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COMPUTABLE CRITERIA FOR NULL CONTROLLABILITY OF IMPULSIVE QUASI - LINEAR FRACTIONAL MIXED VOLTERRA –FREDHOLM- TYPE INTEGRO - DIFFERENTIAL SYSTEMS IN BANACH SPACES WITH MULTIPLE DELAYS IN THE LIMITED CONTROL POWERS

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Abstract

The Impulsive Quasi - Linear Fractional Mixed Volterra - Fredholm – Type Integro – Differential Equations in Banach Spaces with Multiple Delays in the Limited Control Powers is presented for Null Controllability Analysis. Computable Criteria for the System to be Null Controllable are established. Uses were made of the Stability of the free base system and the Properness of the control base system to establish the null controllability of the system. The mild solution of the system is established using the variation of constant formula. The set functions upon which our study hinges are extracted from the mild solution. We also made use of Schauder fixed point theorem to establish results.

Keywords: Multiple-Delays, Null Controllability, Free base system, Control base system, Impulsive Quasi – Linear ,Fractional System, Set functions.

1.0 INTRODUCTION AND PRELIMINARIES

The pioneering work of Vito Volterra on the Integration of the differential equations of dynamics and partial differential dynamical systems published in 1884 gave vent to the conception of integral equation of volterra type[1].It is equally observed in[2]that the mixed initial boundary hyperbolic partial differential equation which arises in the study of lossless transmission lines can be replaced by an associated neutral differential equation. This equivalence has been the basis of a number of investigations of the stability properties of distributed network[3]which study has been extended to compartmental models governed by neutral Volterraintegro-differential equations. Compartmental models have been found in[4] to have numerous applications in Applied Mathematics; these models are used to vividly describe the evolutions of systems, in theoretical epidemiology, physiology, population dynamics chemical reaction kinetics and the analysis of ecosystems[5] . Most of these models can be divided into separate compartments. A paradigm for such systems can be seen as one in which compartments are connected by pipes through which materials pass in definite time. An example of compartmental model is given in [5]as the radio cardiogram where the two compartments correspond to the left and right ventricles of the heart and the pipe between these compartments represent the pulmonary and systemic circulations. Other applications of VolterraIntegro-differential equation arise in tracer kinetics in the modeling of uptake of potassium by red blood cells as well as in modeling the kinetic of lead in a body [4, 5]. The wide application of Volterraintegro-differential equations in Bio-Mathematics and economic models underscores the immense interest the study has generated. Literature on the relative controllability of Volterra equations is still scanty. However, sufficient conditions for the relative controllability of Non-linear neutral Volterra Integro-differential equations have been provided in [2]. However, the systems with delays in the state, investigation into their relative controllability are still attracting attention and interest. The controllability and approximate controllability of delay Volterra systems were investigated byusing fixed point theorem [6] .Thecontrollability and Local null controllability of Nonlinear Integrodifferential Systems and Functional Differential systems in Banach spaces were studied and it was shown that the controllability problem in Banach spaces can be converted into one of a fixed point problem for a single-valued mapping [7].

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Balachandran and co-workers studied the controllability of Sobolev-type Partial Functional Differential Systems in Banach Spaces [8].In [6] the Retarded Functional Differential Systems of Sobolev-Type in Banach Spaces werestudied and it was established that once a system of the Sobolev – type is controllable with a single delay in the control of the system, then it is also controllable with either multiple delays or distributed delays or both multiple and distributed delays in the control .However, necessary and sufficient conditions for the target set of Nonlinear Infinite spaces of Functional Differential systems with Distributed Delays in the control to be on the boundary of the Attainable set of the system have been provided in [9] .It was made clear in [10] that whenever an optimal control is in use to steer the system of interest from the initial point to the target (desired point) , then the target set must be on the boundary of the attainable set of the system. Optimality conditions for the relative controllability of neutral Volterra Integro-differential equation has been provided in [11] ; though there are studies in the optimal controllability of ordinary and functional differential systems. From [12, 13,14]), we gain clarity of meaning and understanding of the conceptual frame work of optimal controllability.

Many processes studied Applied Sciences are represented by differential Equations. However, the situation is quite different in many physical phenomena that have sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flow, population dynamics, theoretical physics, radio physics, pharmacokinetics, mathematical economy, chemical technology, biotechnology and medicine etc. Adequate mathematical models of such processes are systems of differential equation with impulses. The theory of impulsive differential and integro-differential equations is a new and important branch of differential equations [15, 16] .

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of sciences and engineering [17, 18]. There has been significant development in fractional differential equations in recent years [19, 20, and 21] Recently, the existence results for Impulsive fractional differential and Integro-differential equations in Banach spaces were studied using standard fixed point theorems [22].

Controllability is the most important qualitative behavior of any dynamical system. It is well known that the issue of controllability plays an important role in control theory and engineering [23, 24and 25] because they have close connections to structural decomposition, quadratic optimal control, observer design etc. The literature related to controllability of Impulsive fractional Integro-differential equations and controllability of Impulsive Quasi-linear Integro-differential equations is limited, to our knowledge, to the recent works [17,26].The study of controllability of Impulsive Quasi-linear fractional mixed Volterra-Fredholm-type integro-differential equations is presented in [27].

One of the celebrated triumphs ofLaSalle was his solution of the null controllability problem of Linear Ordinary Differential Control System

$$\dot{x} = Ax + Bu \qquad (1)$$

Where A isnxn square constant matrix, B is nxm constant matrix, while the control powers are small. i.e. Control powers are square integrable and lie in the unit cube,

$$C^m = \{ u \in R^m : |u_i| \leq 1 ; i = 1,2, \ldots , n \} \qquad (2)$$

Here, u_i denotes the ith component of u. In his work,[28] "The Time Optimal Control Problem, in Theory of Nonlinear Oscillations", he showed that if system (1) is proper (and this holds if and only if

$$rank[B, AB, A^2B, \ldots , A^{n-1}B] = n \ ) \qquad (3)$$

And if the system

$$\dot{x} = Ax \qquad (4)$$

is stable (i.e., all the Eigen values of A have no positive real part) , then system(1) is null controllable with constraints (i.e. , limited control powers u lie in $\bar{B}_1(0)$).

