_{n+\frac{1}{3}}\ \cdot\ \cdot\ \cdot\ y'_{n+2}\ y''_{n+\frac{1}{3}}\ \cdot\ \cdot\ \cdot\ y''_{n+2}\ \right]^T,$$

$$A^i = \begin{bmatrix}
0&0&0&0&0&1&0&0&0&0&0&\frac{1}{3}h&0&0&0&0&0&\frac{1}{18}h^2\\
0&0&0&0&0&1&0&0&0&0&0&\frac{2}{3}h&0&0&0&0&0&\frac{2}{9}h^2\\
0&0&0&0&0&1&0&0&0&0&0&h&0&0&0&0&0&\frac{1}{2}h^2\\
0&0&0&0&0&1&0&0&0&0&0&\frac{4}{3}h&0&0&0&0&0&\frac{8}{9}h^2\\
0&0&0&0&0&1&0&0&0&0&0&\frac{5}{3}h&0&0&0&0&0&\frac{25}{18}h^2\\
0&0&0&0&0&1&0&0&0&0&0&2h&0&0&0&0&0&2h^2\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&\frac{1}{3}h\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&\frac{2}{3}h\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&h\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&\frac{4}{3}h\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&\frac{5}{3}h\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&2h\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix},$$

$$y_m = \left[\ y_{n-\frac{1}{3}}\ \cdot\ \cdot\ \cdot\ y_n\ y'_{n-\frac{1}{3}}\ \cdot\ \cdot\ \cdot\ y'_n\ y''_{n-\frac{1}{3}}\ \cdot\ \cdot\ \cdot\ y''_n\ \right]^T,$$

$F_m = \left[\ f_n\ \ f_{n+1}\ \ f_{n+2}\right]^T$ *and*

$$B_i = \begin{bmatrix} \frac{23713h^3}{6531840} & \frac{1469h^3}{816480} & \frac{139h^3}{6531840} \\ \frac{479h^3}{25515} & \frac{296h^3}{25515} & \frac{h^3}{7290} \\ \frac{409h^3}{8960} & \frac{19h^3}{672} & \frac{3h^3}{8960} \\ \frac{2152h^3}{25515} & \frac{256h^3}{3645} & \frac{16h^3}{25515} \\ \frac{176125h^3}{1306368} & \frac{25625h^3}{163296} & \frac{1375h^3}{1306368} \\ \frac{h^3}{5} & \frac{8h^3}{35} & \frac{h^3}{210} \\ \frac{14879h^2}{544320} & \frac{143h^2}{8505} & \frac{107h^2}{544320} \\ \frac{1073h^2}{17010} & \frac{328h^2}{8505} & \frac{4h^2}{8505} \\ \frac{661h^2}{6720} & \frac{h^2}{14} & \frac{h^2}{1344} \\ \frac{1136h^2}{8505} & \frac{1664h^2}{8505} & \frac{8h^2}{8505} \\ \frac{18575h^2}{108864} & \frac{500h^2}{1701} & \frac{275h^2}{108864} \\ \frac{47h^2}{210} & \frac{8h^2}{105} & \frac{h^2}{35} \\ \frac{959h}{8640} & \frac{49h}{540} & \frac{h}{960} \\ \frac{169h}{1620} & \frac{16h}{405} & \frac{h}{1620} \\ \frac{103h}{960} & \frac{13h}{60} & \frac{h}{960} \\ \frac{14h}{135} & \frac{64h}{135} & 0 \\ \frac{665h}{5184} & -\frac{25h}{324} & \frac{95h}{5184} \\ \frac{11h}{60} & -\frac{16h}{15} & \frac{11h}{60} \end{bmatrix}.$$

Equation (16) can be explicitly written as follows

$$y_{n+\frac{1}{3}} = y_n + \frac{1}{3}hy'_n + \frac{1}{18}h^2 y''_n + \frac{h^3}{6531840}\left[23713 f_n + 11752 f_{n+1} + 139 f_{n+2}\right]$$

$$y_{n+\frac{2}{3}} = y_n + \frac{2}{3}hy'_n + \frac{2}{9}h^2 y''_n + \frac{h^3}{25515}\left[479 f_n + 296 f_{n+1} + 4 f_{n+2}\right]$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + h^3\left[\frac{409}{8960} f_n + \frac{19}{672} f_{n+1} + \frac{3}{8960} f_{n+2}\right]$$

