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NEW MULTIPLE INTEGRAL COLLOCATION METHODS FOR THE NUMERICAL SOLUTION OF FOURTH-ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

This paper presents new collocation approaches for the numerical solutions of linear and nonlinear foruth-order Volterra integro-differential equations. These approaches involve the use of Chebyshev and Berstein polynomials as basis functions. Approximations to the lower order derivatives of the function through successive integration of Chebyshev and Berstein polynomials to the highest order derivatives are generated. We successfully implemented the new approaches on both linear and nonlinear integro-differential equations. Numerical results show that the new methods are accurate and highly promising in comparison with other numerical methods.

Keywords: Volterra integro-differential equations, Collocation, Chebyshev polynomials, Berstein polynomials, Multiplr integral.

1.0 Introduction

We consider the fourth-order integro-differntial equation of the form:

$$y^{\prime v}(x)= f(x)+\lambda y(x)+\int_0^x \left(g(t)y(t)+h(t)F(y(t))\right)dt \tag{1}$$

With the boundary conditions:

$$y(x_0)=\alpha_0,\quad y'(x_0)=\alpha_1,\ \ y(x_n)=\beta_0, \tag{2}$$

where $x\in(x_0,x_n)$ and F is a real nonlinear continuous function, $\lambda,\ \alpha_0,\ \alpha_1,\ \beta_0$ and β_1 are real constants, and f, g and h are given and can be approximated by Taylor polynomials [1].

Integro-differential equations are usually difficult to solve analytically and these have been of great interest to many researchers. Numerical methods for solution of linear and nonlinear integro-differential equations have been studied by authors [1-9].

A standard method for solving integro-differential equations is the collocation method, where one looks for an approximate solution in a finite dimensional space and determines the approximate solutions by requiring that after substituting the approximate solution into the original equation, the equality would hold at certain points (so-called collocation points [2]). Collocation method is a simple and yet powerful method for solving both linear and nonlinear boundary value problems of ordinary differential equations, partial differential equations and integro-differntial equations.

Collocation method has successful been applied to many boundary value problems. For example, Abubakar and Taiwo [3] solved Fredholm-Volterra integro-differential equation with the method, Venkatesh et. al [4] employed wavelet collocation method for solving nonlinear integro-differential equation while Sweilam et al. [5] used Pseudospectral collocation method for solving fourth-order integro-differential equations.

In this paper, we proposed and applied Multiple Integral Collocation Method (MICM) for solving fourth-order integro-differential equations of the form of equations (1) and (2). The use of the proposed method is justified by the interesting properties of the Chebyshev and Berstein polynomials used as basis functions.

This paper is organized as follows: In Section 2, we describe the properties of Chebyshev and Berstein polynomials. In Section 3, we discuss the solution technique while numerical examples are presented in Section 4. Finally, concluding remarks are given in Section 5.

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2. Chebyshev and Berstein Polynomials

2.1 Chebyshev Polynomials

The Chebyshev polynomial of degree n of the first kind which is valid in the interval $-1 \le x \le 1$ is defined by

$$T_n(x) = \cos(n\cos^{-1} x) \tag{3}$$

with

$$T_0(x) = 1, \quad T_1(x) = x \tag{4}$$

The recurrence relation is given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \; n \ge 1 \tag{5}$$

On a general note, the Chebyshev polynomial of degree n of the first kind which is valid in the interval $a \le x \le b$ is defined by

$$T_n(x) = \cos\left[n\cos^{-1}\left(\tfrac{2x-a-b}{b-a}\right)\right], \; a \le x \le b, \tag{6}$$

and this satisfies the recurrence relation

$$T_{n+1}(x) = 2\left(\tfrac{2x-a-b}{b-a}\right)T_n(x) - T_{n-1}(x), \; n \ge 1 \tag{7}$$

The Chebyshev polynomials of the first kind are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on the interval [-1,1], on the interval [-1,1], that is,

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, \; n \ne m \\ \pi, \; n = m = 0 \\ \frac{\pi}{2}, \; n = m \ne 0. \end{cases} \tag{8}$$