The condition in system (3) is equivalent to the controllability of system (1) when the controls are unlimited "big" in the sense that they are only assumed to be square integrable. This is equivalent to null controllability with square integrable controls. We call such controls unrestrained or unlimited in contrast to the restrained controls or limited controls which lie in a closed and bounded set.

For the delay system

$$\dot{x}(t) = L(t, x_t) + B(t)u(t), t > t_0 \qquad (5)$$

$$x_{t_0} = \phi \in W_2^{(1)}([-h , 0] , R^n) \equiv W_2^{(1)}$$

Null controllability is not equivalent to controllability. For example, all nth order scalar differential difference equations of retarded type are null controllable [29], whereas they are never controllable [29].In[30],it was proved that if system (5) is controllable with unrestrained controls, and if

$$\dot{x}(t) = L(t, x_t)t > t_0 \qquad (6)$$

is uniformly asymptotically stable, then system (5) is null controllable with constrained controls

(Limited control powers).The problem was posed on whether the weaker condition of null controllability with unrestrained control powers and the uniform asymptotic stability assumption was sufficient for restrained null controllability. The issue is affirmatively settled in this paper and extends any other results in the literature.

The nonlinear infinite delay system of the form:

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) + \int_{-\infty}^{0} A(\theta)\, x(t + \theta)d\theta + f\big(t, x_t, u(t)\big), t > t_0 \quad (7)$$

was studied in [31] and established that system (7) is Euclidean null controllability if the linear base system

$$\dot{x}(t) = L(t, x_t) + B(t)u(t), t > t_0 \qquad (8)$$

is proper and the free system

$$\dot{x}(t) = L(t, x_t) + \int_{-\infty}^{0} A(\theta)\, x(t+\theta) d\theta\ , t > t_0 \qquad (9)$$

is uniformly asymptotically stable, provided that f satisfies some growth conditions. An analogous result was obtained in [32]for the delay system

$$\frac{d}{dt} D(t\ , x_t) = L(t, x_t) + B(t)u(t) + \int_{-\infty}^{0} A(\theta)\, x(t+\theta) d\theta + f\big(t, x_t, u(t)\big), t > t_0$$

$$x(t) = \phi(t), t \in (-\infty\ , 0)$$

Oraekieextended this result in[33] to perturbed delay systems with distributed delays in the control.

In this paper, therefore, we shall considerImpulsive Quasi − linear fractional mixed Volterra − Fredholm − type Integro − differential equations in Banach spaces with Multiple delays in the limited control powers of the form:

$$D^q x(t) = A(t\ , x)x(t) + \sum_{j=0}^{m} B_j u\big(t - h_j\big) + \quad f\left(t\ , x(t)\ , \int_0^t g(t\ , s\ , x(s))\, ds\ , \int_0^b k(t\ , s\ , x(s)) ds\right) \qquad (10)$$

$$t \in J = [0\ , b], t \neq t_k\ , k = 1,2,3, \dots , m$$

$$\Delta_x|_{t=t_k} = I_k\big(x({t_k}^{-1})\big)\ , k = 1,2,3, \dots , m \qquad (11)$$

$$x(0) = x_0 \qquad (12)$$

with the main objective, of investigating the null controllabilityof the systems (10)

Here, A is an nxn continuous matrix and $f\left(t\ , x(t)\ , \int_0^t g\big(t\ , s\ , x(s)\big)\, ds\ , \int_0^b k\big(t\ , s\ , x(s)\big) ds\right)$ *is continuous.*

The controls of interest, u, are square integrable with values in the unit cube C^m *, given by*

$$C^m = \{ u \in R^m : |u_i| \leq 1\ ; i = 1,2, \dots , n\ \}$$

Here, the state variable $x(.)$ *takes values in the Banach space X and* $u(.)$ *is a control function , an admissible square integrable m − dimensional vector function, with* $u : J \to C^m$ *as a Banach space. i. e.* $u \in L_2(J\ , C^m)$ *. Here,* $0 < q < 1$*, and* $A(t, x)$ *is a bounded linear operator on a Banach space X .*

Further more,

$$f : JxXxXxX \to X, \qquad g : \Omega x X \to X\ , \qquad k : \Omega x X \to X\ , \qquad I_k : X \to X\ ,$$

$$\Delta_x I_{t=t_k} = x(({t_k}^{+}) - x({t_k}^{-})\ , \qquad for\ all\quad k = 1,2, \dots\ , m\ ;$$

$$0 < t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = t_1\ ;\ \Omega = \{(t\ , s)\ , 0 \leq s \leq t_1\}\ .$$

Thus the control space will be the Lebesgue space of square integrable functions.

The constraint control set C^m *is the closed and bounded subset of* L_2 *.*

Let $h > 0$ *, be given. For a function* $u: [-h\ , t_1] \to X$ *and* $t \in [t_0\ , t_1]$*,*

we use the ssymbol u_t *to denote the function defined on the delay interval* $[-h\ , 0\ ]$ *by* $u_t(s) = u(t + s)$*, for* $s \in [-h\ , 0\ ]$.

Here, we develop sufficient computable criteria for the null controllability of system (10)'

Our objective, therefore, is to study the controllability of the perturbed system described by system (10)-(12) through its linear base control system

$$D^q x(t) = A(t\ , x)x(t) + \sum_{j=0}^{m} B_j u\big(t - h_j\big) \qquad (13)$$

and it free system

$$D^q x(t) = A(t\ , x)x(t) \qquad (14)$$

Definition1.1 (Solution)

A continuous function $x : [t_0 - \gamma\ , t_0 + a] \to R^n$ *is said to be a solution of system*(14) *if there exists* $t_0 \in R$ *,* $a > 0$ *such that* $x \in B([t_0 - \gamma\ , t_0 + a], R^n)\ , t \in (t_0\ , t_0 + a)$ *and x satisfies system*(14) *on* $[t_0\ , t_0 + a]$.

Given $t_0 \in R\ , \phi \in B$ *, we say that* $x(t_0\ , \phi)$ *is a solution of system*(14) *with initial value* $(t_0\ , \phi)$ *if there exists an* $a > 0$ *such that* $x(t_0\ , \phi)$ *is a solution of system*(14) *on* $[t_0 - \gamma\ , t_0 + a]$ *and* $x_{t_0}(t_0\ , \phi) = \phi$.