$$y_{n+\frac{4}{3}} = y_n + \frac{4}{3}hy'_n + \frac{8}{9}h^2 y''_n + \frac{h^3}{25515}\left[2152 f_n + 1792 f_{n+1} + 16 f_{n+2}\right]$$

$$y_{n+\frac{5}{3}} = y_n + \frac{5}{3}hy'_n + \frac{25}{18}h^2 y''_n + \frac{h^3}{1306368}\left[176125 f_n + 205000 f_{n+1} + 1375 f_{n+2}\right]$$

$$y_{n+2} = y_n + 2hy'_n + 2hy''_n + \frac{h^3}{210}\left[42 f_n + 48 f_{n+1} + f_{n+2}\right]$$

$$y'_{n+\frac{1}{3}} = y'_n + \frac{1}{3}hy''_n + \frac{h^2}{544320}\left[14879 f_n + 9152 f_{n+1} + 107 f_{n+2}\right]$$

$$y'_{n+\frac{2}{3}} = y'_n + \frac{2}{3}hy''_n + \frac{h^2}{17010}\left[1073 f_n + 656 f_{n+1} + 8 f_{n+2}\right]$$

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{6720}\left[661 f_n + 480 f_{n+1} + 5 f_{n+2}\right]$$

$$y'_{n+\frac{4}{3}} = y'_n + \frac{4}{3}hy''_n + \frac{h^2}{8505}\left[1136 f_n + 1664 f_{n+1} + 8 f_{n+2}\right]$$

$$y'_{n+\frac{5}{3}} = y'_n + \frac{5}{3}hy''_n + \frac{h^2}{17010}\left[1073 f_n + 656 f_{n+1} + 8 f_{n+2}\right]$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{210}\left[47 f_n + 16 f_{n+1} + 6 f_{n+2}\right]$$

$$y''_{n+\frac{1}{3}} = y''_n + \frac{h}{8640}\left[959 f_n + 784 f_{n+1} + 9 f_{n+2}\right]$$

$$y''_{n+\frac{2}{3}} = y''_n + \frac{h}{1620}\left[169 f_n + 64 f_{n+1} + f_{n+2}\right]$$

$$y''_{n+1} = y''_n + \frac{h}{960}\left[103 f_n + 208 f_{n+1} + f_{n+2}\right]$$

$$y''_{n+\frac{4}{3}} = y''_n + \frac{h}{135}\left[14 f_n + 64 f_{n+1}\right]$$

$$y''_{n+\frac{5}{3}} = y''_n + \frac{h}{5184}\left[665 f_n - 400 f_{n+1} + 95 f_{n+2}\right]$$

$$y''_{n+2} = y''_n + \frac{h}{60}\left[11 f_n - 64 f_{n+1} + 11 f_{n+2}\right].$$

The four-point hybrid block method in equation (16) is a zero-stable block method of order six with error constants

$$C_9 = \begin{pmatrix} -4.390\times10^{-6}, -2.998\times10^{-5}, -6.628\times10^{-5}, -1.166\times10^{-4}, -3.806\times10^{-5}, -6.702\times10^{-4}, \\ -8.328\times10^{-6}, -1.952\times10^{-5}, -3.031\times10^{-5}, -2.601\times10^{-5}, -1.952\times10^{-5}, -6.702\times10^{-4}, \\ -8.864\times10^{-6}, -4.827\times10^{-6}, -4.464\times10^{-5}, -1.537\times10^{-4}, 1.027\times10^{-3}, 1.905\times10^{-3}, \end{pmatrix}^T.$$

4. Numerical Experiments

Problem 1

$y''' = e^x,\ y(0)=3,\ y'(0)=1,\ y''(0)=5,\ h=0.1$

Theoretical solution: $y(x) = 2 + 2x^2 + e^x$

Table 1: Comparison of absolute error in Problem 1 using the New block Method (16) with [12]

	Exact Solution	Numerical Solution	Error in [12], $p=6, k=2.$	Error in 4PHBM $p=6, k=2.$
0.1	3.1251709180756477	3.1251709180756505	6.34270e-13	2.84630e-15
0.2	3.3014027581601697	3.3014027581601869	2.32882e-12	1.70451e-14
0.3	3.5298588075760033	3.5298588075761317	5.44348e-12	1.28555e-13
0.4	3.8118246976412706	3.8118246976416542	9.85317e-12	3.83913e-13
0.5	4.1487212707001282	4.1487212707010162	1.59974e-11	3.01208e-13
0.6	4.5421188003905097	4.5421188003922070	2.37223e-11	1.69799e-12
0.7	4.9937527074704775	4.9937527074734183	3.35679e-11	2.94179e-12
0.8	5.5055409284924695	5.5055409284971565	4.53443e-11	4.68892e-12
0.9	6.0796031111569526	6.0796031111640457	5.97084e-11	7.09603e-12
1.0	6.7182818284590482	6.7182818284692931	2.64322e-11	1.02479e-11

Table 1 shows the exact, numerical solution and the maximum absolute error of the new block method for Problem 1 compared with that of [12]. It is observed that the new block method performs better.