2.2 Berstein Polynomials

The general form of Berstein polynomials of degree m on interval $[a,b]$ is defined as (see [10])

$$B_{i,m}(x) = 2\binom{n}{i}\frac{(x-a)^i(b-x)^{m-i}}{(b-a)^m}, i = 0,1,\ldots,m, \tag{9}$$

where the binomial coefficients

$$\binom{n}{i} = \frac{\mathrm{m}}{\mathrm{i(m-i)}}, \; m \ge 1.$$

These $(m+1)$ B-polynomials of degree m form a complete basis over the interval $[a,b]$. The B-polynomial can be generated by a recursive relation:

$$B_{i,m}(x) = \frac{b-x}{b-a}B_{i,m-1}(x) + \frac{x}{b-a}B_{i-1,m-1}(x). \tag{10}$$

The derivatives of the degree m B-polynomials are combinations of B-polynomials of degree $m-1$, which can be formulated as

$$B'_{i,m}(x) = \frac{m}{b-a}\left(B_{i-1,m-1}(x) - B_{i,m-1}(x)\right), \tag{11}$$

$$B''_{i,m}(x) = \frac{m(m-1)}{(b-a)^2}\left(B_{i-2,m-2}(x) - 2B_{i-1,m-2}(x) + B_{i,m-2}(x)\right), \tag{12}$$

$$B'''_{i,m}(x) = \frac{m(m-1)(m-2)}{(b-a)^3}\left(B_{i-3,m-3}(x) - 3B_{i-2,m-3}(x) + 3B_{i-1,m-3}(x) - B_{i,m-3}(x)\right), \tag{13}$$

where the set $B_{i,m} = 0$ if $i < 0$ or $i > m$.

3. Description of Numerical Solution Technique

In this section, we described the development of the new integral collocation methods for the numerical solution of fourth-order integro-differential equations of the form of equations (1) and (2).

3.1 Chebyshev-Multiple Integral Collocation Method

The construction process of Chebyshev-Multiple Integral Collocation Method (CMICM) starts with fourth order integration of equation (1) to obtain

$$y(x) = \int\int\int\int f(x)dxdxdxdx + \lambda\int\int\int\int y(x)dxdxdxdx$$
$$+\int\int\int\int\left[\int_0^x \left(g(t)y(t) + h(t)F(y(t))\right)dt\right]dxdxdxdx + \sum_{i=0}^{3}\frac{1}{i!}k_{i+1}x^i \tag{14}$$

Following [11], the function $y(x)$ and other derivatives are obtained through successive integration of the fourth-order derivative.

Here, we assume that the $y'^{v}(x)$ and its derivatives have truncated series expansion of the form:

$$y'^{v}(x) = \sum_{n=0}^{N} a_n T_n(x) = \sum_{n=0}^{N} a_n \phi_n'^{v}(x) \tag{15}$$

$$y'''(x) = \sum_{n=0}^{N} a_n \phi_n'''(x) + c_1 \tag{16}$$

$$y''(x) = \sum_{n=0}^{N} a_n \phi_n''(x) + c_1 x + c_2 \tag{17}$$

$$y'(x) = \sum_{n=0}^{N} a_n \phi_n'(x) + \tfrac{1}{2} c_1 x^2 + c_2 x + c_3 \tag{18}$$

$$y(x) = \sum_{n=0}^{N} a_n \phi_n(x) + \tfrac{1}{6} c_1 x^3 + \tfrac{1}{2} c_2 x^2 + c_3 x + c_4, \tag{19}$$

where $\phi_n^{(j)}(x) = \int \phi_n^{(j+1)}(x)dx,\ j = 0,1,2,3.$

Substituting equation (15) - (19) into (14) gives a new trial function. That is

$$y_{ntf}(x) = \iiiint f(x)dxdxdxdx$$

$$+\lambda \iiiint \left(\sum_{n=0}^{N} a_n \phi_n(x) + \tfrac{1}{6} c_1 x^3 + \tfrac{1}{2} c_2 x^2 + c_3 x + c_4 \right) dxdxdxdx$$

$$+\iiiint \left[\int_0^x \left(g(t) \left(\sum_{n=0}^{N} a_n \phi_n(t) + \tfrac{1}{6} c_1 t^3 + \tfrac{1}{2} c_2 t^2 + c_3 t + c_4 \right) \right.\right.$$

$$\left.\left. + h(t) F\left(\left(\sum_{n=0}^{N} a_n \phi_n(t) + \tfrac{1}{6} c_1 t^3 + \tfrac{1}{2} c_2 t^2 + c_3 t + c_4 \right) \right) \right) dt \right] dxdxdxdx + \sum_{i=0}^{3} \frac{1}{i} k_i x^i \tag{20}$$