Definition 1.2. (*complete state*)

The complete state for system(10)*is given by the set*

$z(t) = \{x, u_t\}$

Definition 1.3.

The system(10) is said to be relatively controllable on the interval $[t_0, t_1]$ *if for every initial complete state* $z(0)$ *and* $x_1 \in X$, *there exists a control function* $u(t)$ *defined on* $[t_0, t_1]$ *such that the solution* $x(.)$ *of the system* (10)*satisfies* $x(t_1) = x_1$

2.0. Basic Definitions of Fractional Calculus

Let X *be a Banach space and* $R^+ = [0, \infty)$. *Suppose* $f \in L_1(R^+)$. *Let* $C(J, X)$ *be the Banach space of continuous functions* $x(t)$ *with* $x(t) \in X$ *for* $t \in J = [t_0, t_1]$ *and*
$\|x\|_{C(J,X)} = \max_{t\in J}\|x(t)\|$.

Let $B(X)$*denote the Banach space of bounded linear operators from* X *into* X *with the norm* $\|A\|_{B(X)} = sup\{\|A(y)\| : \|y\| = 1\}$

Also consider the Banach space $PC(J, X) = \{x : J \to X : x \in C\{[t_k, t_{k+1}], X)\}$, $k = 0,1,2, \dots, m$ *and there exists* $x(t_k^-)$ *and* $x((t_k^+)$ *with* $x((t_k^-) = x(t_k)$ *with the norm* $\|x\|_{PC} = \operatorname{Sup}_{t\in J}\|x(t)\|$.

Definition 2.0.1.

The Riemann – Liouville fractional integral operator of order $\alpha > 0$, *of function* $f \in L_1(R^+)$ *is defined as*

$$\int_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds$$

where $\Gamma(.)$ *is the Euler Gamma function.*

Definition 2.0.2.

The Riemann – Liouville fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in N$ *is defined as* : $D_{0^+}{}^{\alpha}\mathrm{f}(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}\mathrm{f}(s)\mathrm{ds}$,

Where the function $f(t)$*has absolutely continuous derivatives up to order* $(n-1)$.

Definition 2.0.3

The Caputo fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in N$ *is defined as*:

$$D_{0^+}{}^{\alpha}\mathrm{f}(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\mathrm{f}^n(s)\mathrm{ds},$$

Where the function $f(t)$ *has absolutely continuous derivatives up to order* $(n-1)$.

If $n = 1$, *then*, $n-1 < \alpha < n = 0 < \alpha < 1$, *implies that*

$$D_{0^+}{}^{\alpha}\mathrm{f}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{1-\alpha-1}\mathrm{f}^1(s)\mathrm{ds} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\mathrm{f}^1(s)\mathrm{ds}$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-s)^{\alpha}}\mathrm{f}^1(s)\mathrm{ds} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\mathrm{f}^1(s)}{(t-s)^{\alpha}}\mathrm{ds}$$

Where , $\mathrm{f}^1(s) = Df(s) = \frac{df(s)}{ds}$ *and* f *is an abstract function with values in* X.

2.1. *Variation of Constant Formula*

From the works of[21]and [28] ,we have the Mild solution of systems (10) – (12) as the following integral equation :

$$\begin{aligned}
x(t) = x_0 &+ \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s,x)x(s)ds + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s,x)x(s)ds \\
&+ \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\left[\sum_{j=0}^{m}B_j(s)u(t-h_j)\right]ds \\
&+ \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right)ds \\
&+ \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\sum_{j=0}^{m}B_j(s)u(s-h_j) + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right)ds \\
&+ \sum_{0<t_k<t} I_k(x(t_k^-)) \qquad (15)
\end{aligned}$$

One may assume without loss of generality that ,
$h_m > h_{m-1} > h_{m-2} > \ldots > h_1 > h_0 = 0$
The initial control $u_0(t)$ *is given on* $\left[t_0 - h_m , t_0\right]$, $t_0 = 0$
The solution of the system(10) *for* $t > t_0 + h_m$ *is given by*

$$x(t , x_0 , u) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s , x)x(s)ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} A(s , x)x(s)ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \sum_{j=0}^{m} B_j(s)u(t - h_j)\, ds$$
$$+ \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s , x(s) , \int_0^s g(s , \tau , x(\tau))d\tau , \int_0^b k(s , \tau , x(\tau))d\tau \right) ds$$
$$+ \frac{1}{\Gamma(q)} \sum_{j=0}^{m} B_j\, (s)u(s - h_j) \int_{t_k}^{t} (t - s)^{q-1} ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f\left(s , x(s) , \int_0^s g(s , \tau , x(\tau))d\tau , \int_0^b k(s , \tau , x(\tau))d\tau \right) ds$$
$$+ \sum_{0<t_k<t} I_k\left(x\,(t_k^-)\right) \tag{16}$$

$$\Rightarrow x(t , x_0 , u) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s , x)x(s)ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} A(s , x)x(s)ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_0}^{t_1} \left(t_k - s + h_j\right)^{q-1} \sum_{j=0}^{m} B_j\left(s + h_j\right)u\left(t - h_j + h_j\right) ds, for\ k = 1$$
$$+ \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s , x(s) , \int_0^s g(s , \tau , x(\tau))d\tau , \int_0^b k(s , \tau , x(\tau))d\tau \right) ds$$
$$+ \sum_{j=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_j}}^{t-h_j} \left(t - s + h_j\right)^{q-1} B_j\left(s + h_j\right)u\left(t - h_j + h_j\right)ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f\left(s , x(s) , \int_0^s g(s , \tau , x(\tau))d\tau , \int_0^b k(s , \tau , x(\tau))d\tau \right) ds$$
$$+ \sum_{0<t_k<t} I_k\left(x\,(t_k^-)\right) \tag{17}$$