Problem 2 x

$y''' = 3\sin x,\ y(0)=1,\ y'(0)=0,\ y''(0)=-2,\ h=0.1$

Theoretical solution: $y(x) = 3\cos x + \dfrac{x^2}{2} - 2.$

Table 2: Comparison of absolute error in Problem 2 using the new block Method (16) with [12] and [6]

x	Exact Solution	Numerical Solution	Error in [12], $p=6, k=2.$	Error in [6], $p=6, k=2$	Error in 4PHBM, $p=6, k=2$
0.1	0.9900124958340770	0.9900124958340767	1.7282e-12	9.88099e-15	6.11595e-16
0.2	0.9601997335237251	0.9601997335237210	6.3179e-12	2.01506e-13	3.91341e-15
0.3	0.9110094673768181	0.9110094673767834	1.4295e-11	1.11977e-12	3.46333e-14
0.4	0.8431829820086554	0.8431829820085459	2.5020e-11	3.70648e-12	1.09373e-13
0.5	0.7577476856711178	0.7577476856708200	3.8928e-11	9.27203e-12	2.89056e-13
0.6	0.6560068447290348	0.6560068447283941	5.5360e-11	1.95163e-11	6.40742e-13
0.7	0.5395265618534650	0.5395265618522155	7.4644e-11	3.64735e-11	6.24976e-12
0.8	0.4101201280414957	0.4101201280393092	9.6128e-11	6.25218e-11	2.18703e-12
0.9	0.2698299048119925	0.2698299048083906	1.2002e-10	1.00344e-10	3.60282e-12
1.0	0.1209069176044184	0.1209069175988407	1.4570e-10	1.52909e-10	5.57843e-12

Table 2 above presents the exact, numerical solution and the maximum absolute error of the new block method for Problem 2 is compared with that of [12] implemented in block mode (BM) and [6] implemented in Predictor-Corrector Mode (PCM) of same order of accuracy. It is observed that the new block method performs better than these methods.

Problem 3.

$$y''' = -6(y)^4,\ \ y(1) = -1,\ \ y'(1) = -1,\ y''(1) = -2,\ \ h = \frac{1}{20}.$$

Theoretical solution: $y(x) = \dfrac{1}{x-2}$

Table 3: Comparison of absolute error in Problem 3 using the new block Method (16) with [6]

x	Exact Solution	Numerical Solution	Error in [6], $p=6, k=2$	Error in 4PHBM, $p=6, k=2$
1.05	1.0526315789473684	1.0526315789134564	1.017989e-09	3.39120e-11
1.10	1.1111111111111112	1.1111111113454312	2.573804e-08	2.34320e-10
1.15	1.1764705882352944	1.1764705881246332	1.217663e-07	2.22831e-10
1.20	1.2500000000000002	1.2499996367894231	3.624816e-07	4.78942e-09
1.25	1.3333333333333337	1.33333246458223601	8.645410e-07	3.12490e-09
1.30	1.4285714285714290	1.4285696173562405	1.819265e-06	1.12151e-09
1.35	1.5384615384615392	1.5384572157689081	3.551325e-06	3.22692e-08
1.40	1.6666666666666676	1.6666604235674526	6.632889e-06	2.43099e-08
1.45	1.8181818181818195	1.8181696026814550	1.211591e-05	2.15500e-08
1.50	2.0000000000000018	2.0000005423879468	2.203049e-05	5.42210e-07

In Table 3 above, the exact, numerical solution and the maximum absolute error of the new block method for Problem 2 is compared with that of [6] implemented in PCM of the same order of accuracy with the new block method. It is observed that the new block method performs better.

5. Conclusion

This paper presented accurate four-point hybrid block method for the solution of general third order ordinary differential equations directly. The introduction of many points of interpolation as against block methods in literature has reduced function evaluation and enhanced order of accuracy of the new developed method. The block method obtained was applied on some linear and non-linear third order ODE problems to test its usability and accuracy. The results in the Tables above revealed that the developed method performed better than the existing methods.

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