Now, substituting equation (20) into equation (1), we obtain

$$y_{ntf}^{iv}(x) = f(x) + \lambda y_{ntf}(x) + \int_0^x \left(g(t) y_{ntf}(t) + h(t) F(y_{ntf}(t)) \right) dt \tag{21}$$

which yields the following residual function

$$R(x) = \tilde{y}_{ntf}^{iv}(x) - f(x) - \lambda \tilde{y}_{ntf}(x) - \int_0^x \left(g(t) \tilde{y}_{ntf}(t) + h(t) F(\tilde{y}_{ntf}(t)) \right) dt \tag{22}$$

Collocating equation (22) at point $x = x_j$, we have

$$R(x_j) = \tilde{y}_{ntf}^{iv}(x_j) - f(x_j) - \lambda \tilde{y}_{ntf}(x_j) - \int_0^{x_j} \left(g(t) \tilde{y}_{ntf}(t) + h(t) F(\tilde{y}_{ntf}(t)) \right) dt \tag{23}$$

where

$$x_j = a + \frac{(b-a)j}{N+6}, \quad j = 1, 2, \cdots, N+5, \tag{24}$$

and the subscript $'ntf'$ means 'new trial function'. Thus, equation (23) constitutes (N+5) linear or nonlinear algebraic system of equations in $(N+9)$ unknowns. Extra 4 equations are obtained from the given boundary conditions. Altogether, we have linear or nonlinear algebraic system of $(N+9)$ equations. These equations can be solved by using Guassian elimination for the linear case which Newon's method can be employed for nonlinear equations. The values of constants so obtained are substituted into equation (20) which eventually gives the approximate solution.

3.2 Bernstein-Multiple Integral Collocation Method

Bernstein approximations are adopted here to approximate the solution of equation (1). We start with Bernstein approximation for the fourth-order derivative and generate approximations to the third, second, first derivatives and function y itself. Thus we have

$$y'^{v}(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) = \sum_{i=0}^{m} c_i \varphi_i''''(x) \tag{25}$$

Similarly, successive integration of (25) gives

$$y'''(x) = \sum_{i=0}^{m} c_i \varphi_i'''(x) + q_1 \tag{26}$$

$$y''(x)=\sum_{i=0}^{m} c_i\varphi_i''(x)+q_1x+q_2 \quad (27)$$

$$y'(x)=\sum_{i=0}^{m} c_i\varphi_i'(x)+\tfrac{1}{2}q_1x^2+q_2x+q_3 \quad (28)$$

$$y(x)=\sum_{i=0}^{m} c_i\varphi_i(x)+\tfrac{1}{6}q_1x^3+\tfrac{1}{2}q_2x^2+q_3x+q_4, \quad (29)$$

where $\varphi_i^{(k)}(x)=\int\varphi_i^{(k+1)}(x)dx,\ k=0,1,2,3.$

Similarly, substituting equations (25)-(29) into (14) gives a new trial function. That is

$$y_{ntf}(x)=\iiiint f(x)dxdxdxdx$$

$$+\lambda\iiiint\left(\sum_{i=0}^{m}c_i\varphi_i(x)+\tfrac{1}{6}q_1x^3+\tfrac{1}{2}q_2x^2+q_3x+q_4\right)dxdxdxdx$$

$$+\iiiint\left[\int_0^x\left(g(t)\left(\sum_{i=0}^{m}c_i\varphi_i(t)+\tfrac{1}{6}q_1t^3+\tfrac{1}{2}q_2t^2+q_3t+q_4\right)\right.\right.$$

$$\left.\left.+h(t)F\left(\left(\sum_{i=0}^{m}c_i\varphi_i(t)+\tfrac{1}{6}q_1t^3+\tfrac{1}{2}q_2t^2+q_3t+q_4\right)\right)\right)dt\right]dxdxdxdx+\sum_{i=0}^{3}\frac{1}{i!}k_{i+1}x^i \quad (30)$$