$$\Rightarrow, x(t , x_0 , u) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s , x)x(s)ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} A(s , x)x(s)ds$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{0-h_j}}^{t_0} \left(t_0 - s + h_j\right)^{q-1} B_j\left(s + h_j\right)u_0\left(s - h_j + h_j\right)ds , for\ k = 1$$
$$+ \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \sum_{0<t_1<t} \int_{t_{1-h_j+1}}^{t_{1-h_j}} \left(t_1 - s + h_j\right)^{q-1} B_j\left(s + h_j\right)u\left(s - h_j + h_j\right)ds$$
$$+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s , x(s) , \int_0^s g(s , \tau , x(\tau))d\tau , \int_0^b k(s , \tau , x(\tau))d\tau \right) ds$$
$$+ \sum_{j=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_j}}^{t-h_j} \left(t - s + h_j\right)^{q-1} B_j\left(s + h_j\right)u\left(t - h_j + h_j\right)ds$$

$$+ \quad \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1} f\left(s\,, x(s)\,, \int_0^s g\big(s\,,\tau\,,x(\tau)\big)d\tau\,, \int_0^b k\big(s\,,\tau\,,x(\tau)\big)d\tau\right)ds$$

$$+ \sum_{0<t_k<t} I_k\big(x\,(t_k^{\,-})\big) \qquad (18)$$

$$\Rightarrow, x(t\,,x_0\,,u) \;=\; x_0 \;+\; \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s\,,x)x(s)ds \quad + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s\,,x)x(s)ds$$

$$+ \quad \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{0-h_j}}^{t_0}\big(t_0-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u_0\big(s-h_j+h_j\big)ds\,, for\; k=1$$

$$+ \quad \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{1-h_j+1}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u\big(s-h_j+h_j\big)ds$$

$$+ \quad \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1} f\left(s\,, x(s)\,, \int_0^s g\big(s\,,\tau\,,x(\tau)\big)d\tau\,, \int_0^b k\big(s\,,\tau\,,x(\tau)\big)d\tau\right)ds$$

$$+ \quad \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t-h_j}\big(t-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u\big(s-h_j+h_j\big)ds$$

$$+ \quad \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1} f\left(s\,, x(s)\,, \int_0^s g\big(s\,,\tau\,,x(\tau)\big)d\tau\,, \int_0^b k\big(s\,,\tau\,,x(\tau)\big)d\tau\right)ds$$

$$+ \sum_{0<t_k<t} I_k\big(x\,(t_k^{\,-})\big) \qquad (19)$$

Put

$$F\big(s\,,x(s),x_s\,,u(s)\big) = f\left(s\,, x(s)\,, \int_0^s g\big(s\,,\tau\,,x(\tau)\big)d\tau\,, \int_0^b k\big(s\,,\tau\,,x(\tau)\big)d\tau\right),$$

system(19) *becomes*

$$x(t\,,x_0\,,u) \;=\; x_0 \;+\; \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s\,,x)x(s)ds \quad + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s\,,x)x(s)ds$$

$$+ \quad \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{0-h_j}}^{t_0}\big(t_0-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u_0(s)ds$$

$$+ \quad \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{1-h_j+1}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds$$

$$+ \quad \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}F\big(s\,,x(s),x_s\,,u(s)\big)\,ds$$

$$+ \quad \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t-h_j}\big(t-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds$$

$$+ \quad \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}G(s)\,ds + \sum_{0<t_k<t} I_k\big(x\,(t_k^{\,-})\big) \qquad (20)$$

For brevity, let

$$\mu(t) \;=\; x_0 \;+\; \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s\,,x)x(s)ds \quad + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s\,,x)x(s)ds$$

$$+ \quad \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{0-h_j}}^{t_0}\big(t_0-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u_0(s)ds \qquad (21)$$

$$\beta(t) \;=\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}G(s)\,ds \;+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}F\big(s\,,x(s),x_s\,,u(s)\big)\,ds$$

$$+\sum_{0<t_k<t} I_k\big(x\,(t_k^{\,-})\big) \tag{22}$$

$$Z(t,s)u(s)ds = \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\sum_{0<t_1<t}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds$$
$$+\ \sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t-h_j}\big(t-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds \tag{23}$$

$\Rightarrow$

$$z(t,s)=\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\left[\sum_{0<t_1<t}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}+\int_{t_{1-h_{j+1}}}^{t-h_j}\big(t-s+h_j\big)^{q-1}\right]B_j\big(s+h_j\big)$$

Then,

$$z(t_1,s)=\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\left[\sum_{0<t_1<t}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}+\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}\right]B_j\big(s+h_j\big)$$

$$\Rightarrow\ z(t_1,s)=\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds+\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds$$

$$\Rightarrow\ z(t_1,s)=2\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds \tag{24}$$

Substituting equations (21), (22) *and* (24) *in* (20) *, we have the mild solution*:

$$x(t\ ,x_0\ ,u\ )\ =\ \mu(t)+\beta(t)+\ z(t_1,s)$$
$$=\ \mu(t)+\beta(t)+\ 2\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds \tag{25}$$

2.2 . Basic Set Function and Properties

We shall define the set functions upon which our study hinges

Definition 2.2.1. (Reachableset)

The reachable set $R(t_1,t_0)$ *of the systems* (10) − (12) *is given as*

$$R(t_1,t_0)=\left\{\begin{matrix} 2\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)u(s)ds\ : \\ |u_j|\le 1\ , for\ every\ jand\ u_j\in U=C^m\subseteq L_2(J\ ,X^m);\ j=1\ ,2\ ,\dots,m\end{matrix}\right\}$$

Definition 2.2.2. (Attainable set)

The attainable set $A(t_1,t_0)$ *of the systems* (10) − (12) *is given as*

$$A(t_1,t_0)=\left\{x(t\ ,x_0\ ,u\ ):u\in U\right\},$$

where $U=\{u\in C^m\subseteq L_2([t_0\ ,t_1]\ ,X^m):|u_j|\le 1\ ;\ j=1\ ,2\ ,\dots,m\}$

Definition 2.2.3.(Targetset)

The target set $G(t_1,t_0)$ *of the systems* (10) − (12) *is given as*

$G(t_1,t_0)=\{x(t\ ,x_0\ ,u):t\ge\tau>t_0=0\ , for\ fixed\ \tau\ and\ u\in U=C^m\subseteq L_2([t_0\ ,t_1]\ ,X^m)\}$.

Definition 2.2.4. (Controllability grammian)

The controllability grammian $W(t_1,t_0)$ *of the systems* (10) − (12) *is given as*

$$W(t_1,t_0)=2\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\Big[\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)\Big]\Big[\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)\Big]^T ds$$
$$=2\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\Upsilon(t_1,s)\Upsilon^T(t_1,s)\ \ ds\ ,$$

Thus,

$$W^{-1}\big((t_1,t_0)\big)=\frac{1}{W(t_1,t_0)}=\frac{1}{2\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\Upsilon(t_1,s)\Upsilon^T(t_1,s)ds}$$

where T denotes matrix transpose and

$$\Upsilon(t_1,s)=\big(t_1-s+h_j\big)^{q-1}B_j\big(s+h_j\big)\ .$$

2.3. Relationship between the set functions
We shall first establish the relationship between the attainable set and the reachable set to enable us see that once a property has been proved for one set, and then it is applicable to the other.
From the equation (20), *we have the attainable set* $A(t_1, t_0)$ *as*:
$A(t_1, t_0) = [\,\eta(t) + R(t_1, t_0)]\,, for\ u \in U\,, t \in [t_0\,, t_1]\,, where\ \ \eta(t) = \mu(t)\ + \beta(t).$

This means that the attainable set is the translation of the reachable set through the origin $\eta \in X^n$. *Using the attainable set, therefore, it is easy to show that the set functions possess the prperties of conexity, closedness, boundedness,, and compactness. Also, the set functions are continuous on* $[0\,,\infty)$ *to the metric space of compact subsets of* R^n.
[12] *and* [29] *gave the impetus for adoption of the proofs of these properties for systms*(10) − (12).
Definition 2.3.1. (*Relative controllability*)
System(10) *is relatively controllable on the interval* $[t_0\,, t_1]$ *if*
$A(t_1, t_0) \cap G(t_1, t_0) \neq \emptyset\,; t_1 > t_0$
***Definition* 2.3.2. (*Properness*)**
System(10) *is proper in* X^n *on* $[t_0\ , t_1]$ *if span* $R(t_1, t_0) = X^n$
i.e.

$$2C^T \sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} \left(t_1 - s + h_j\right)^{q-1} B_j\left(s + h_j\right) u(s) ds = 0\ \ a.e\ \ , \Rightarrow C = 0, C \in R^n\ .$$

Definition 2.3.3. (Null controllability)
The system(10)*is said to be null controllable on* $[t_0, t_1]$ *if for each* $\phi \in B([-\gamma\,,0]\,, R^n)$, *there exists a* $t_1 > t_0, u \in L_2([t_0, t_1]\,, P)$, P *a compact convex subset of* R^m, *such that the solution* $x(t\,, t_0\,, \phi\,, u\,, f)$ *of system*(10)*satisfies*
$x_{t_0}(t_0\,, \phi\,, u\,, f) = \phi\,, and\ x(t_1\,, t_0\,, \phi\,, u\,, f) = 0.$

Definition 2.3.4. (Controllability)
The system(10)*is said to be null controllable on*$[t_0, t_1]$ *if for each function* ϕ *and every* $x_1 \in R^n$, there exists an admissible control function, u, such that a solution of *system*(10) satisfies $x(t_1) = x_1$.

3. MAIN RESULT
The issue of controllability of Neutral Volterra Integrodifferential Equations have been settled in [2,3]*as contained in* [9].
From the results of these studies the following equivalent statements emerge.
Theorem 3.1.(Necessary conditions)
Consider the system

$$D^q x(t) = A(t\,, x)x(t) + \sum_{j=0}^{m} B_j u\left(t - h_j\right) + \quad f\left(t\,, x(t)\,, \int_0^t g(t\,, s\,, x(s))\, ds\,, \int_0^b k(t\,, s\,, x(s)) ds\right) \quad (3.1)$$

$t \in J = [0\,, b], t \neq t_k\,, k = 1,2,3, \dots, m$

$$\Delta_x|_{t=t_k} = I_k\left(x({t_k}^{-1})\right), k = 1,2,3, \dots, m \quad (3.2)$$

$$x(0) = x_0 \quad (3.3)$$

with the same conditions on the system'*sparameters as in the system*(10), *then the following statements are equivalent* :
(*i*). *System*(3.1)*is relatively controllable on the interval* $J = [t_0\,, t_1]$.
(*ii*). *The controllability grammian* $W(t\,, t_0)$ *of system*(3.1) *is non − singular*.
(*iii*). *System*(3.1) *is proper on the interval* $J = [t_0\,, t_1]$.

PROOF:
$\big((i) = (ii).\big)$
Recall: *The controllability grammian* $W(t\,, t_0)$ *of the system*(3.1) *is non − singular, is equivalent to saying that* $W(t\,, t_0)$ *is positive definite, which in turn is equivalent to saying that* C^T *times the controllability index of the system*(3.1) *is equal to zero almost everywhere on the interval* $[t_0\,, t_1]$, *implying that* $C = 0.$; T *denotes matrix transpose*
i.e.

$$2C^T\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds=0\ \ a.e\ \ ,\Rightarrow C=0,C\in R^n\ .$$

$which\ is\ properness\ of\ the\ system(3.1)\ since\ the\ integral\ is\ non-negative.$
$This\ ,therefore, showsed\ that\ (i)\ is\ equivalent\ to\ (ii), or\ (i)=(ii).$

$To\ show\ that\ (ii)\ and\ (iii)\ are\ equivalent.$
$By\ \ the\ definition\ \ of\ properness\ of\ the\ system(3.1), we\ have\ (ii)\ given\ as:$

$$2C^T\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds=0\ \ a.e\ \ ,\Rightarrow C=0,C\in R^n\ .$$

$implies\ that\ C=0\ ,C\in X^n\ \ ,for\ each\ s\in[t_0\ ,t_1]\ ,\ \ then$

$$\sum_{]=0}^{m}\frac{1}{\Gamma(q)}2C^T\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds=0\ \ a.e\ \ ,\Rightarrow C=0,C\in R^n\ .$$

$$=2C^T\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds=0\ \ a.e\ \ ,\Rightarrow C=0,C\in R^n$$

$implies\ that\ C=0\ ,C\in X^n\ \ ,for\ each\ s\in[t_0\ ,t_1]\ ,$ (3.2)