Following the same procedure as discussed in Method 1, we obtain a residual function and collocating it at point $x=x_j$, we have

$$R(x_j)=\tilde{y}_{ntf}^{iv}(x_j)-f(x_j)-\lambda\tilde{y}_{ntf}(x_j)-\int_0^{x_j}\left(g(t)\tilde{y}_{ntf}(t)+h(t)F(\tilde{y}_{ntf}(t))\right)dt \quad (31)$$

where

$$x_j=a+\frac{(b-a)j}{m+6},\quad j=1,2,\cdots,m+5. \quad (32)$$

Similarly, equation (31) constitutes $(m+9)$ linear or nonlinear algebraic system of equations in $(m+9)$ unknowns. Extra 4 equations are obtained from the given boundary conditions. Altogether, we have a system of $(m+9)$ linear or nonlinear algebraic equations. The solution of these equations can be obtained either by Guassian elimination method for linear case and Newton's method for nonlinear algebraic equations. The values of the unknown constants obtained are then substituted in (30) to get the approximate solution.

4. Numerical Examples

Example 1: We consider the integro-differential equation [1,6]

$$y^{iv}(x)=x(1+e^x)+3e^x+y(x)-\int_0^x y(t)dt,\qquad 0<x<1 \quad (33)$$

subject to the boundary conditions

$$y(0)=1,\quad y'(0)=1,\quad y(1)=1+e,\quad y'(1)=2e \quad (34)$$

The exact solution to this problem is $y(x)=1+xe^x$.

Method 1: Chebyshev-Multiple Integral Collocation Method

Using the proposed method (CMCM) for case $N=6$, we obtain the following trial solution

$$y_6(x)=1+k_3x+k_4+\frac{1}{24}c_4x^4-\frac{1}{40320}c_1x^7(x-8)+\frac{1}{2}x^2(-1+k_2)+\tfrac{1}{6}x^3(-2+k_1)+\frac{1}{120}x^5(1-c_4+c_3)$$

$$+e^x(x-1)-\frac{1}{1816214400}x^8\left[5005a_0(x-9)+a_6(2048x^7-46080x^6-45045+365365x\right.$$

$$-876876x^2+1054872x^3-710528x^4+263424x^5)+5a_5(256x^6+9009-51051x+85085x^2$$

$$-68432x^3+28392x^4-5376x^5)+7a_4(128x^5-6435+23595x-25168x^2+12064x^3-2496x^4)$$

$$+91a_3(8x^4+495-1045x+627x^2-144x^3)+91a_2(8x^3-495+495x-132x^2)$$

$$\left.+1001a_1(x^2+45-15x)\right]-\frac{1}{5040}x^6(7c_3+c_2x-7c_2)\ . \quad (35)$$

Substituting equation (35) into equation (33), we obtain a residual equation which is collocated at points

$$x_j = \frac{j}{12}, \quad j = 1, 2, \cdots, 11. \tag{36}$$

Using the boundary conditions (34), we obtain extra four equations which are solved together with equations obtained from the residual equation having collocated it at $x = x_j$ using (36). Thus, we get the following results

$a_0 = 8.1028421774115924438,$ $a_1 = 4.7297512699363044175,$
$a_2 = 0.68821725084416654003,$ $a_3 = 0.065682572218730633685,$
$a_4 = 0.0046325795763991388327,$ $a_5 = 0.00025600286566246282460,$
$a_6 = 0.000011788136510570500289,$ $c_1 = 2.9999993294688326025,$
$c_2 = 1.0000000000255297166,$ $c_3 = 0.99999999910054226743,$
$c_4 = 1.0000000000255297166,$ $k_1 = 2.9999999999994229266,$
$k_2 = 2.0000000000000127918,$ $k_3 = 1,\ k_4 = 1.$

Substituting these values into equation (35) we obtain the following approximate solution

$$\begin{aligned} y_6(x) = {} & 2 + x + 0.000099206695212003253452x^8 + 0.041666666667730404858x^4 \\ & -0.000074404745274524618118x^7(x-8) + 0.50000000000000639590x^2 \\ & +0.16666666666657048777x^3 + 0.0083333333256251045902x^5 \\ & +0.0000027550312310351806796x^9 + 2.7656984063547698082*10^{-7}x^{10} \\ & +2.4022027933391708003*10^{-8}x^{11} + 2.8395061908711169232*10^{-9}x^{12} \\ & -2.0631503836064248252*10^{-10}x^{13} + 1.1866091490032031461*10^{-10}x^{14} \\ & -1.3292540557793393225*10^{-11}x^{15} + e^x(x-1) \\ & -\frac{1}{5040}x^6(-7.0000001945036057540 + 2.0000000268867716609x) \end{aligned} \tag{37}$$