It follows from this last equation (3.2) that C is orthogonal to the reachable set

$$R(t,t_0)=\left\{\begin{matrix} 2\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds\ \ : \\ \left|u_j\right|\le 1\ for\ every\ jand\ u_j\in U=C^m\subseteq L_2(J\ ,X^m); j=1\ ,2\ ,\dots,m \end{matrix}\right\}$$

$If\ we\ assume\ the\ relative\ controllability\ of\ the\ system(3.1)\ now, R(t,t_0)=R^n\ ,sothat$
$C=0, showing\ that\ (iii)\ implies\ (ii). Or\ (i)\ is\ equivalent\ to\ (ii)\ and$
$(ii)\ is\ equivalent\ to\ (iii)\ and\ vis-a-vis\ (iii)\ is\ equivalent\ to\ (ii)\ is\ equivalent\ to\ (i).$
$Conversely, assume\ that\ system(3.1)\ is\ not\ controllable, sothat\ \ the$
$reachable\ R(t,t_0)\neq R^n\ for\ \ t>t_0\ .Then\ ,there\ exists\ C\neq 0\ ,C\in R^n\ \ ,such\ that$
$C^TR(t,t_0)=0\ .$
$It\ follows\ \ that\ for\ all\ admissible\ controls\ \ u\in L_2\ \ that$

$$0=2C^T\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds$$

$$=\sum_{]=0}^{m}\frac{1}{\Gamma(q)}2C^T\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds$$

Hence,

$$2C^T\sum_{]=0}^{m}\frac{1}{\Gamma(q)}\int_{t_{1-h_{j+1}}}^{t_{1-h_j}}\left(t_1-s+h_j\right)^{q-1}B_j\left(s+h_j\right)u(s)ds=0\ \ ae\ \ \ \Rightarrow\ C\neq 0\ ,C\in X^n\ \ ,$$

$for\ each\ s\in[t_0\ ,t_1]\ \ .$
$By\ definition\ of\ properness\ ,it\ implies\ that\ the\ system(3.1)\ is\ not\ proper\ ,since$
$C\neq 0\ .Hence\ the\ system(3.1)\ is\ relatively\ controllable. And\ or\ controllable.$

THEOREM 3.2 (Sufficient condition)
Assume for system (10) that:
$(i). The\ constraint\ set\ U\ is\ an\ arbitrary\ compact\ subset\ of\ R^n.$
$(ii). The\ system(12)\ is\ uniformly\ asymptotically\ stable\ sothat\ the\ solution\ of$
$system(12)\ satisfies$
$\|x(t,t_0\ ,\phi\ ,0,0)\|\le Me^{-a(t-t_0)}\|\phi\|\ ,for\ some\ a>0\ ,M>0.$
$(iii). The\ linear\ control\ system(11)\ is\ proper\ in\ R^n.$
$(iv). The\ continuous\ function\ F\ satisfies$
$\left|F\left(s\ ,x(s),x_s\ ,u(s)\right)\right|\le exp(-bt)\pi\left(x(s)\ ,u(s)\right)$
$for\ all\left(t\ ,x(.)\ ,u(.)\right)\in[t_0\ ,\infty)xBxL_2\ ,where\int_{t_0}^{\infty}\pi\left(x(s)\ ,u(s)\right)ds\le k<\infty\ \ and$
$b-a\ge 0$,*then* system (10) $is\ \ null\ controllable.$

PROOF

By theorem 3.1 *above*, (ii) *and* (iii), $W^{-1}(t_1, t_0)$ *exists for each* $t_1 > t_0$.

Suppose the pair of functions x, u *form a solution pair to the set of integral equations*

$$x(t, x_0, u) = \mu(t) + \beta(t) + z(t_1, s)$$

$$= \mu(t) + \beta(t) + 2\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} (t_1 - s + h_j)^{q-1} B_j(s + h_j)\, u(s) ds \qquad (3.3)$$

$$u(t) = \frac{[-Y^T(t_1, s) W^{-1}(t_1, t_0)][\mu(t) + \beta(t)]}{2} \qquad (3.4)$$

Then, u *is square integrable on* $[t_0, t_1]$ *and* x *is a solution of system*(10)*corresponding to* u *with initial state* $x(t_0) = \phi$. *Now*,

$$x(t_1, x_0, u) = \mu(t_1) + \beta(t_1) + z(t_1, s)$$

$$= \mu(t_1) + \beta(t_1) + 2\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} (t_1 - s + h_j)^{q-1} B_j(s + h_j)\, u(s) ds$$

$$= \mu(t_1) + \beta(t_1) + 2\left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} (t_1 - s + h_j)^{q-1} B_j(s + h_j)\right] x \left[\frac{[-Y^T(t_1, s) W^{-1}(t_1, t_0)][\mu(t) + \beta(t)]}{2}\right] ds \qquad (3.5)$$

$$\Rightarrow x(t_1, x_0, u) = \mu(t_1) + \beta(t_1) + 2\left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} Y(t_1, s)\right] x \left[\frac{[-Y^T(t_1, s) W^{-1}(t_1, t_0)][\mu(t) + \beta(t)]}{2}\right] ds$$

$$\Rightarrow x(t_1, x_0, u) = \mu(t_1) + \beta(t_1) + 2\left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} Y(t_1, s)\right] x \left[\frac{[-Y^T(t_1, s)][\mu(t) + \beta(t)]}{2W(t_1, t_0)}\right] ds$$

$$\Rightarrow x(t_1, x_0, u) = \mu(t_1) + \beta(t_1) + 2\left[\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} Y(t_1, s)\right] x \left[\frac{[-Y^T(t_1, s)]}{2W(t_1, t_0)}\right] ds. [\mu(t) + \beta(t)]$$

$$\Rightarrow x(t_1, x_0, u) = \mu(t_1) + \beta(t_1) + \left[\frac{-2\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_j+1}}^{t_{1-h_j}} [Y(t, s)][Y^T(t_1, s)]\, ds.}{2\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_j+1}}^{t_{1-h_j}} [Y(t, s)][Y^T(t_1, s)]\, ds}\right] [\mu(t) + \beta(t)]$$

$$\Rightarrow x(t_1, x_0, u) = \mu(t_1) + \beta(t_1) + [-1][\mu(t) + \beta(t)]$$

$$\Rightarrow x(t_1, x_0, u) = \mu(t_1) + \beta(t_1) - \mu(t_1) - \beta(t_1) = 0.$$

Then, equation(3.5) *is steered to zero or the origin in finite time* $t_1 \in [t_0, t_1]$.