Method 2: Bernstein-Multiple Integral Collocation Method

Similarly, using the proposed BMICM for case $m = 6$ and following the same procedure as discussed in Method 1, we obtain the following approximate solution:

$$\begin{aligned} y_6(x) = {} & 2 + x + 0.041666666667730404858x^4 - 0.000074404745274524618118x^7(x-8) \\ & +0.50000000000000639590x^2 + 0.16666666666657048777x^3 \\ & +0.0083333333256251045902x^5 + e^x(x-1) - \frac{1}{1816214400}x^8\big(-502.3101271678584446x^2 \\ & -43.62915304982826179x^3 - 5.157520327492662083x^4 \\ & +0.3747123436071511372x^5 - 0.2155136623591362378 3x^6 \\ & +0.02414210357364838 4592x^7 - 5003.727394255822054x \\ & -1.8018062842045136177*10^5\big) - \frac{1}{5040}x^6\big(-7.0000001945036057540 \\ & +2.0000000268867716609x\big) \end{aligned}$$

Example 2: Consider the nonlinear fourth-order integro-differential equation [1,6]

$$y^{iv}(x) = 1 + \int_0^x e^{-t}y^2(t)dt, \qquad 0 < x < 1 \tag{39}$$

subject to the boundary conditions

$$y(0) = 1, \quad y'(0) = 1, \quad y(1) = e, \quad y'(1) = e \tag{40}$$

The exact solution to this problem is $y(x) = e^x$.

Using the proposed methods and following the same procedures as discussed in Example 1, we obtain the results presented in Table 2.

5. Conclusion

In this work, we have proposed and employed powerful new collocation techniques, known as Multiple Integral Collocation Methods to solve fourth-order linear and nonlinear Volterra integro-differential equations. The MICM's results are compared with results obtained using method of weighted residual and optimal homotopy asymptotic method. The results produced by the new methods are in excellent agreement with the exact solutions. The solutions obtained are valid in the given domain and for better results; using large numbers N and m are recommended.

Table 1: Comparison of Absolute Errors for Example 1

X	Exact Solution	CMICM	BMICM	OHAM[1]	MWR[6]
0	1.000000000	0	0	0	0
0.1	1.110517091	2.500E-17	2.500E-17	7.565E-12	4.824E-04
0.2	1.244280551	7.360E-17	7.360E-17	2.203E-11	1.489E-03
0.3	1.404957642	1.418E-16	1.418E-16	1.280E-11	2.500E-03
0.4	1.596729879	2.244E-16	2.242E-16	2.349E-11	3.174E-03
0.5	1.824360635	3.168E-16	3.168E-16	5.993E-11	3.339E-03
0.6	2.093271280	4.142E-16	4.140E-16	9.569E-11	2.976E-03
0.7	2.409626895	5.124E-16	5.122E-16	1.001E-10	2.199E-03
0.8	2.780432742	6.066E-16	6.065E-16	6.603E-11	1.229E-03
0.9	3.213642800	6.923E-16	6.923E-16	1.875E-11	3.736E-04
1.0	3.718281825	4.441E-16	4.441E-16	4.574E-14	4.390E-10

Table 2: Comparison of Absolute Errors for Example 2

X	Exact Solution	CMICM	BMICM	OHAM[1]	MWR[6]
0.0	1.000000000	0	0	0	0
0.1	1.105170918	3.476E-13	3.476E-13	2.692E-08	3.448E-02
0.2	1.221402758	1.223E-12	1.223E-12	7.684E-08	1.216E-03
0.3	1.349858808	2.375E-12	2.375E-12	1.124E-07	2.038E-03
0.4	1.491824698	3.552E-12	3.552E-12	1.174E-07	2.588E-03
0.5	1.648721271	4.503E-12	4.503E-12	9.530E-08	2.721E-03
0.6	1.822118800	4.977E-12	4.977E-12	5.861E-08	2.426E-03
0.7	2.013752707	4.732E-12	4.732E-12	2.653E-08	1.793E-03
0.8	2.225540928	3.568E-12	3.568E-12	8.508E-09	1.002E-03
0.9	2.459603111	1.552E-12	1.552E-12	1.843E-09	3.044E-04
1.0	2.718281828	4.441E-16	4.441E-16	4.590E-10	4.590E-10

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