It remains to prove that the function $u: [t_0, t_1] \to C^m$ *is in the arbitrary compact constraint subset of* R^m, *that is* $|u| \le \rho_1$, *for some constant* $\rho_1 > 0$.

By part (ii) *of theorem*3.2 *above*, $|Y^T(t_1, s) W^{-1}(t_1, t_0)| \le \lambda_1$, for some constant $\lambda_1 > 0$.

Thus,

$$|u(t)| \le \lambda_1[\lambda_2 \exp(-\rho(t_1 - t_0))] \int_{t_{1-h_j+1}}^{t_{1-h_j}} \lambda_3 \exp[-\rho(t_1 - s) exp(-ks)\pi(x(s), u(s)) ds]$$

It follows that

$$|u(t)| \le \lambda_1[\lambda_2 \exp(-\rho(t_1 - t_0))] + \lambda\lambda_3 \exp(-\rho t_1)(3.6)$$

since $k - \rho \ge 0$ *and* $t \ge t_0 \ge 0$. *Thus, by taking* t, *sufficiently large, we have* $|u(t)| \le \rho_1$; $t \in [t_0, t_1]$, *showing that* u *is an admissible control*.

Now, we need to prove the existence of a solution pair of the integral equations (3.3) and (3.4).

Let N *be the Banach space of all functions* $(x, u): [t_0 - h, t_1] x [t_0 - h, t_1] \to R^n R^m$, *where* $x \in N([t_0 - h, t_1], R^n)$; $u \in L_2([t_0 - h, t_1], R^m)$ *with the norm defined by*

$\|(x,u)\| = \|x\|_2 + \|u\|_2$

$$where, \quad \|x\|_2 = \left[\int_{t_{1-h_j+1}}^{t_{1-h_j}} |x(s)|^2 \, ds\right]^{1/2} \; ; \; \|u\|_2 = \left[\int_{t_{1-h_j+1}}^{t_{1-h_j}} |u(s)|^2 \, ds\right]^{1/2}.$$

Define the operator $T: N \to N$ *by* $T(x,u) = (y,v)$, *where*

$$\mathrm{v(t)} = -Y^T(t_1, s)W^{-1}(t_1, t_0)[\mu(t_1) + \beta(t_1)] \quad (3.7)$$

And $v(t) = u(t), t \in [t_0 - h, t_0]$.

$$y(t) = x(t, x_0, u) = \mu(t) + \beta(t) + z(t_1, s)$$

$$= \mu(t) + \beta(t) + 2\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} (t_1 - s + h_j)^{q-1} B_j(s + h_j) u(s) ds \quad (3.8)$$

And $y(t) = \phi(t)$, *for* $t \in [t_0 - h, t_0]$.

It was already shown that

$|v(t)| \leq \rho_1$, *for* $t \in [t_0, t_1]$ *and also, for the function* $v: [t_0 - h, t_0] \to C^m$,

we have

$v(t) \leq \rho_1$.

Thus, we have

$$\|v(t)\|_2 \leq \rho_1(t_1 + h - t_0)^{1/2} = k_0$$

$$And, |y(t)| \leq \lambda_2 exp[-\rho(t - t_0)] + \lambda_4 \int_{t_{1-h_j+1}}^{t_{1-h_j}} |v(s)| \, ds + \lambda\lambda_3 exp(-\rho t)$$

$$Put \quad \lambda_4 = sup \left| 2\sum_{]=0}^{m} \frac{1}{\Gamma(q)} \int_{t_{1-h_{j+1}}}^{t_{1-h_j}} (t_1 - s + h_j)^{q-1} B_j(s + h_j) \right|.$$

Since $\rho > 0; t \geq t_0 \geq 0$, *we have it that for* $t \in [t_0, t_1]$

$|y(t)| \leq \lambda_2 + \lambda_4 \rho(t_1 - t_0) + \lambda\lambda_3 = k_1$

And for $t \in [t_0 - h, t_0]$, *we have,* $|y(t)| \leq sup|\phi(t)| = \delta$

Thus, if $\beta = max[k_1, \delta]$, *then* $\|y(t)\|_2 \leq \beta(t_1 + h - t_0)^{1/2} = k_2 < \infty$.

Now, Let, $r = max\{k_0, k_2\}$. *Then if we put*

$G(r) = \{(x,u) \in N: \|x\|_2 \leq r, \|u\|_2 \leq r\}$,

we have $T: G(r) \to G(r)$.

Since $G(r)$ *is bounded, closed and convex, by Riesz theorem as contained in*[33], *it is relatively compact under the transformation* T.

Thus, the Schauders' fixed point theorem implies that the function T *has a fixed point.*

Hence, system(10) *is null controllable*(*Euclidean null controllable*).

4. CONCLUSION

We investigated system(10) *for null controllability through its linear base control and its free system*

We established necessary and sufficient conditions for Impulsive Quasilinear Fractional Mixed Volterra – Fredholm Type Integrodifferential Equations in Banach spaces with Multiple Delays in the Control, to be null controllable. The computable criteria are reported. The existence and form of the null control of the system have been established. We have also established the relationship between the Relative Controllability of our system and the Intersection of its two set functions namely, Attainable set and Targes set showing that if.

$A(t_1, t_0) \cap G(t_1, t_0) \neq \emptyset$ *for* $t \in [t_0, t_1]$,

then the system is relatively controllable and a null control for the system exists.

5. REFERENCES

[1] Robertson, E.F and J.J. O'Connor (2005), Biography of Vito Volterra (3/5/1860- 11/10/1940), Publication of the School of Mathematics and Statistics, University of St. Andrew, Scotland.

[2] Balachandran, K and Dauer (1989), Relative Controllability of Perturbations of Nonlinear Systems, Journal of Optimization Theory and Applications.vol.6, pp51-56.

[3] Balachandran, K and Dauer (1997), Asymptotic Neutral Volterra Integro-differential Systems, Journal of Mathematical Systems' Estimation, vol.7, N02, pp1-4.

[4] Burton, T.A. (1983),Volterra Integral and Differential Eqations, Academic Press, New York.
[5] Gyori, I and Wu (1991), Neutral Equation Arising from Compartmental Systems with Pipes, Journal of Dynamics and Differential Equations; 3, pp289-311.
[6] Oraekie, P.A. (2016); Controllability Results of Retarded Functional Differential Systems of Sobolev-Type in Banach Spaces with Multiple Delays in the Control, Journal of the Nigerian Association of Mathematical Physics ,Vol34,pp13-20.
[7] Chukwu, E.N. (1988),The Time Optimal Control theory of Linear Differential Equations of Neutral Type, Journal Computer Mathematics and Applications, vol.16, pp851-866.
[8] Brill (1977); A Semi linear Sobolev Equation in Banach Spaces, Journal of Differential Equations, 24, pp412-425.
[9] Oraekie, P.A. (2017); Necessary and Sufficient Conditions for the Target set of a Nonlinear Infinite Space of Neutral Functional Differential Systems with Distributed Delays in the Control to be on the Boundary of the Attainable set, Journal Computer Mathematics and Applications, vol.41, pp21-26.
[10] Oraekie, P.A. (2015); Location of the Target set on Semilnear Dynamical Systems with Multiple Delays in the Control, Reiko International Journal of Science and Technology, Vol. 6, N2A, pp62-65.
[11] Oraekie, P.A. (20I3), The Relative Controllability of Neutral VolterraIntegro-differential Systems with zero in the Interior of the Reachable set, African Journal of Sciences, Vol.14, N01; pp3271-3282.
[12] Chukwu, E.N. (1989), The Time Optimal Control theory of Linear Differential Equations of Neutral Type, Journal Computer Mathematics and Applications, vol.16, pp851-866.
[13] Hmanmed, A. (1986), Stability Conditions of Delay Differential Systems, International Journal of Control, vol.43, N02, pp455-463.
[14] Klamka,J. (1976) ,Relative Controllability of Nonlinear Systems with distributed Delays in Control, International Journal of Control, 28, pp307 – 312.
[15] Rogovchenko, Y.V. (1997), Nonlinear Impulsive Evolution Systems and Applications to Population Models, Journal of Mathematical Analysis and Applications, Vol.2007, N02, pp300 – 315.
[16] Hernandez E. (2002),A Second Order Impulsive Cauchy Problem, International Journal of Mathematics, Science, 31, N08, pp451 – 461.
[17] Banila, B., RiveroM. Rodriquez-Germa L and J .J Trujillo (2007), Fractional Differential Equations as Alternative Models to Nonlinear Differential Equations, Applied Mathematics Computation 87, pp79 – 88.
[18] Agarwal, R.P., Belmekki and M.Benchohra (2009), A survey on Semi-linear Differential Equations and Inclusions involving Riemann – Liouville Fractional Derivative, Advances in Differential Equations, Article ID 981728, 47 pages.
[19] Miller, K. S. and B. Ross (1993), an Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, Inc. , New York.
[20] Lakshmikatham,V and A.S.Vatsala (2008), Basic Theory of Fractional Differential Equations, Nonlinear Analysis 69,pp2677 – 2682.
[21] Balachandran K, Kiruthika and J.J.Trujillo (2010) ,Existence Results for Fractional Impulsive Integro –differential Equations in Banach Spaces, Commune Nonlinear Sci. Numer. Simulat, doi;10.1016/J.cnsns.2010.08.005.1, 2.3,2,2,4.
[22] Balachandran K, Park,J.Y and S.H.Park (2010) ,Controllability of Nonlocal Impulsive Quassi-linear Integrodifferential Systems in Banach Spaces, Reports on Mathematical Physics.65; 2, pp247-257.
[23] Benchohra and Ouahab (2005) ,Controllability Results for Fractional Semi-linear Differential Inclusions in Frechet Spaces, Nonlinear Analysis, 65; pp405 – 423.
[24] Balachandran, K and J.H.Kim (2006) ,Remarks on the paper, Controllability of Second Order Differential Inclusion in Banach Spaces, Journal of Mathematical Analysis and Applications; 285, pp537 – 550.
[25] Chang, Y.K, Nieto J.J and W.S.Li (2009), Controllability of Semi-linear Differential Systems with Nonlocal Initial Conditions in Banach Spaces, Journal of Optimization Theory and Applications 142; pp267 – 273.
[26] Tai, Z and X. Wang (2009), Controllability of Fractional-order Impulsive Neutral Functional Infinite Delay Integro-differential Systems in Banach Spaces, Applied Mathematics Letters, 22; pp1760 – 1765.
[27] Kavitha V and M. MallikaArjunan (2011) , Controllability of Impulsive Quasi-linear Fractional Mixed Volterra-Fredholm-Type Integro-differential Equations in Banach Spaces, Journal off Nonlinear Science and Application, vol.4, pp152- 169.
[28] J.P. LaSalle (1959),The Time Optimal Control Problem, in "Theory of Nonlinear Oscillations", Princeton Univ. Press, N,J. vol.5 , pp.1-24
[29] H.T. Banks, M.Q. Jacobs and C.E. Lange hop (1975),Characterization of the controlled states in ${W_2}^{(1)}$of linear hereditary systems,SIAM J. Control Ootim.13. , pp611-649.
[30] Chukwu, E.N. (1997); Function Space Null Controllability of Delay Systems with Limited Power, J. of Mathematical Analysis and Applications, 124, pp 293-304.
[31] Sinba, A.S.C (1985); Null Controllability of Nonlinear Infinite Delay systems with Restrained Control, International Journal of Control, 42,pp735-741.
[32] Onwuatu, J.U (1993); Null Controllability of Nonlinear Infinite Neutral Systems, KYBERNETIKA,Vol..29, N04, pp325-336.
[33] Oraekie, P.A (2018); Euclidean Null Controllability of Nonlinear Neutral Systems with Multiple Delays in Control,Journal of the Nigerian Association of Mathematical Physics, Vol.44, pp1-8.